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ON SOME INTRINSIC PROPERTIES OF THE INVERSE NAKAGAMI–m CUMULATIVE DISTRIBUTION FUNCTION

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ABSTRACT: F. Louzada, P. Ramos and D. Nasimento, *IEEE Trans. on Reliability*, (2018) presented some mathematical properties of the new inverse Nakagami–m distribution.

The aim of this note is to study "saturation" of the inverse Nakagami–m cumulative distribution function [1]:

$$M(t) = \frac{1}{\Gamma(a)}\Gamma(a, \frac{a}{\lambda t^2})$$

to the horizontal asymptote with respect to Hausdorff distance.

We prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t_0}(t)$ by means of the family $M(t)$.

Numerical examples on real datasets (1. Approximating cdf of the number of Bitcoin received per address [11] and 2. experimental growth data (mean height) of sunflower plants [12]) using *CAS Mathematica*, illustrating our results are given.

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Key Words: inverse Nakagami–m cumulative distribution function, Heaviside function, Hausdorff distance, Upper and lower bounds

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1. INTRODUCTION AND PRELIMINARIES

The Nakagami distribution [2] plays an important role in the general area of communications engineering.

Definition 1. *The new inverse Nakagami–m cumulative distribution function (IN-CDF) is given by [1]:*

$$M(t) = \frac{1}{\Gamma(a)}\Gamma(a, \frac{a}{\lambda t^2}) \tag{1}$$

where

$$\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}dt$$

is the upper incomplete gamma function and $a, \lambda \in R^+$.

The author's study [1] of inverse distributions has provided a better comprehension of standard distributions and contributed to adding more flexibility for fitting data.

For other results, see [3]–[8].

Definition 2. *The shifted Heaviside step function is defined by*

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0. \end{cases} \tag{2}$$

Definition 3. *[9] The Hausdorff distance (the H–distance)* $\rho(f,g)$ *between two interval functions* f,g *on* $\Omega \subseteq \mathbb{R}$*, is the distance between their completed graphs* $F(f)$ *and* $F(g)$ *considered as closed subsets of* $\Omega \times \mathbb{R}$.

More precisely,

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \tag{3}$$

wherein $||.||$ *is any norm in* $\mathbb{R}^2$*, e. g. the maximum norm* $||(t,x)|| = \max\{|t|,|x|\}$*; hence the distance between the points* $A = (t_A, x_A)$*,* $B = (t_B, x_B)$ *in* $\mathbb{R}^2$ *is* $||A-B|| = max(|t_A - t_B|, |x_A - x_B|)$.

2. MAIN RESULTS.

When studying the intrinsic properties of the family $M(t)$, it is also appropriate to study the "saturation" to the horizontal asymptote.

In this Section we give upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t_0}(t)$ by means of family (1).

Let t_0 is the unique positive root of the nonlinear equation $M(t_0) - \frac{1}{2} = 0$.

The one–sided Hausdorff distance d satisfies the relation

$$M(t_0 + d) = 1 - d. \tag{4}$$

The following theorem gives upper and lower bounds for d

Theorem. Let

$$\begin{aligned} p &= -\tfrac{1}{2}, \\ q &= 1 + \frac{2ae^{-\frac{a}{\lambda t_0^2}}\left(\frac{a}{\lambda t_0^2}\right)^{a-1}}{\lambda t_0^3 \Gamma(a)} \\ s &= 2.1q. \end{aligned} \tag{5}$$

Let $s > e^{1.05}$. With some constraints imposed on the parameters a and λ which we will not explore here, for the one–sided Hausdorff distance d between $h_{t_0}(t)$ and the cumulative function (1) the following inequalities hold:

$$d_l = \frac{1}{s} < d < \frac{\ln s}{s} = d_r. \tag{6}$$

The proof follows the ideas given in [10] and will be omitted.

We note that the function $G(d) = p + qd$ approximates $F(d) = M(t_0 + d) - 1 + d$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).

For some experiments, see Fig. 2–Fig. 3.

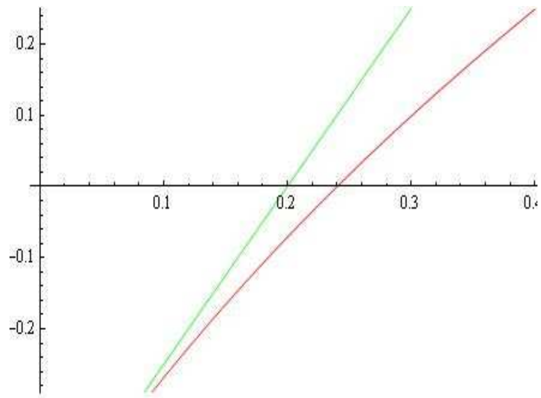

Figure 1: The functions $F(d)$ and $G(d)$ for $a = 1.9$ and $\lambda = 2.6$.

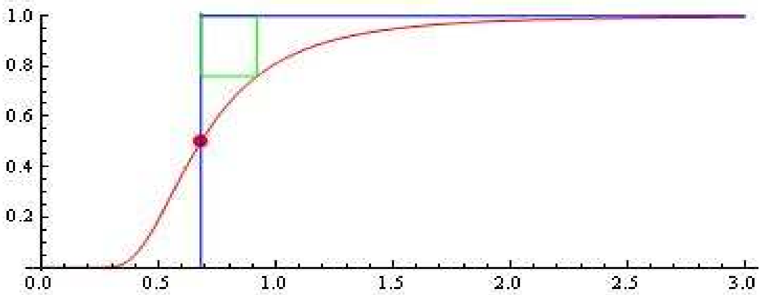

Figure 2: The cumulative function (1) for $a = 1.9$; $\lambda = 2.6$; $t_0 = 0.680285$; Hausdorff distance $d = 0.240132$; $d_l = 0.190354$; $d_r = 0.315773$.

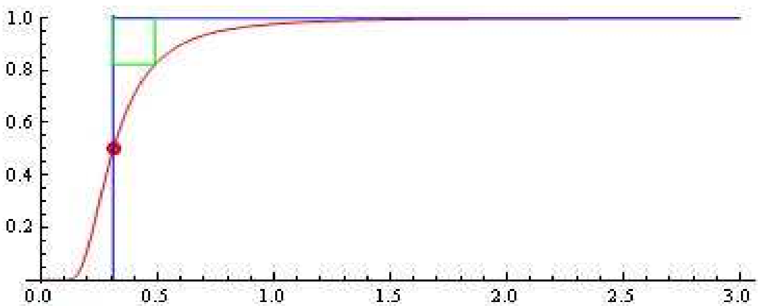

Figure 3: The cumulative function (1) for $a = 1.6$; $\lambda = 13$; $t_0 = 0.30987$; Hausdorff distance $d = 0.178368$; $d_l = 0.119571$; $d_r = 0.25395$.

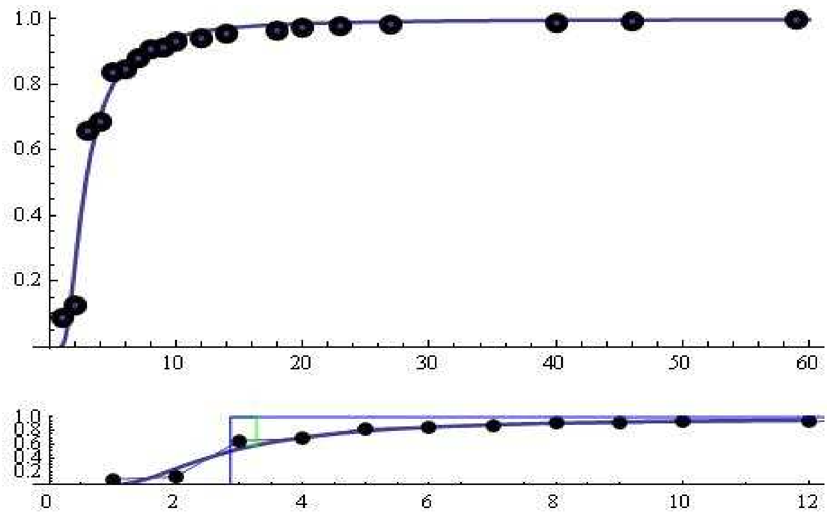

Figure 4: The fitted model (1) for approximation of the data: "cdf of the number of Bitcoin received per address" [11]. The "saturation" on the level "median".

3. NUMERICAL EXAMPLES

We examine the following "specific datasets":

1. Approximating cdf of the number of Bitcoin received per address [11]

We consider the following data (see, [11]):

$$data\_CDF\_of\_Bitcoin\_received\_(inransoms)\_per\_address\_in\_C_{CL}$$
$$:= \{\{1, 0.0857\}, \{2, 0.1238\}, \{3, 0.6571\}, \{4, 0.6854\}, \{5, 0.8381\},$$
$$\{6, 0.8476\}, \{7, 0.8810\}, \{8, 0.9095\}, \{9, 0.9143\}, \{10, 0.9333\},$$
$$\{12, 0.9429\}, \{14, 0.9571\}, \{18, 0.9667\}, \{20, 0.9762\}, \{23, 0.9810\},$$
$$\{27, 0.9857\}, \{40, 0.9905\}, \{46, 0.9952\}, \{59, 0.9981\}\}.$$

The model (1) for $a = 1$, $\lambda = 0.176462$ is visualized on Fig. 4.

2. Approximating the "growth data (mean height) of sunflower plants"

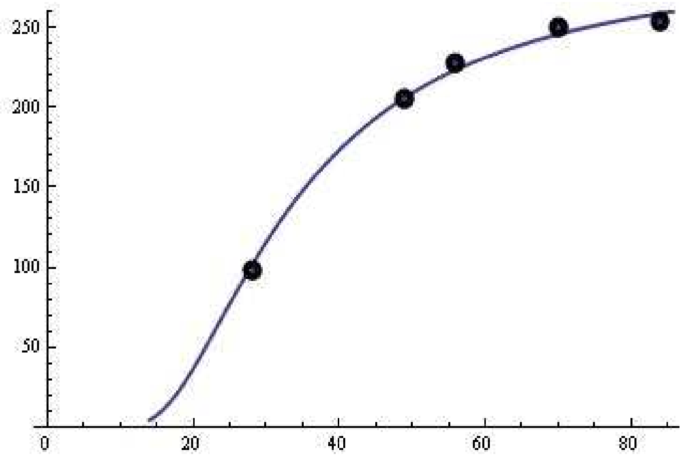

Figure 5: The fitted model $M^*(t)$.

We analyze experimental growth data (mean height) of sunflower plants (DSP) (see, for example [12]):

$$data\_DSP$$
$$:= \{\{28, 98.1\}, \{49, 205.5\}, \{56, 228.3\}, \{70, 250.5\}, \{84, 254.5\}\}.$$

For $a = 1$, $\lambda = 0.00120642$ and $\omega = 291$ we obtain the fitted model $M^*(t) = \omega M(t)$ (see, Fig. 5).

4. CONCLUSION.

The aim of this note is to study "saturation" of the inverse Nakagami–m cumulative distribution function [1].

The data from Examples 1 and 2 have been approximated over the past 4-5 years with known baseline c.d.f. and the generalized results lead us to conclude that "the choice of right approximation model means abandoning perfection...".

In concrete case, the conducted analysis in the article shows that the authors of [1] are right when they claim that IN distribution produces very good results when "specific data" are fitted.

For other results, see [13]–[26].

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