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A NOTE ON THE APPLICATIONS OF THE FOUR–PARAMETER MARSHALL–OLKIN GENERALIZED BURR XII CUMULATIVE DISTRIBUTION FUNCTION

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ABSTRACT: In [1] Yousof et al. defined the c.d.f. of the Marshall–Olkin generalized –G (MOG–G) family by:

$$F(t) = \frac{1 - (1 - G(t))^a}{1 - (1 - \lambda)(1 - G(t))^a}$$

where $G(t)$ is the baseline c.d.f. and $a, \lambda \in R^+$.

In [2] the authors considered the following new four–parameter Marshall–Olkin generalized Burr XII (MOGBXII) c.d.f.:

$$M(t) = \frac{1 - (1 + t^\alpha)^{-a\beta}}{1 - (1 - \lambda)(1 + t^\alpha)^{-a\beta}},$$

where $\alpha, \beta \in R^+$.

Also of interest to the specialists is the task of approximating the Heaviside function $h_{t_0}(t)$ where t_0 is the median with the new cumulative function in the Hausdorff sense.

We define a new family of recurrence generated transmuted Marshall–Olkin generalized Burr XII (TMOGBXII) c.d.f.

$$M_{i+1}(t) = M_i(t)(\mu_{i+1} + 1 - \mu_{i+1} M_i(t)),$$

$$i = 0, 1, 2, \ldots, \quad \mu_i \in [0, 1); \;\; M_0(t) = M(t).$$

Some properties and applications to the specific data are given.

Numerical examples, illustrating our results are presented using programming environment *CAS Mathematica*.

AMS Subject Classification: 41A46

Key Words: four–parameter Marshall–Olkin generalized Burr XII (MOGBXII) c.d.f., family of recurrence generated transmuted Marshall–Olkin generalized Burr XII (TMOGBXII) c.d.f., Heaviside step–function $h_{t_0}(t)$, Hausdorff distance

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1. INTRODUCTION AND PRELIMINARIES

Definition 1. *In [2] the authors considered the following new four–parameter Marshall–Olkin generalized Burr XII (MOGBXII) c.d.f.:*

$$M(t) = \frac{1 - (1 + t^{\alpha})^{-a\beta}}{1 - (1 - \lambda)(1 + t^{\alpha})^{-a\beta}}, \tag{1}$$

where $(a, \alpha, \beta, \lambda) \in R^{+}$, $t > 0$.

Definition 2. *The shifted Heaviside step function is defined by*

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases} \tag{2}$$

Definition 3. *[3] The Hausdorff distance (the H–distance) $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f,g) = \max\{\sup_{A\in F(f)} \inf_{B\in F(g)} ||A - B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A - B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = max(|t_A - t_B|, |x_A - x_B|)$.

Definition 4. *We define a new family of recurrence generated transmuted Marshall–Olkin generalized Burr XII (TMOGBXII) c.d.f. by*

$$\begin{aligned} &M_{i+1}(t) = M_i(t)(\mu_{i+1} + 1 - \mu_{i+1} M_i(t)), \\ &i = 0, 1, 2, \ldots, \quad \mu_i \in [0, 1); \;\; M_0(t) = M(t). \end{aligned} \tag{3}$$

For some generalized family of distributions, see [8]–[17].

In this note we study the Hausdorff approximation of the Heaviside function $h_{t_0}(t)$ by the family $M_i(t)$.

2. MAIN RESULTS

2.1. A NOTE ON THE NEW FOUR–PARAMETER MARSHALL–OLKIN GENERALIZED BURR XII (MOGBXII) C.D.F. (1)

The investigation of the characteristic ”supersaturation” of the model (1) to the horizontal asymptote is important.

Sensitive analysis for the ”saturation in the Hausdorff sense”.

In this Section we study the Hausdorff approximation of the Heaviside step function $h_{t_0}(t)$ by families of the type (1) where t_0 is the ”median”.

The quantile function is defined by [2]:

$$Q(u) = \left(\left(\frac{1-u}{1-(1-\lambda)u} \right)^{-\frac{1}{a\beta}} - 1 \right)^{\frac{1}{\alpha}}; \quad 0 \le u \le 1 \tag{4}$$

(see, Fig. 1 for $a = 1.9$, $\alpha = 2.6$, $\beta = 40.7$, $\lambda = 0.9$).

The median is obtained by substituting $u = 0.5$ in (3).

Let t_0 is the value for which $M(t_0) = \frac{1}{2}$, i.e. $t_0 = Q(0.5)$.

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the (cdf) $M(t)$ (1) satisfies the relation

$$M(t_0 + d) = 1 - d. \tag{5}$$

For given $a, \alpha, \beta, \lambda$ and t_0, the nonlinear equation (4) has unique positive root – d.

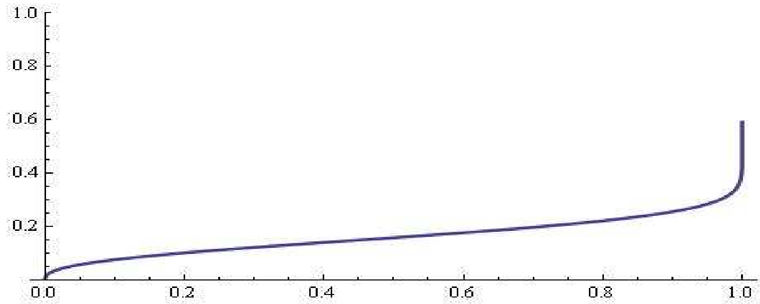

Figure 1: The function $Q(u)$ for $a = 1.9$, $\alpha = 2.6$, $\beta = 40.7$, $\lambda = 0.9$.

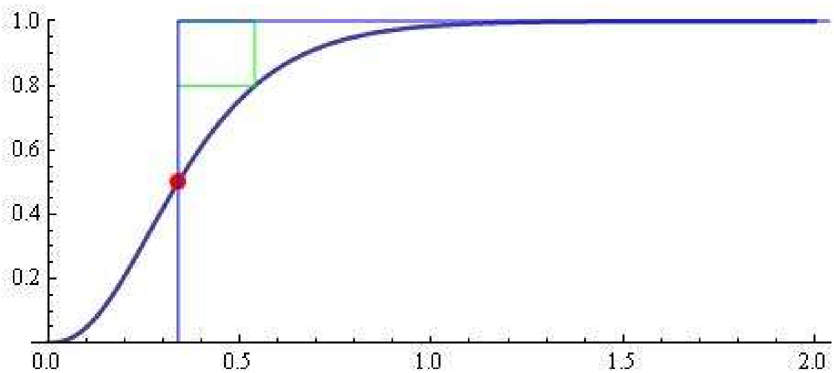

Figure 2: The model (1) for $a = 1.1$, $\alpha = 2.3$, $\beta = 4.6$, $\lambda = 0.5$ and $t_0 = 0.339632$; H–distance $d = 0.200236$.

The model (1) for $a = 1.1$, $\alpha = 2.3$, $\beta = 4.6$, $\lambda = 0.5$ and $t_0 = 0.339632$ is visualized on Fig. 2.

From the nonlinear equation (4) we have: $d = 0.200236$.

The model (1) for $a = 1.6$, $\alpha = 2.3$, $\beta = 10.6$, $\lambda = 0.7$ and $t_0 = 0.22324$ is visualized on Fig. 3.

From the nonlinear equation (4) we have: $d = 0.145662$.

The model (1) for $a = 1.9$, $\alpha = 2.6$, $\beta = 40.7$, $\lambda = 0.9$ and $t_0 = 0.158615$ is visualized on Fig. 4.

From the nonlinear equation (4) we have: $d = 0.0982747$.

Example 1. Application of the new cumulative sigmoid for analysis of the ”cancer data” [4]–[5]

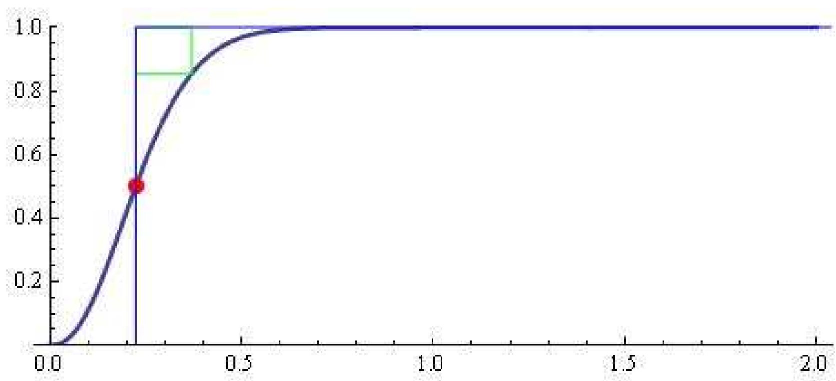

Figure 3: The model (1) for $a = 1.6$, $\alpha = 2.3$, $\beta = 10.6$, $\lambda = 0.7$ and $t_0 = 0.22324$; H–distance $d = 0.145662$.

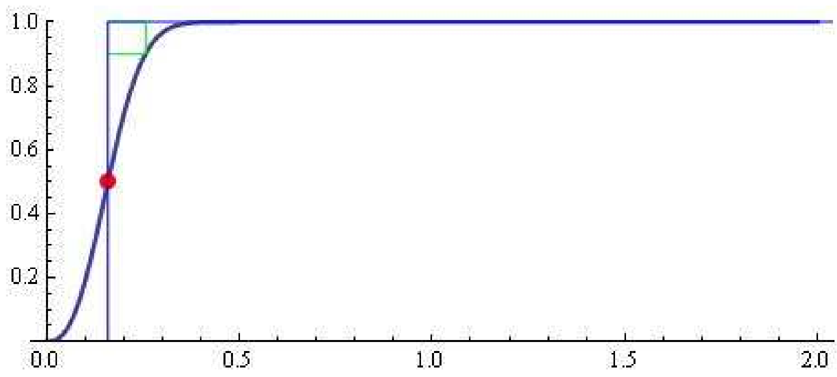

Figure 4: The model (1) for $a = 1.9$, $\alpha = 2.6$, $\beta = 40.7$, $\lambda = 0.9$ and $t_0 = 0.158615$; H–distance $d = 0.0982747$.

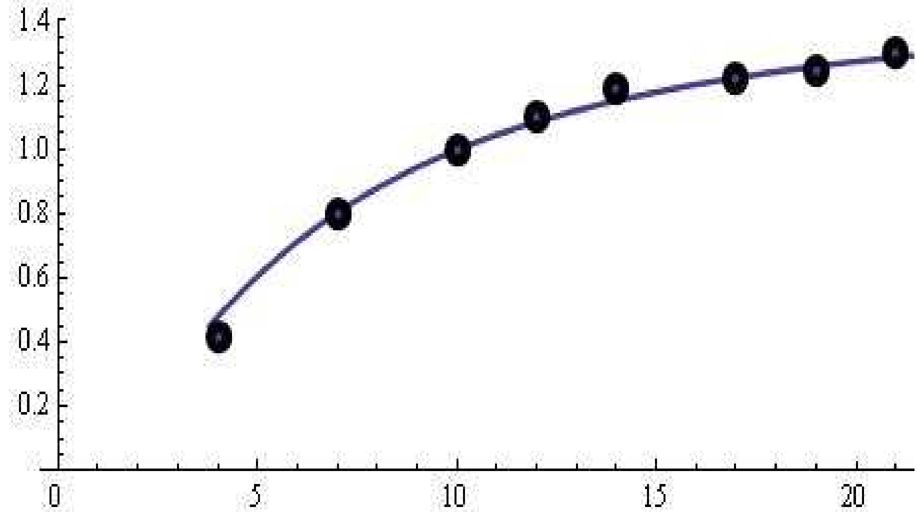

Figure 5: The model $M^*(t)$ based on the "cancer data".

We will illustrate the advances of the new family for approximation and modelling of "cancer data" (for some details see, [4]–[5]).

days	4	7	10	12	14	17	19	21
$R(t)$	0.415	0.794	1.001	1.102	1.192	1.22	1.241	1.3

Table 1: The "cancer data" [4]–[5]

The model $M^*(t) = \omega M(t)$ based on the data from Table 1 for the estimated parameters:

$$\omega = 1.5;\ \lambda_1 = 14.1;\ \beta = 0.152603;\ \alpha = 7.99;\ a = 1.2$$

is plotted on Fig. 5.

Example 2. Analysis of data "growth of the cumulative number of TREZ publications" [6], [7]

$data\_Journal$

$:= \{\{1.1, 5\}, \{2, 37\}, \{3, 107\}, \{4, 201\}, \{5, 298\}, \{6, 439\},$

$\{7, 617\}, \{8, 773\}, \{9, 936\}, \{10, 1121\}, \{11, 1316\},$

$\{12, 1451\}, \{13, 1563\}, \{14, 1629\}, \{15, 1722\}, \{16, 1788\}\};$

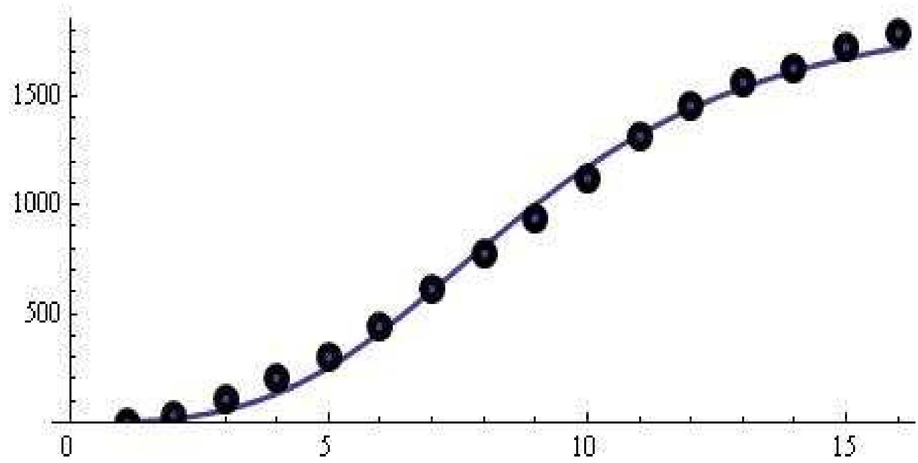

Figure 6: The fitted model $M^*(t)$.

After that using the model $M^*(t) = \omega M(t)$ for $a = 3.40038$, $\alpha = 0.407809$, $\beta = 3.51872$ $\lambda = 250000$ and $\omega = 1920$ we obtain the fitted model (see, Fig. 6).

2.2. A NOTE ON THE NEW FAMILY OF RECURRENCE GENERATED TRANSMUTED MARSHALL–OLKIN GENERALIZED BURR XII (TMOGBXII) C.D.F. (3)

Sensitive analysis for the "saturation in the Hausdorff sense".

For $i = 0$ from (3) we have:

$$M_1(t) = M(t)(\mu_1 + 1 - \mu_1 M(t)) \tag{6}$$

Let t_1 is the value for which $M_1(t_1) = \frac{1}{2}$.

The one–sided Hausdorff distance d_1 between the function $h_{t_1}(t)$ and the (cdf) $M_1(t)$ (5) satisfies the relation

$$M_1(t_1 + d_1) = 1 - d_1. \tag{7}$$

For example, for fixed $a = 1.9$, $\alpha = 2.6$, $\beta = 40.7$, $\lambda_1 = 0.9$, $\mu_1 = 0.99$ and $t_1 = 0.121085$ for the one–sided H–distance we find $d_1 = 0.0807279$ (see, Fig. 7).

Some comparisons between $M(t)$ and $M_1(t)$ are depicted on Fig. 7.

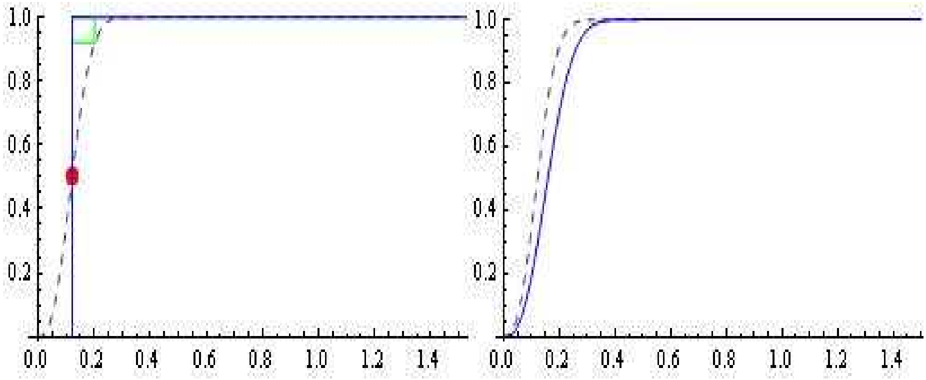

Figure 7: (a) The model $M_1(t)$ for $a = 1.9$, $\alpha = 2.6$, $\beta = 40.7$, $\lambda_1 = 0.9$, $\mu_1 = 0.99$; Hausdorff distance $d_1 = 0.0807279$. b) Comparisons between $M_1(t)$ (dashed) and $M(t)$ (blue); Hausdorff distance $d = 0.0982747$.

Obviously, the new model (3) can also be used for approximating of some "specific data".

It can be seen that the "supersaturation" by the (cdf) $M_i(t)$ is faster.

Evidently, $\{d_i\}_1^\infty \to 0$.

For other approximation and modelling results, see [8]–[17], [18]–[40].

We hope that the results will be useful for specialists in this scientific area.

2.3. CONCLUSIONS

Example 3. Analysis of "*data_Nicotine*" [2]

$$data\_Nicotine :=$$

$$\{\{0.11, 0.021\}, \{0.21, 0.053\}, \{0.31, 0.063\}, \{0.41, 0.105\}, \{0.51, 0.2\},$$

$$\{0.61, 0.274\}, \{0.71, 0.358\}, \{0.81, 0.495\}, \{0.91, 0.632\}, \{1.01, 0.726\},$$

$$\{1.11, 0.832\}, \{1.21, 0.905\}, \{1.31, 0.942\}, \{1.41, 0.958\}, \{1.51, 0.974\},$$

$$\{1.61, 0.979\}, \{1.71, 0.989\}, \{1.81, 1\}, \{1.9, 1\}, \{2, 1\}\};$$

After that using the model $M(t)$ for $a = 2.184684259592676$, $\alpha = 1.8384302383783595$, $\beta = 2.4308549362757175$, $\lambda = 14.1$ we obtain the fitted model (see, Fig. 8).

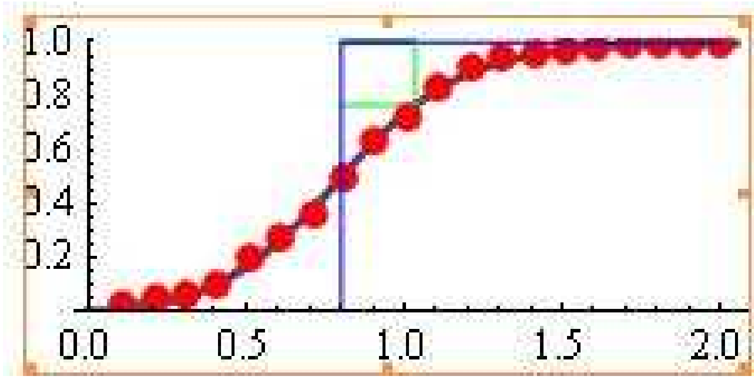

Figure 8: The fitted model $M(t)$ for $a = 2.184684259592676$, $\alpha = 1.8384302383783595$, $\beta = 2.4308549362757175$, $\lambda = 14.1$, $t_0 = 0.802464$; Hausdorff distance $d = 0.233486$.

Approximation of "Nicotine" data set with the proposed model at values $a, \alpha, \beta, \lambda$ gives good results.

Research on "supersaturation" at the "median" level t_0 (the magnitude of the one–sided Hausdorff distance d) is also an indication of the reliability of the used model.

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