Neural, Parallel, and Scientific Computations, **27**, No. 2 (2019), 85-92 ISSN: 1061-5369

INVESTIGATIONS ON A NEW PARAMETRIC FAMILY OF SIGMOIDAL FUNCTIONS WITH POSSIBLE APPLICATION TO GENERATE A NEW CLASS OF FUZZY OPERATORS

NIKOLAY KYURKCHIEV AND ANTON ILIEV

Faculty of Mathematics and Informatics
University of Plovdiv Paisii Hilendarski
24, Tzar Asen Str., 4000 Plovdiv, BULGARIA

ABSTRACT: In this article, following Kertesz, Dombi and Benyi's ideas [1], we define a new parametric family of sigmoidal functions that could possibly be used by specialists to generate a new class of fuzzy operators. In particular, we will study the function

$$M(t) = \frac{1}{1 + \left(\left(\frac{\nu}{1-\nu}\right)^2 \frac{1-t}{t}\right)^\lambda}$$

and we prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t^*}(t)$ by means of this new family.

Similar results are also obtained for the sigmoidal function class proposed in [1].

The estimates can be used in practice as one possible additional criterion in "saturation" study.

Numerical examples using *CAS Mathematica*, illustrating our results are given.

AMS Subject Classification: 41A46

Key Words: new family of sigmoidal functions, Hausdorff distance, upper and lower bounds

Received: March 23, 2019 ; **Accepted:** July 2, 2019 ;
Published: July 9, 2019. **doi:** 10.12732/npsc.v27i2.3
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/npsc

1. INTRODUCTION

The Pliant system is a type of fuzzy theory that is similar to a fuzzy system [3]. In these systems a so-called distending function, which represents a soft inequality are used [1]. In this system we usually have a generator function. Using this function it can be created aggregation operator, conjunctive operator or disjunctive operator which are widely used in area of cloud computing.

For more details see [1]–[5].

Definition 1. The operators of the Pliant system are [1]–[4]:

$$c(x) = \frac{1}{1 + \left(\sum_{i=1}^{n} \omega_i \left(\frac{1-x_i}{x_i} \right)^{\alpha} \right)^{\frac{1}{\alpha}}}, \tag{1}$$

$$d(x) = \frac{1}{1 + \left(\sum_{i=1}^{n} \omega_i \left(\frac{1-x_i}{x_i} \right)^{-\alpha} \right)^{-\frac{1}{\alpha}}}, \tag{2}$$

$$a_\nu(x) = \frac{1}{1 + \left(\frac{1-\nu}{\nu} \right) \prod_{i=1}^{n} \left(\frac{1-x_i}{x_i} \frac{1-\nu}{\nu} \right)^{\omega_i}}, \tag{3}$$

$$n(x) = \frac{1}{1 + \left(\frac{1-\nu}{\nu} \right)^2 \frac{x}{1-x}}, \tag{4}$$

where $\nu \in [0, 1]$.

Definition 2. Kertesz, Dombi and Benyi [1] defined the following "Kappa" function:

$$\kappa_\nu^\lambda(x) = \frac{1}{1 + \left(\frac{\nu}{1-\nu} \frac{1-x}{x} \right)^\lambda} \tag{5}$$

and the following aggregation operator:

$$a_{\nu,\nu_0}(x) = \frac{1}{1 + \frac{1-\nu_0}{\nu_0} \frac{\nu}{1-\nu} \prod_{i=1}^{n} \frac{1-x_i}{x_i}}. \tag{6}$$

For other results, see [2]–[4].

Definition 3. The shifted Heaviside step function is defined by

$$h_{t^*}(t) = \begin{cases} 0, & \text{if} \quad t < t^*, \\ [0, 1], & \text{if} \quad t = t^*, \\ 1, & \text{if} \quad t > t^*. \end{cases} \tag{7}$$

Definition 4. [6] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \tag{8}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A-t_B|,|x_A-x_B|)$.

Definition 5. Combining parts of constructions of (4) and (5) we define a new parametric family of sigmoidal function for $\lambda > 0$, $\nu \in [0,1]$ and $t \in [0,1)$

$$M(t) = \frac{1}{1+\left(\left(\frac{\nu}{1-\nu}\right)^2 \frac{1-t}{t}\right)^{\lambda}}. \tag{9}$$

2. MAIN RESULTS

In this Section we prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t^*}(t)$ by means of families (9).

Let

$$t^* = \frac{1}{\left(\frac{1-\nu}{\nu}\right)^2 + 1}.$$

Evidently, for the "median" we have $M(t^*) = \frac{1}{2}$.

The one–sided Hausdorff distance d satisfies the relation

$$M(t^*+d) = 1-d. \tag{10}$$

The following theorem gives upper and lower bounds for d

Theorem 1. *Let*

$$\begin{aligned} p &= -\tfrac{1}{2}, \\ q &= 1+\tfrac{\lambda\nu^2}{4(1-\nu)^2}\left(1+\tfrac{(1-\nu)^2}{\nu^2}\right)^2 \\ \gamma &= 2.1q. \end{aligned} \tag{11}$$

Let $\gamma > e^{1.05}$. For the one–sided Hausdorff distance d between $h_{t^*}(t)$ and the function (9) the following inequalities hold:

$$d_l = \frac{1}{\gamma} < d < \frac{\ln\gamma}{\gamma} = d_r. \tag{12}$$

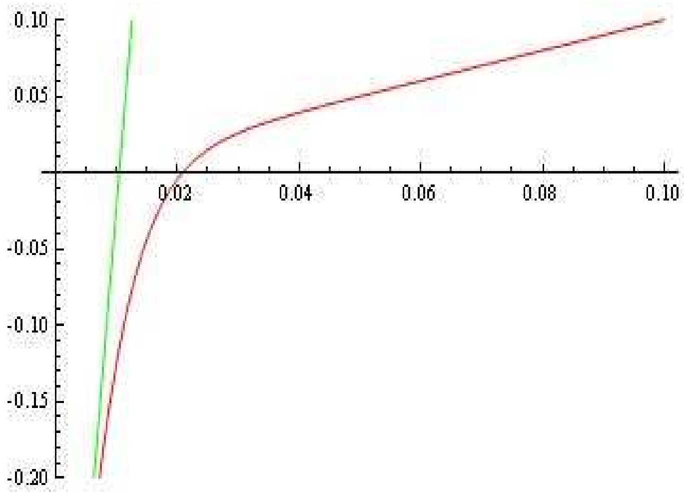

Figure 1: The functions $F(d)$ and $G(d)$ for $\lambda = 40$; $\nu = 0.4$.

Proof. Let us examine the function:

$$F(d) = \frac{1}{1 + \left(\left(\frac{\nu}{1-\nu}\right)^2 \frac{1-(t^*+d)}{t^*+d}\right)^\lambda} - 1 + d. \tag{13}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{14}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).

In addition $G'(d) > 0$ and the function G is also increasing.

Further, for $\gamma > e^{1.05}$ we have

$$G(d_l) < 0; \quad G(d_r) > 0.$$

This completes the proof of the theorem.

Approximations of the $h_{t^*}(t)$ by model (9) for various λ and ν are visualized on Fig. 2–Fig. 3.

Some computational examples using relations (10) and (12) are presented in Table 1.

For example, the aggregation operator $A(x_1, x_2)$ is visualized on Fig. 4.

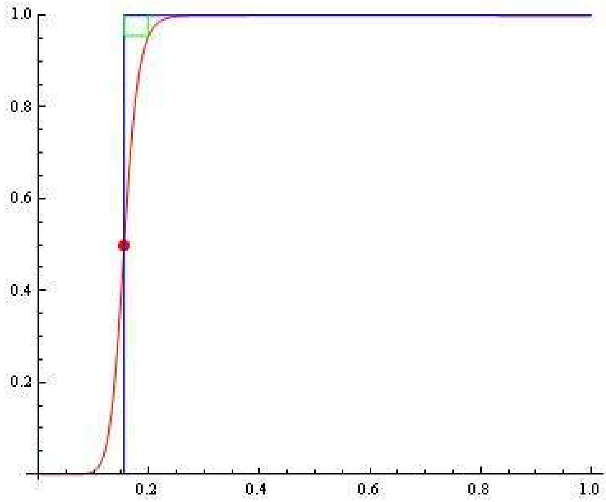

Figure 2: The model (9) for $\lambda = 10$; $\nu = 0.3$; $t^* = 0.155172$; Hausdorff distance $d = 0.044549$; $d_l = 0.0237261$; $d_r = 0.0887637$.

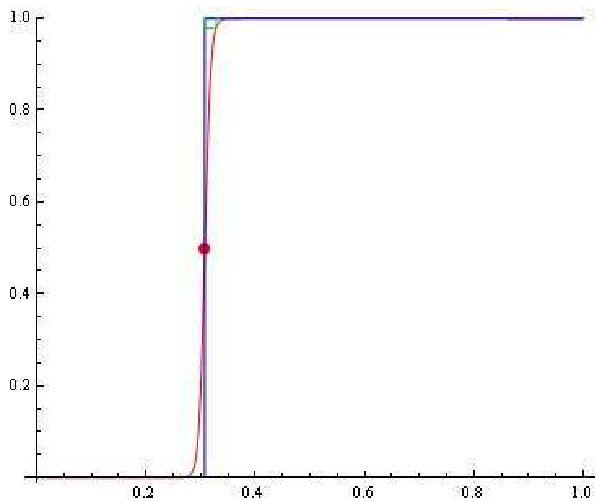

Figure 3: The model (9) for $\lambda = 40$; $\nu = 0.4$; $t^* = 0.307692$; Hausdorff distance $d = 0.0208654$; $d_l = 0.00993213$; $d_r = 0.0458068$.

Remark. We will restart the $\kappa_\nu^\lambda(t)$ function (5):

$$\kappa_\nu^\lambda(t) = \frac{1}{1 + \left(\frac{\nu}{1-\nu}\frac{1-t}{t}\right)^\lambda}$$

λ	ν	d_l	$H - distance$	d_r
10	0.3	0.0237261	0.044549	0.0887637
40	0.4	0.00993213	0.0208654	0.0458068
60	0.5	0.0078064	0.0169195	0.037883
80	0.65	0.00411295	0.00987668	0.022595
100	0.75	0.00170814	0.00471499	0.0108848
130	0.8	0.000809805	0.00249939	0.00576477
350	0.8	0.000301106	0.0010722	0.00244139

Table 1: Bounds for d computed by (10) and (12) for various λ and ν

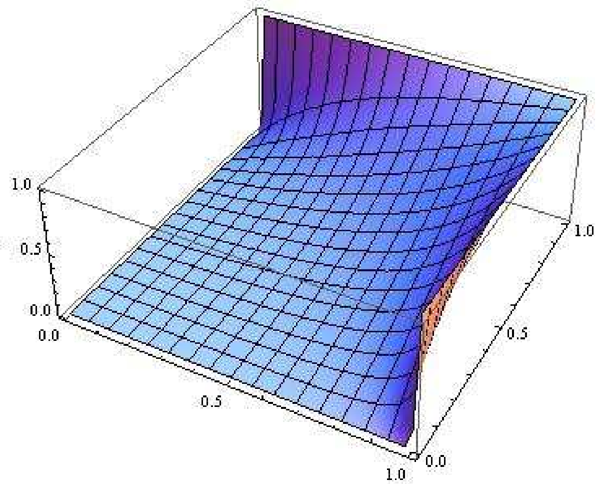

Figure 4: The aggregation operator $A(x_1, x_2)$

Evidently $\kappa_\nu^\lambda(\nu) = \frac{1}{2}$.

The following theorem is valid

Theorem 2. *Let*

$$\alpha = -\tfrac{1}{2},$$

$$\beta = 1 + \tfrac{\lambda}{4\nu(1-\nu)} \tag{15}$$

$$\delta = 2.1\beta.$$

Let $\delta > e^{1.05}$. *For the one–sided Hausdorff distance* d *between the Heaviside function*

$h_\nu(t)$ and the function (5) the following inequalities hold:

$$d_l = \frac{1}{\delta} < d < \frac{\ln \delta}{\delta} = d_r. \tag{16}$$

The proof follows the ideas given in this paper and will be omitted.

The operators of the Pliant systems (such as (9)) are used for reducing the energy consumption of cloud datacenters and for further research we let the word to the specialists in this area.

3. CONCLUSIONS

In this paper we study a new parametric family of sigmoidal functions with applications to Fuzzy Sets Theory.

We prove upper and lower estimates for the one–sided Hausdorff approximation of the shifted Heaviside function $h_{t^*}(t)$ by means of this new family.

The estimates can be used in practice as one possible additional criterion in "saturation" study.

Various results related to the approximation of some classical functions and point sets in the plane with various classes of sigmoidal functions in respect to the Hausdorff distance and they find the application in different scientific fields, see for example [7]–[15].

ACKNOWLEDGMENTS

This paper is supported by the National Scientific Program "Information and Communication Technologies for a Single Digital Market in Science, Education and Security (ICTinSES)", financed by the Ministry of Education and Science.

REFERENCES

[1] A. Kertesz, J. Dombi, A. Benyi, A Pliant–based Virtual Machine Scheduling Solution to Improve the Energy Efficiency of IaaS Clods, *J. of Grid Computing*, **14**, No. 1 (2016), 41–53.

[2] J. Dombi, Pliant system, *IEEE International Conference on Intelligent Engineering System Proceedings*, Budapest, Hungary, (1997).

[3] J. Dombi, A general class of fuzzy operators, the de Morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators, *Fuzzy Sets and Systems*, **8** (1982).

[4] J. Dombi, A. Kertesz, Advances Scheduling Techniques with the Pliant System for High–Level Grid Broking, *Communications in Computer and Information Science (CCIS)*, **129**, Springer–Verlag Berlin Heidelberg, (2011), 173–185.

[5] K. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, Studies in Fuzziness and Soft Computing, **35**, Springer-Verlag, Berlin Heidelberg (1999).

[6] B. Sendov, *Hausdorff Approximations*, Kluwer, Boston (1990).

[7] N. Kyurkchiev, S. Markov, On the Hausdorff distance between the Heaviside step function and Verhulst logistic function, *J. Math. Chem.*, **54**, No. 1 (2016), 109–119.

[8] N. Kyurkchiev, S. Markov, *Sigmoid functions: Some Approximation and Modelling Aspects*, LAP LAMBERT Academic Publishing, Saarbrucken (2015), ISBN: 978-3-659-76045-7.

[9] N. Kyurkchiev, A. Iliev, S. Markov, *Some techniques for recurrence generating of activation functions*, LAP LAMBERT Academic Publishing, (2017), ISBN: 978-3-330-33143-3.

[10] N. Kyurkchiev, A. Iliev, A. Rahnev, *Some Families of Sigmoid Functions: Applications to Growth Theory*, LAP LAMBERT Academic Publishing, (2019), ISBN: 978-613-9-45608-6.

[11] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, *Some software reliability models: Approximation and modeling aspects*, LAP LAMBERT Academic Publishing, (2018), ISBN: 978-613-9-82805-0.

[12] A. Iliev, N. Kyurkchiev, A. Rahnev, T. Terzieva, *Some models in the theory of computer viruses propagation*, LAP LAMBERT Academic Publishing, (2019), ISBN: 978-620-0-00826-8.

[13] N. Kyurkchiev, A Note on the Volmer's Activation (VA) Function, *C. R. Acad. Bulg. Sci.*, **70**, No. 6 (2017), 769–776.

[14] N. Kyurkchiev, Comments on the Yun's algebraic activation function. Some extensions in the trigonometric case, *Dynamic Systems and Applications*, **28**, No. 3 (2019), 533–543.

[15] N. Kyurkchiev, A. Andreev, Synthesis of slot aerial grids with Hausdorff-type directive patterns - implementation in programming environment MATHEMATICA, *Compt. rend. Acad. bulg. Sci.*, **66**, No. 11 (2013), 1521–1528.