SMOOTHNESS OF THE FUNDAMENTAL MATRIX OF LINEAR FRACTIONAL SYSTEM WITH VARIABLE DELAYS

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ABSTRACT: In the present work, a linear fractional system with variable delays and incommensurate order derivatives in Caputo sense is considered. Sufficient conditions for the smoothness of its fundamental matrix are obtained.

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1. INTRODUCTION

Nowadays the fractional calculus and respectively the fractional differential equations have lot of applications in various fields of the science. For a good introduction on fractional calculus theory and fractional differential equations see the monographs of Kilbas et al. [12], Kiryakova [13], Podlubny [26], Feckan et al. [9] and Abbas et al. [1]. For distributed order fractional differential equations see [11], for an application-oriented exposition Diethelm [8] and for fractional evolution equations in Banach spaces Bajlecova [4]. The impulsive differential and functional differential equations with fractional derivatives and some applications are considered in the monograph of Stamova and Stamov [28].

Also it is worth noting some new interesting results for fractional differential equations and systems obtained in [2], [20],[29], [35], [41] and [42]. It may be noted that fractional systems of retarded and neutral type with distributed delays are studied (basically existence and stability) in [14], [21], [31]–[34], [38] for single order fractional derivatives and in [6] for Caputo-type distributed order fractional derivatives.

It is well known that the integral representation (variation of constants formula) of the solutions for linear fractional differential equations and/or systems (ordinary or with delay) is an evergreen theme for research. This explains that a lot of papers are devoted to different aspects of this problem. For linear fractional ordinary differential equations and systems, we refer to the works [5] , [12], [24], [26] and the references therein. The variation of constants formula for linear fractional differential systems with delay is treated in [7], [10], [15], [36], [37], [39], [40].

In the present work, we consider linear fractional systems with variable delays and incommensurate order derivatives in Caputo sense. The aim of the work is to obtain sufficient conditions for the smoothness of their fundamental matrix. As far as we know, there are no results concerning the smoothness of the fundamental matrix for fractional differential systems with variable delays.

The results obtained in this article would be a good basis for building models of different processes from the real world. Good examples of new studies with application in modeling are [3], [16]–[19],[22], [23], [25],[27], [30].

The paper is organized as follows. In Section 2, we recall some needed definitions of Riemann-Liouville and Caputo fractional derivatives, as well as some their properties and the problem statement. Section 3 is devoted to the existence and the uniqueness of the solutions of the Cauchy problem for linear fractional systems with variable delays and incommensurate order derivatives in Caputo sense and special type discontinuous initial function. As a corollary of this result we obtain the smoothness of the fundamental matrix.

2. PRELIMINARIES AND PROBLEM STATEMENT

For convenience and to avoid possible misunderstandings, below we recall only the definitions of Riemann-Liouville and Caputo fractional derivatives and some needed their properties. For details and other properties we refer to [12, 13, 26].

Let $\alpha \in (0,1)$ be an arbitrary number and denote by $L_1^{loc}(\mathbb{R},\mathbb{R})$ the linear space of all locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}$. Then for each $t > a, a \in \mathbb{R}$ and $f \in L_1^{loc}(\mathbb{R},\mathbb{R})$ the left-sided fractional integral operator and the corresponding left side Riemann-Liouville and Caputo fractional derivatives of order α are defined by

$$(D_{a+}^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$

$$ {}_{RL}D_{a+}^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(D_{a+}^{-(1-\alpha)} f(t)\right) $$

$$ {}_{C}D_{a+}^{\alpha} f(t) = {}_{RL}D_{a+}^{\alpha}[f(s) - f(a)](t) = {}_{RL}D_{a+}^{\alpha} f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} $$

respectively. Everywhere below in this work we will denote with $D_{a+}^{\alpha} =_C D_{a+}^{\alpha}$ the left side Caputo fractional derivative. We will use also the following relations (see [12]):

(i) $(D_{a+}^{0} f)(t) = f(t)$;

(ii) ${}_{C}D_{a+}^{\alpha} D_{a+}^{-\alpha} f(t) = f(t)$;

(iii) $D_{a+}^{-\alpha}\ {}_{C}D_{a+}^{\alpha} f(t) = f(t) - f(a)$.

We will use the following notations: $\mathbb{R}_+ = (0, \infty), \bar{\mathbb{R}_+} = [0, \infty)$, $J_a = [a, \infty), a \in \mathbb{R}, J_{s+M} = [s, s+M], s \in J_a, M \in \mathbb{R}_+, \langle n \rangle = \{1, 2, \ldots, n\}, \langle m \rangle_0 = \langle m \rangle \cup 0, n, m \in \mathbb{N}, I, \Theta \in \mathbb{R}^{n \times n}$ denote the identity and zero matrix respectively and $I^k, k \in \langle n \rangle$ denotes the k-th column of the identity matrix. For $W(t) = (w_1(t), \ldots, w_n(t))^T : J_a \to \mathbb{R}^n, \beta = (\beta_1, \ldots, \beta_n), \beta_k \in [-1, 1], k \in \langle n \rangle$ we will use the notation $I_\beta(W(t)) = diag((w_1(t))^{\beta_1}, \ldots, (w_n(t))^{\beta_n})$ and for $Y(t) = \{y_{kj}(t)\}_{k,j=1}^{n} : J_a \to \mathbb{R}^{n \times n}$ we denote $|Y(t)| = \sum\limits_{k,j=1}^{n} |y_{kj}(t)|, t \in J_a$.

For arbitrary fixed $s \in J_a$ consider the homogeneous linear fractional system of incommensurate type with variable delays in the following general form

$$ D_{a+}^{\alpha} X(t, s) = \sum_{i=0}^{m} A^i(t) X(t - \sigma_i(t), s), \tag{1} $$

or described in more detailed form

$$ D_{a+}^{\alpha_k} x_k(t, s) = \sum_{i=0}^{m} (\sum_{j=1}^{n} a_{kj}^{i}(t) x_j(t - \sigma_i(t), s)), $$

where $a \in \mathbb{R}, h \in \mathbb{R}_+, k \in \langle n \rangle, \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_k \in (0, 1)$,
$\sigma_i \in C(J_a, [0, h]), A^i(t) = \{a_{kj}^{i}(t)\}_{k,j=1}^{n} \in L_1^{loc}(J_a, \mathbb{R}^{n \times n}), i \in \langle m \rangle_0$,
$X(t, s) = (x_1(t, s), \ldots, x_n(t, s))^T : J_a \times J_a \to \mathbb{R}^n$,
$D_{a+}^{\alpha} X(t, s) = (D_{a+}^{\alpha_1} x_1(t, s), \ldots, D_{a+}^{\alpha_n} x_n(t, s))^T$.

Let $s \in J_a$ be an arbitrary fixed number and define the matrix valued function $\Phi_*(t, s) = \{\phi_{kj}^{*}(t, s)\}_{k,j=1}^{n} : (-\infty, s] \to \mathbb{R}^{n \times n}$ as follows

$$ \Phi_*(t, s) = \begin{cases} I, & t = s \\ \Theta, & t < s \end{cases} $$

Let denote $\Phi_*^j(t,s) = (\phi_{1j}(t,s),\dots,\phi_{nj}(t,s))^T, j \in \langle n\rangle$ and for each $j \in \langle n\rangle$ consider the following initial conditions for the system (1):

$$X(t,s) = \Phi_*^j(t,s), t \in [s-h,s]. \tag{2}$$

We say that the conditions (H) are fulfilled if the following conditions hold:

(H1) For every $t \in J_a$ and $i \in \langle m\rangle_0$ the matrices $A^i(t) = \{a_{kj}^i(t)\}_{k,j=1}^n \in L_1^{loc}(J_a, \mathbb{R}^{n\times n})$ are locally bounded and the delay $\sigma_0(t) \equiv 0$.

(H2) For every $s \in J_a$ and $i \in \langle m\rangle$ the sets
$S_s^i = \{t \in J_a | t - \sigma_i(t) \in \{s\}\}$ do not have limit points.

Everywhere below in our considerations we will assume that the conditions (H) hold.

Remark 1. Note that the Condition (H2) is ultimately fulfilled in the case of constant delays and also when all delays $\sigma_i(t)$ are monotonic functions.

Definition 1. For arbitrary fixed $s \in J_a$ and $j \in \langle n\rangle$ the vector function $X(t,s)$ is a continuous solution of the initial problem IP (1), (2) in $J_{s+M}(J_s)$ if $X|_{J_{s+M}} \in C(J_{s+M}, \mathbb{R}^n)(X|_{J_s} \in C(J_s, \mathbb{R}^n))$ satisfies the system (1) for $t \in (s, s+M](t \in (s,\infty))$ and the initial condition (2) for $t \in [s-h, s]$ too.

Let $s \in J_a$ be an arbitrary fixed number and consider the following matrix IP

$$D_{a+}^{\alpha} C(t,s) = \sum_{i=0}^{m} A^i(t) C(t-\sigma_i(t), s), \quad t \in (s,\infty) \tag{3}$$

$$C(t,s) = \Phi_*(t,s), \quad t \in [s-h,s]. \tag{4}$$

Definition 2. The matrix valued function
$t \to C(t,s) = (C^1(t,s),\dots,C^n(t,s)) = \{c_{kj}(t,s)\}_{k,j=1}^n$ is called a solution of the IP (3), (4) for $s \in J_a$ if $C(\cdot,s) : [s,\infty) \to \mathbb{R}^{n\times n}$ is continuous for $t \in [s,\infty)$ and satisfies the matrix equation (3) on $t \in (s,\infty)$, as well as the initial condition (4) too.

Remark 2. Really in the condition (4) are used only the values of $\Phi_*(\cdot,s)$ in $[s-h,s]$, but for convenience we define $C(t,s) = \Theta$ for $t \in (-\infty, s-h)$. Then $C(t,s)$ is prolonged as continuous in t function on $(-\infty,s)$. The matrix $C(t,s)$ will be called fundamental (or Cauchy) matrix for the homogeneous system (1).

Remark 3. Note that for every $t \in J_a$ the function $C(t,\cdot) : [a,t] \to \mathbb{R}^{n\times n}$ is locally bounded and Lebesgue measurable (see [33] and [37]).

3. MAIN RESULTS

Let $s \in J_a$ and $j \in \langle n \rangle$ be arbitrary and consider the following integral equation

$$C^j(t,s) = \Phi_*^j(s,s) + \\ + I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\eta) \sum_{i=0}^m A^i(\eta) C^j(\eta - \sigma_i(\eta), s) \mathrm{d}\eta \tag{5}$$

or described in detailed form for every $k \in \langle n \rangle$

$$c_{kj}(t,s) = \phi_{kj}^*(s,s) + \\ + \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-\eta)^{\alpha_k - 1} \sum_{i=0}^m (\sum_{q=1}^n a_{kq}^i(\eta) c_{qj}(\eta - \sigma_i(\eta), s)) \mathrm{d}\eta \tag{6}$$

where $\Gamma(\alpha) = (\Gamma(\alpha_1), \ldots, \Gamma(\alpha_n))^T$.

Then if the matrix $C(t,s)$ is the fundamental matrix for the homogeneous system (1), in virtue of Lemma 4 in [37] it follows that the j-th column $C^j(t,s) = (c_{1j}(t,s), \ldots, c_{nj}(t,s))^T$ of $C(t,s)$ for every $j \in \langle n \rangle$ satisfies the initial condition (2) for $t \in [s-h, s]$ with initial function $\Phi_*^j(t,s)$ and so is the unique continuous solution of the integral equation (5) for every $j \in \langle n \rangle$ satisfying the initial condition (2) for $t \in [s-h, s]$ and vice versa.

For every fixed $s \in J_a$ by $BVC(\mathbb{R} \times \{s\}, \mathbb{R}^n)$ we will denote the linear space of all vector valued functions $G(\cdot, s) : \mathbb{R} \times s \to \mathbb{R}^n$, which are with bounded variation in t on every compact subinterval of $\mathbb{R}$ and $G(\cdot, s) \in C(J_s, \mathbb{R}^n)$.

For every fixed $s \in J_a$ and $j \in \langle n \rangle$ we define the sets

$$E(s, \Phi_*^j) = \{G \in BVC(\mathbb{R} \times \{s\}, \mathbb{R}^n) | G(t,s) = \Phi_*^j(t,s), t \in (-\infty, s]\}$$

$$E_M(s, \Phi_*^j) = \{G_M(\cdot, s) = G(\cdot, s)|_{[s-h, s+M]} | G \in E(s, \Phi_*^j), M \in (0, h]\}$$

and for every $M \in (0, h]$ in the sets $E_M(s, \Phi_*^j)$ we define a metric function $d_M : E_M(s, \Phi_*^j) \times E_M(s, \Phi_*^j) \to \bar{\mathbb{R}_+}$ with

$$d_M(G_M, G_M^*) = \sum_{q=1}^n \sup_{t \in [s-h, s+M]} |g_q(t,s) - g_q^*(t,s)|$$

for arbitrary $G_M, G_M^* \in E_M(s, \Phi_*^j)$.

Since J_{s+M} is a compact interval for every $s \in J_a, M \in \mathbb{R}_+$ and $G_M(s,s) = G_M^*(s,s)$ for arbitrary $G_M, G_M^* \in E_M(s, \Phi_*^j)$ then there exists a well-known result which guaranties that $E_M(s, \Phi_*^j)$ is a complete metric space in respect to the metric

$$d^{Var}(G_M, G_M^*) = Var_{[s, s+M]}(G_M(\cdot, s) - G_M^*(\cdot, s)) = \\ = \sum_{k=1}^n Var_{[s, s+M]}(g_k(\cdot, s) - g_k^*(\cdot, s)).$$

The aim of the next simple lemma is to establish the same result in respect to the metric d_M for every $s \in J_a$ and $j \in \langle n \rangle$. It was a big surprise for us, that we could not find a result from which this statement directly follows. That's why for completeness we will sketch the proof.

Lemma 1. *For every fixed numbers $s \in J_a, j \in \langle n \rangle$ and $M \in (0, h]$ the set $E_M(s, \Phi_*^j)$ is a complete metric space concerning the metric d_M.*

Proof. Let $s \in J_a, j \in \langle n \rangle$ and $M \in (0, h]$ be arbitrary fixed numbers and consider an arbitrary Cauchy sequence $\{G_M^l(t,s) = (g_1^l(t,s), \dots, g_n^l(t,s))^T\}_{l=1}^{\infty} \subset E_M(s, \Phi_*^j)$, i.e. $\lim\limits_{l,r\to\infty} d_M(G_M^l(t,s), G_M^r(t,s)) = 0$. It is clear that there exists $G_M^0(t,s) = (g_1^0(t,s), \dots, g_n^0(t,s))^T \in C(J_s, \mathbb{R}^n), G_M^0(t,s) = \Phi_*^j), t \in (-\infty, s]$ such that $\lim\limits_{l\to\infty} d_M(G_M^l(t,s), G_M^0(t,s)) = 0$. Introduce the notations

$$V^l = Var_{[s-h,s+M]} G_M^l(\cdot, s) = \sum_{k=1}^{n} Var_{[s-h,s+M]} g_k^l(\cdot, s) = \sum_{k=1}^{n} V_k^l$$

and let $P_m = \{t_0, \dots, t_m\}, t_0 = s, t_m = s + M$ be an arbitrary partition of $[s, s+M]$. Then for every $l \in \mathbb{N}$ and arbitrary fixed $k \in \langle n \rangle$ we have that

$$\sum_{i=0}^{m-1} |g_k^l(t_{i+1}, s) - g_k^l(t_i, s)| \le V_k^l. \tag{7}$$

Since $\lim\limits_{l,r\to\infty} d_M(G_M^l(t,s), G_M^r(t,s)) = 0$ then for every $\varepsilon > 0$ there exists a number l_0 such that for each $p \in \mathbb{N}$ and $i \in \langle m-1 \rangle_0$ the inequality $|g_k^{l_0}(t_i, s) - g_k^{l_0+p}(t_i, s)| < \frac{\varepsilon}{2m}$ holds for $t_i \in [s, s+M]$ which implies the inequalities

$$\begin{aligned} g_k^{l_0}(t_{i+1}, s) - \frac{\varepsilon}{2m} < g_k^{l_0+p}(t_{i+1}, s) < g_k^{l_0}(t_{i+1}, s) + \frac{\varepsilon}{2m}, \\ - g_k^{l_0}(t_i, s) - \frac{\varepsilon}{2m} < -g_k^{l_0+p}(t_i, s) < -g_k^{l_0}(t_i, s) + \frac{\varepsilon}{2m}. \end{aligned} \tag{8}$$

From (8) it follows that for each $p \in \mathbb{N}$ and $i \in \langle m-1 \rangle_0$ we have

$$|g_k^{l_0+p}(t_{i+1}, s) - g_k^{l_0+p}(t_i, s)| < |g_k^{l_0}(t_{i+1}, s) - g_k^{l_0}(t_i, s)| + \frac{\varepsilon}{m}$$

and hence from the last inequality and from (7) for each $p \in \mathbb{N}$ we obtain the following estimation

$$\begin{aligned} \sum_{i=0}^{m-1} |g_k^{l_0+p}(t_{i+1}, s) - g_k^{l_0+p}(t_i, s)| < \sum_{i=0}^{m-1} |g_k^{l_0}(t_{i+1}, s) - g_k^{l_0}(t_i, s)| + \varepsilon < \\ < V_k^{l_0} + \varepsilon \end{aligned}$$

Therefore the sequence $\{V^l\}_{l=1}^{\infty}$ is bounded from above and let denote $V = \sup_{l\in\mathbb{N}}\{V^l\}_{l=1}^{\infty}$.

We will prove that $G_M^0(\cdot,s)$ has bounded variation on $[s,s+M]$ and hence $G_M^0(\cdot,s) \in E_M(s,\Phi_*^j)$.

Since $\lim_{l\to\infty} d_M(G_M^l(t,s),G_M^0(t,s)) = 0$ then for every $\varepsilon > 0$ there exists a number l_0 such that for each $l \geq l_0$ and arbitrary $t_i \in P_m = \{t_0,\dots,t_m\}$ the inequality $|g_k^l(t_i,s) - g_k^0(t_i,s)| < \frac{\varepsilon}{2m}$ holds. Then analogically way we can prove that

$$\sum_{i=0}^{m-1} |g_k^0(t_{i+1},s) - g_k^0(t_i,s)| < \sum_{i=0}^{m-1} |g_k^l(t_{i+1},s) - g_k^l(t_i,s)| + \varepsilon < \\ < V^l + \varepsilon \leq V + \varepsilon$$

for $l \geq l_0, k \in \langle n\rangle$ and hence $G_M^0(\cdot,s) \in E_M(s,\Phi_*^j)$. □

Let $s \in J_a$ be an arbitrary fixed point and $M \in (0,h]$ be an arbitrary number. Then for each fixed $j \in \langle n\rangle$ and for every $G_M = (g_1,\dots,g_n)^T \in E_M(s,\Phi_*^j)$ using (6) we define for $t \in [s,s+M]$ and $k \in \langle n\rangle$ the operators $\Re_k g_k(t,s)$ with

$$\Re_k g_k(t,s) = \phi_{kj}^*(s,s) + \\ + \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-\eta)^{\alpha_k - 1} \sum_{i=0}^{m} (\sum_{q=1}^{n} a_{kq}^i(\eta) g_q(\eta - \sigma_i(\eta),s)) \mathrm{d}\eta \tag{9}$$

and the condition $\Re_k g_k(t,s) = 0$ for $t < s$.

Theorem 1. *Let the conditions (H) hold.*

Then for every fixed $s \in J_a$ and $j \in \langle n\rangle$ there exists $M^0 \in (0,h]$ such that in the complete metric space $E_M(s,\Phi_*^j)$ the operator $(\Re G_{M_0})(t,s) = (\Re_1 g_1(t,s),\dots,\Re_n g_n(t,s))^T$ has a unique fixed point, i.e. the IP (5), (2) with initial function $\Phi_*^j(t,s)$ has a unique solution $C^j(t,s)$ with interval of existence J_{s+M_0}.

Proof. Let $s \in J_a$ be an arbitrary fixed point, $M \in (0,h]$ and $j \in \langle n\rangle$ are arbitrary numbers. Substitute in (5) an arbitrary function $G_M = (g_1,\dots,g_n)^T \in E_M(s,\Phi_*^j)$. Then for every $k \in \langle n\rangle$ from (9) and the conditions (H) it follows that the all addends in the right side of (9) are continuous functions in t on the interval $t \in [s,s+h]$ and absolutely continuous functions on $t \in (-\infty,s+h]$ and hence they are with bounded variation in t on every compact subinterval of $(-\infty,s+h]$. Since for $k \in \langle n\rangle$ we have that $\Re_k g_k(t,s) = 0$ for every $t < s$ and $\Re_k g_k(s,s) = \phi_{kj}^*(s,s)$ for $t = s$ we can conclude that $\Re G_M \in E_M(s,\Phi_*^j)$ for each $G_M \in E_M(s,\Phi_*^j)$, i.e. $\Re(E_M(s,\Phi_*^j)) \subset E_M(s,\Phi_*^j)$.

The rest part of the proof is analogical of the proof of Theorem 5 in [37] but again for completeness of our exposition we will sketch it too.

Let $G_M, G^*_M \in E_M(s, \Phi^j_*)$ be arbitrary, $A^h = \sum\limits_{i=0}^{m} \sup\limits_{\eta \in J_{s+h}} |A^i(\eta)|$. Then from (9) for $t \in [s, s+M]$ we have

$$|\Re_k g_k(t,s) - \Re_k g^*_k(t,s)| \leq \frac{1}{\Gamma(\alpha_k)} \int\limits_a^t (t-\eta)^{\alpha_k - 1} \sum_{i=0}^{m} (\sum_{q=1}^{n} |a^i_{kq}(\eta)|\ |g_q(\eta - \sigma_i(\eta), s) - g^*_q(\eta - \sigma_i(\eta), s)|) \mathrm{d}\eta \tag{10}$$

Then from (10) it follows that

$$\begin{aligned} &|\Re_k g_k(t,s) - \Re_k g^*_k(t,s)| \leq \\ &\leq \frac{(t-s)^{\alpha_k} A^h}{\Gamma(1+\alpha_k)} \sum_{q=1}^{n} \sup_{\eta \in [s-h, s+M]} |g_q(\eta, s) - g^*_q(\eta, s)| \end{aligned} \tag{11}$$

Let $M^0 = (\Gamma(1+\alpha_k))^{\frac{1}{\alpha_k}} (2nA^h)^{-\frac{1}{\alpha_k}}$ and then for every $t \in [s, s+M^0]$ the following inequality holds

$$\frac{(t-s)^{\alpha_k} A^h}{\Gamma(1+\alpha_k)} \leq \frac{1}{2n} \tag{12}$$

Therefore from (11) and (12) it follows that

$$d_{M^0}(\Re G_{M^0}, \Re G^*_{M^0}) \leq \frac{1}{2} d_{M^0}(G_{M^0}, G^*_{M^0})$$

i.e. the operator $\Re$ is contractive in $E_M(s, \Phi^j_*)$. □

Theorem 2. *Let the conditions (H) hold.*

Then for every fixed $s \in J_a$ *and* $j \in \langle n \rangle$ *the IP* (5), (2) *with initial function* $\Phi^j_*(t,s)$ *has a unique solution* $C^j(t,s)$ *with interval of existence* J_s.

Proof. According Theorem 1 for every $s \in J_a$ and $j \in \langle n \rangle$ the IP (5), (2) with initial function $\Phi^j_*(t,s)$ has a unique solution $C^j(t,s)$ with interval of existence J_{s+M^0}. Let denote by
$C^j_{max}(t,s) = (c^{1j}_{max}(t,s), \ldots, c^{nj}_{max}(t,s))^T$ the maximal solution of the IP (5), (2) with interval of existence J_{max} with left end the point s, i.e. $C^j_{max}(t,s)$ is a continuation of every other solution of the IP (5), (2).

Let assume that $J_{max} \neq [s, \infty)$, i.e. the right site of J_{max} is a finite point. If in addition we suppose that $J_{max} = [s, s+M_{max})$, then obviously the right side of (5) can be prolonged as continuous function at $t = s + M_{max}$ and hence (5) holds for $t = s + M_{max}$ too. Thus we obtain one prolongation of the solution $C^j_{max}(t,s)$ which contradicts to our assumption that $C^j_{max}(t,s)$ is the maximal solution and hence we obtain that $J_{max} = [s, s+M_{max}]$.

Now consider a new initial problem for the system (5) with initial interval $s-h, s+M_{max}]$ and initial function $C^j_{max}(t,s)$ which is well defined in this interval. Then the initial condition has the form:

$$C^j(t,s) = C^j_{max}(t,s),\ t \in [s-h, s+M_{max}] \tag{13}$$

It must be noted that the introduced new IP (5), (13) is a different kind as the IP (5), (2) since the initial point of the IP (5), (13) does not coincide with the initial point of the IP (5), (2) and the jump points of the initial functions have different positions - for the IP (5), (2) it is the right side of the initial interval but for the IP (5), (13) it is an inner point of the initial interval.

Although this difference be essential in several cases, the proof of the existence and uniqueness of the solution of the IP (5), (13) for every fixed $s \in J_a$ is similar to the corresponding one for the IP (5), (2) , but for completeness we will sketch them.

As above introduce the sets

$$\begin{aligned} E_\mu(s, C^j_{max}) = &\{G_m u(\cdot, s) = G|_{[s-h, s+M_{max}+\mu]}| \\ &\qquad G \in E(s, \Phi^j_*), \mu \in (0,h], G|_{[s-h,s+M_{max}]} = C^j_{max}(t,s)\} \end{aligned}$$

with metric

$$d_\mu(G_\mu, G^*_\mu) = \sum_{q=1}^{n} \sup_{t\in[s-h, s+M_{max}+\mu]} |g_q(t,s) - g^*_q(t,s)|$$

and for each $G_\mu = (g_1, \ldots, g_n)^T \in E_\mu(s, C^j_{max})$ consider the operator $(\Re G_\mu)(t,s) = (\Re_1 g_1(t,s), \ldots, \Re_n g_n(t,s))^T$, where the operators $\Re_k g_k(t,s)$ are defined with (9) for all $t \in (s+M_{max}, s+M_{max}+\mu]$ and $k \in \langle n \rangle$. In addition we assume that $\Re G_\mu)(t,s) = C^j_{max}$ for $t \in [s-h, s+M_{max}]$.

Let $G_\mu \in E_\mu(s, C^j_{max})$ be arbitrary. Then for every $k \in \langle n \rangle$ obviously $\Re_k g_k(t,s)$ is a continuous function. Moreover for $k \in \langle n \rangle$ from conditions (H) it follows that the addends in the right side of (9) are at least absolutely continuous functions in $J_{s+M_{max}+\mu}$ and hence they are with bounded variation on $J_{s+M_{max}+\mu}$. Note that since $\Re G_\mu)(t,s) = C^j_{max}$ for $t \in [s-h, s+M_{max}]$ then we can conclude that $\Re G_\mu \in E_\mu(s, C^j_{max})$ for each $G_\mu \in E_\mu(s, C^j_{max})$, i.e. $\Re G_\mu(E_\mu(s, C^j_{max})) \subset E_\mu(s, C^j_{max})$.

Let $G_\mu, G^*_\mu \in E_\mu(s, C^j_{max})$ be arbitrary and let denote $A^* = \sum\limits_{i=0}^{m} \sup\limits_{\eta \in J_{s+M_{max}+h}} |A^i(\eta)|$. Then from (9) it follows that the relation (10) holds for $t \in (s+M_{max}, s+M_{max}+\mu]$. Then for every $t \in (s+M_{max}, s+M_{max}+\mu]$ from (10) it follows that

$$\begin{aligned} &|\Re_k g_k(t,s) - \Re_k g^*_k(t,s)| \le \\ &\le \frac{(t-(s+M_{max}))^{\alpha_k} A^*}{\Gamma(1+\alpha_k)} \sum_{q=1}^{n} \sup_{\eta\in[s-h, s+M_{max}+\mu]} |g_q(\eta,s) - g^*_q(\eta,s)| \end{aligned} \tag{14}$$

Choosing $\mu^0 = (\Gamma(1+\alpha_k))^{\frac{1}{\alpha_k}}(2nA^*)^{-\frac{1}{\alpha_k}}$ we obtain that for every $t \in (s+M_{max}, s+M_{max}+\mu^0]$ the inequality $\frac{(t-(s+M_{max}))^{\alpha_k}A^*}{\Gamma(1+\alpha_k)} \leq \frac{1}{2n}$ holds and taking into account (14) we conclude that

$$d_{\mu^0}(\Re G_{\mu^0}, \Re G^*_{\mu^0}) \leq \frac{1}{2} d_{\mu^0}(G_{\mu^0}, G^*_{\mu^0})$$

i.e. the operator $\Re$ is contractive in $E_{\mu^0}(s, C^j_{max})$.

Thus we obtain one prolongation of $C^j_{max}(t,s)$ which contradicts to our assumption that $C^j_{max}(t,s)$ is a maximal solution and therefore $J_{max} = [s, \infty)$. □

Corollary 1. *Let the conditions (H) hold.*

Then the system (1) *for every* $s \in J_a$ *has a unique fundamental matrix* $C(t,s)$ *, which is the unique solution of the IP* (3), (4). *Moreover the fundamental matrix* $C(t,s)$ *is absolutely continuous in* t *on every compact subinterval of* $(a,s) \cup (s,\infty)$.

Proof. According Theorem 2 for every $s \in J_a$ and $j \in \langle n \rangle$ the IP (5), (2) with initial function $\Phi^j_*(t,s)$ has a unique solution $C^j(t,s)$ with interval of existence J_s and hence the matrix
$C(t,s) = (C^1(t,s), \ldots, C^n(t,s))$ is the unique solution of IP (3), (4).

Taking into account that $C^j(t,s) \in E(s, \Phi^j_*)$ for every fixed $s \in J_a$ and $j \in \langle n \rangle$, then we can conclude that $C^j(\cdot, s)$ is absolutely continuous in t on every compact subinterval of $(a,s) \cup (s,\infty)$. □

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