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A STUDY ON A HYPER–POWER–LOGISTIC MODEL. SOME APPLICATIONS

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ABSTRACT: In this paper we prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t^*}(t)$ by means of a new Hyper–Power–Logistic family. Some applications of the new cumulative sigmoid for analysis of the "cancer data" [46]–[47] and for approximating cdf of the number of Bitcoin received per address [45] are considered. Numerical examples, illustrating our results are given.

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1. INTRODUCTION

Sigmoidal functions find multiple applications to population dynamics, neural networks, machine learning, debugging theory, computer viruses propagation theory and others [1]–[36], [38]–[44].

In a series of papers, we have explored the interesting task of approximating the Heaviside function $h(t)$ with some logistic functions.

The task is important in the treatment of questions related to the study of the "super saturation" - the object of the research in various fields.

In this paper we prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t^*}(t)$ by means of a new Hyper–Power–Logistic family.

The proposed model can be successfully used to approximating data from Population Dynamics, Debugging Theory and Computer Viruses Propagation Theory.

2. PRELIMINARIES

Definition 1. The shifted Heaviside step function is defined by

$$h_{t^*}(t) = \begin{cases} 0, & \text{if} \quad t < t^*, \\ [0,1], & \text{if} \quad t = t^*, \\ 1, & \text{if} \quad t > t^* \end{cases} \tag{1}$$

Definition 2. [37] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \tag{2}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

3. MAIN RESULTS.

3.1. A NEW HYPER–POWER–LOGISTIC MODEL

The Verhulst model can be considered as a prototype of models used in bioreactor modelling.

There, especially in the case of continuous bioreactor, the nutrient supply is considered as an input function $s(t)$ as follows:

$$\frac{dy(t)}{dt} = ky(t)s(t)$$

where s is additional specified.

In [3], the author consider the following hyper–logistic equation:

$$\frac{dy(t)}{dt} = ky(t)\frac{2e^{-pt}}{1+e^{-pt}}$$

$$y(t_0) = y_0,$$

where $k > 0$ and $p > 0$ with general solution:

$$y(t) = y_0 e^{2k(t-t_0)+\frac{2k}{p}\ln(1+e^{pt_0})-\frac{2k}{p}\ln(1+e^{pt})}.$$

Following the ideas given in [3] we consider the following new hyper–power–logistic equation:

$$\begin{aligned} &\frac{dy(t)}{dt} = ky(t)\frac{1}{(1+t)^p} \\ &y(t_0) = y_0. \end{aligned} \tag{3}$$

The general solution of this differential equation is of the form:

$$y(t) = y_0 e^{\frac{k}{1-p}(1+t)^{1-p}-\frac{k}{1-p}(1+t_0)^{1-p}}. \tag{4}$$

It is important to study the characteristic - "super saturation" of the model to the horizontal asymptote.

In this Section we prove upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t^*}(t)$ by means of families (4).

Without loss of generality, we consider the following class of this family (for $t_0 = 0$; $y_0 = e^{\frac{k}{1-p}}$):

$$M(t) = e^{\frac{k}{1-p}(1+t)^{1-p}}. \tag{5}$$

The function $M(t)$ and the "input function" $s(t)$ are visualized on Fig. 1.

Let

$$t^* = \left(\frac{p-1}{k}\ln 2\right)^{\frac{1}{1-p}} - 1. \tag{6}$$

Evidently, $M(t^*) = \frac{1}{2}$.

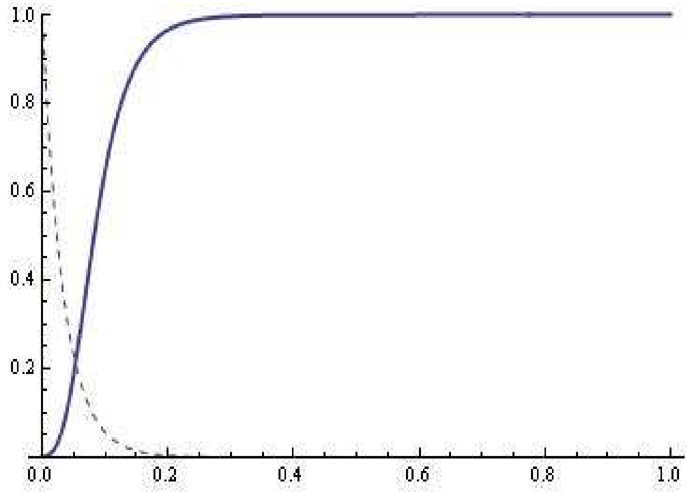

Figure 1: The functions $M(t)$) and "input function" $s(t)$ for $k = 200$; $p = 30$.

3.2. APPROXIMATION RESULTS

The one–sided Hausdorff distance d between the function $h_{t^*}(t)$ and the sigmoid - (5) satisfies the relation

$$M(t^* + d) = 1 - d. \tag{7}$$

The following theorem gives upper and lower bounds for d

Theorem 1. Let

$$\begin{aligned} &\alpha = -\tfrac{1}{2}, \\ &\beta = 1 + \tfrac{k}{2}\left(\tfrac{(p-1)\ln 2}{k}\right)^{-\frac{p}{1-p}} \\ &\gamma = 2.1\beta. \end{aligned} \tag{8}$$

For the one–sided Hausdorff distance d between $h_{t^*}(t)$ and the sigmoid (5) the following inequalities hold for $\gamma > e^{1.05}$:

$$d_l = \frac{1}{\gamma} < d < \frac{\ln \gamma}{\gamma} = d_r. \tag{9}$$

Proof. Let us examine the function:

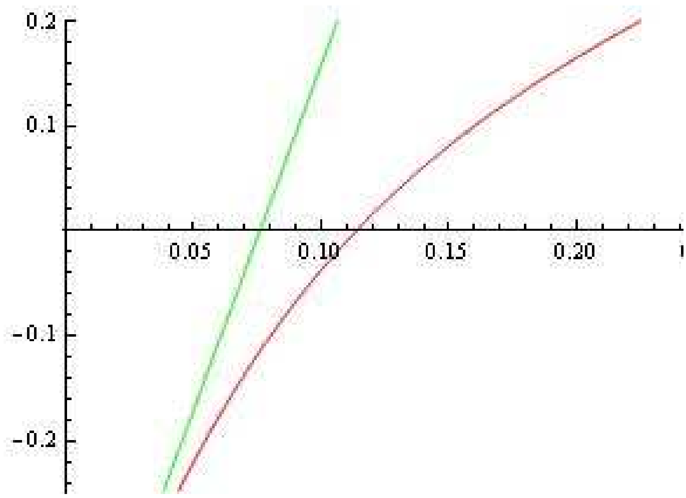

Figure 2: The functions $F(d)$ and $G(d)$ for $k = 300$; $p = 20$.

$$F(d) = M(t^* + d) - 1 + d. \tag{10}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = \alpha + \beta d. \tag{11}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 2).
In addition $G'(d) > 0$.

Further, for $\gamma > e^{1.05}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

Approximations of the $h_{t^*}(t)$ by model (5) for various k and p are visualized on Fig. 3–Fig. 4.

From the graphic it can be seen that the "saturation" is faster.

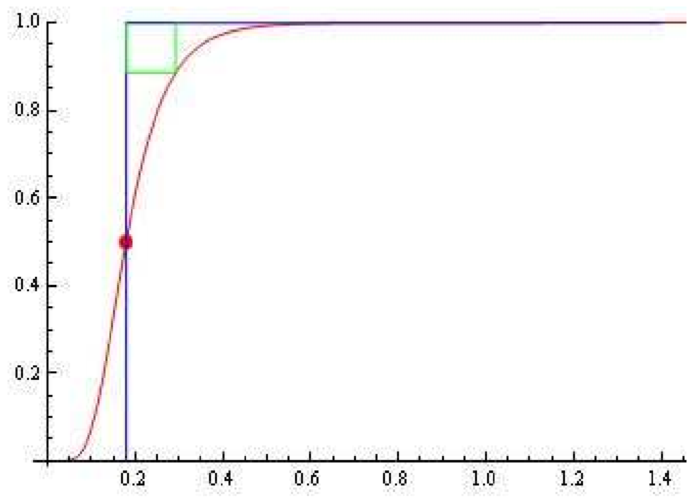

Figure 3: The model (5) for $k = 300$; $p = 20$; $t^* = 0.178826$; Hausdorff distance $d = 0.113634$; $d_l = 0.0723036$; $d_r = 0.189933$.

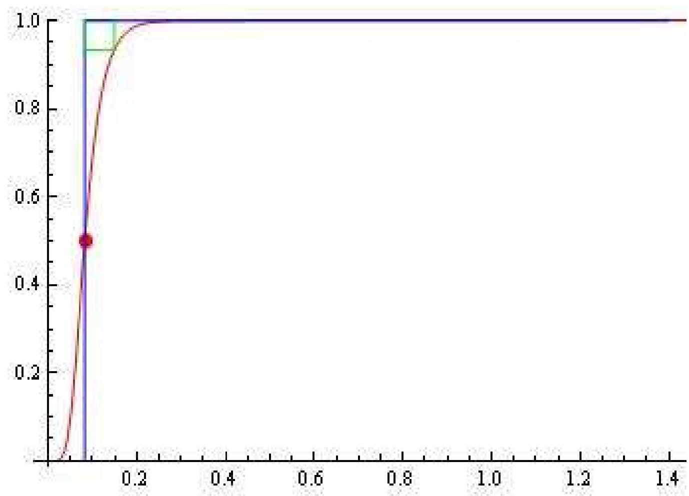

Figure 4: The model (5) for $k = 600$; $p = 40$; $t^* = 0.0827284$; Hausdorff distance $d = 0.0662007$; $d_l = 0.0353162$; $d_r = 0.118077$.

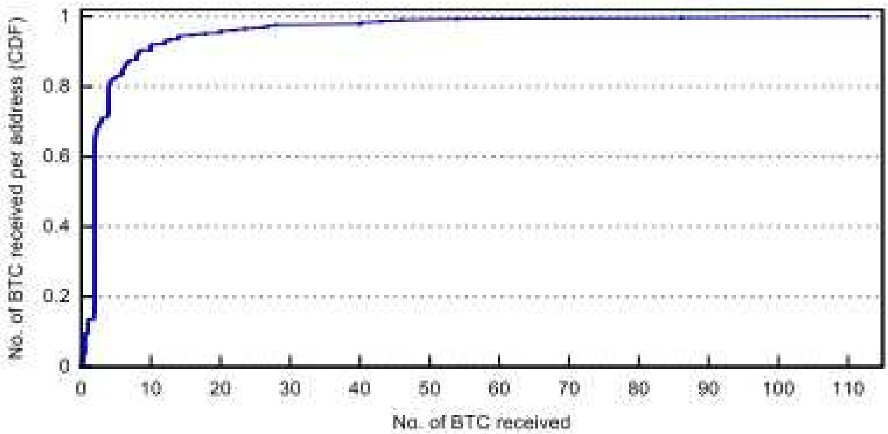

Figure 5: CDF of Bitcoin received (in ransoms) per address in C_{CL} [45].

4. SOME APPLICATIONS

The proposed model can be successfully used to approximating data from Population Dynamic, Biostatistics, Debugging Theory and Computer Viruses Propagation Theory.

4.1. APPROXIMATING CDF OF THE NUMBER OF BITCOIN RECEIVED PER ADDRESS

We consider the following data (see, [45]:

$$data\_CDF\_of\_Bitcoin\_received\_(inransoms)\_per\_address\_in\_C_{CL}$$

$$:= \{\{1, 0.0857\}, \{2, 0.1238\}, \{3, 0.6571\}, \{4, 0.6854\}, \{5, 0.8381\},$$

$$\{6, 0.8476\}, \{7, 0.8810\}, \{8, 0.9095\}, \{9, 0.9143\}, \{10, 0.9333\},$$

$$\{12, 0.9429\}, \{14, 0.9571\}, \{18, 0.9667\}, \{20, 0.9762\}, \{23, 0.9810\},$$

$$\{27, 0.9857\}, \{40, 0.9905\}, \{46, 0.9952\}, \{59, 0.9981\}\}.$$

Fig. 5 show cdf of the number of Bitcoin received per address respectively [45].

After that using the model $M(t)$ for $p = 3$, $k = 19.9786$ we obtain the fitted model (see, Fig. 6).

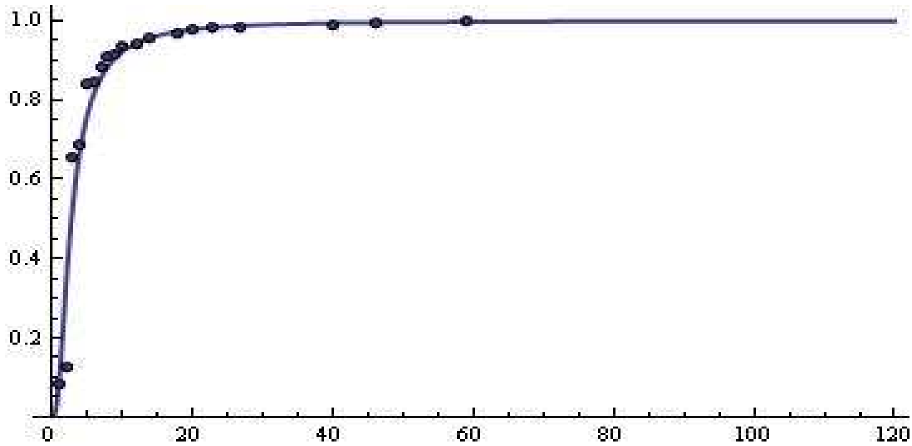

Figure 6: The fitted model $M(t)$ (5).

4.2. APPLICATION OF THE NEW CUMULATIVE SIGMOID FOR ANALYSIS OF THE "CANCER DATA"

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We will illustrate the advances of the new Hyper–Logistic model for approximation and modelling of "cancer data" (for some details see, [46]–[47]).

days	4	7	10	12	14	17	19	21
$R(t)$	0.415	0.794	1.001	1.102	1.192	1.22	1.241	1.3

Table 1: The "cancer data" [46]–[47]

Consider the model

$$M(t) = \omega e^{\frac{k}{1-p}(1+t)^{1-p}}.$$

The model $M(t)$ based on the data from Table 1 for the estimated parameters:

$$\omega = 1.35;\ p = 2.95;\ k = 56.1341$$

is plotted on Fig. 7.

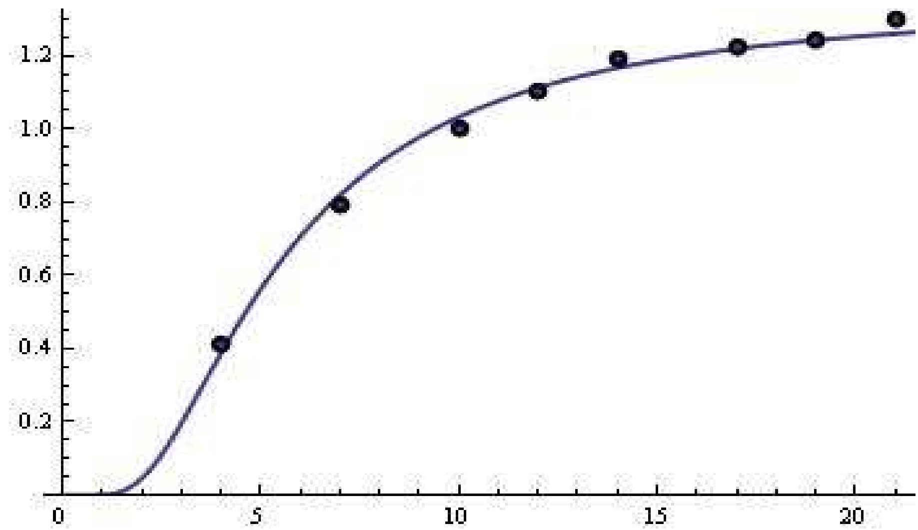

Figure 7: The model $M(t)$ based on the "cancer data".

5. CONCLUSION.

A new Hyper–Power–Logistic population model is introduced.

We prove upper and lower estimates for the Hausdorff approximation of the Heaviside function by means of this new class of functions.

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered family of functions.

The module offers the following possibilities:

- calculation of the H-distance between the h_{t^*} and the function $M(t)$
- software tools for animation and visualization.

Finally, we will point out that when the data accepted in the literature as basis are approximated when compare existing and new models the results obtained with the model in this article are satisfactory.

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