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INVESTIGATIONS ON THE ODD–BURR–III–WEIBULL CUMULATIVE SIGMOID. SOME APPLICATIONS

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ABSTRACT: In this paper we study the one–sided Hausdorff approximation of the shifted Heaviside step function by a family of Odd Burr III Weibull (ODBW) cumulative sigmoid. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study. Application for the approximating cdf of the number of Bitcoin received per address [17] is also discussed.

Numerical examples, illustrating our results are presented using programming environment *CAS Mathematica*.

AMS Subject Classification: 68N30, 41A46

Key Words: odd Burr III Weibull (ODBW) cumulative sigmoid, Heaviside function, Hausdorff approximation, upper and lower bounds

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1. INTRODUCTION

The Weibull distribution has been widely used in survival and reliability analyses.

Some modifications, properties and applications of Weibull and Weibull–R families of distributions can be found in [6]–[15].

The Burr III distribution is well known and widely used in many problems related to forestry, weather forecasting, mechanical factors, reliability quality control, risk analysis, consumer prices and many other areas of research [1]–[5].

In [5], the authors proposed the following Odd Burr III Weibull (ODBW) cumulative sigmoid:

$$M^*(t) = \left(1 + \left(\frac{e^{-\left(\frac{t}{\beta}\right)^\alpha}}{1 - e^{-\left(\frac{t}{\beta}\right)^\alpha}}\right)^c\right)^{-k}, \qquad (1)$$

where $t > 0$, $c > 0$, $k > 0$, $\alpha > 0$, $\beta > 0$.

Definition 1. The *shifted Heaviside step function* is defined by

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases}$$

Definition 2. The Hausdorff distance [16] (the H–distance) $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$.

More precisely,

$$\rho(f, g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t, x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

We study the Hausdorff approximation [16] of the *shifted Heaviside step function* by the family of type (1).

2. MAIN RESULTS

We consider the following class of this family:

$$M(t) = \left(1 + \left(\frac{e^{-\left(\frac{t}{\beta}\right)^\alpha}}{1 - e^{-\left(\frac{t}{\beta}\right)^\alpha}}\right)^c\right)^{-k} \qquad (2)$$

with

$$t_0 = \beta \left(-\ln \frac{\left(0.5^{-\frac{1}{k}} - 1\right)^{\frac{1}{c}}}{1 + \left(0.5^{-\frac{1}{k}} - 1\right)^{\frac{1}{c}}} \right)^{\frac{1}{\alpha}} \tag{3}$$

and $M(t_0) = \frac{1}{2}$.

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid - (2)–(3) satisfies the relation

$$M(t_0 + d) = 1 - d. \tag{4}$$

The following theorem gives upper and lower bounds for d

Theorem. Let

$$\begin{aligned} &p = -\tfrac{1}{2}, \\ &q = 1 + \tfrac{ck\alpha}{\beta} \left(0.5^{-\frac{1}{k}} - 1\right) 0.5^{\frac{k+1}{k}} \left(\left(0.5^{-\frac{1}{k}} - 1\right)^{\frac{1}{c}} + 1 \right) \\ &\times \left(-\ln \tfrac{\left(0.5^{-\frac{1}{k}} - 1\right)^{\frac{1}{c}}}{1 + \left(0.5^{-\frac{1}{k}} - 1\right)^{\frac{1}{c}}} \right)^{\frac{\alpha-1}{\alpha}} \\ &r = 2.1q. \end{aligned} \tag{5}$$

For the one–sided Hausdorff distance d between $h_{t_0}(t)$ and the sigmoid (2)–(3) the following inequalities hold for $q > \dfrac{e^{1.05}}{2.1}$:

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{6}$$

Proof. Let us examine the function:

$$F(d) = M(t_0 + d) - 1 + d. \tag{7}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{8}$$

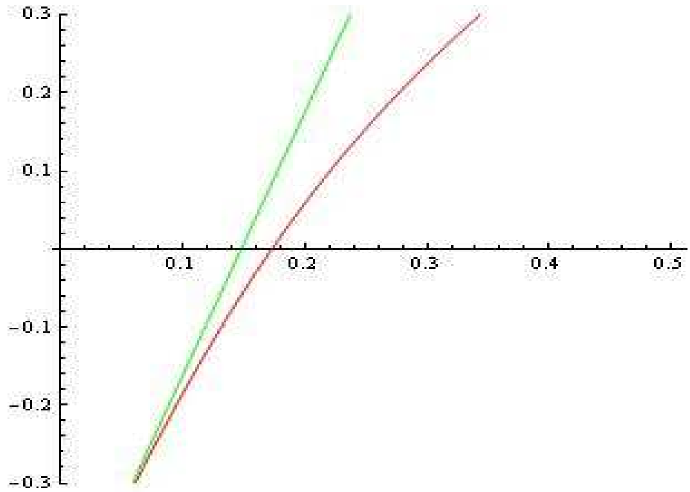

Figure 1: The functions $F(d)$ and $G(d)$ for $\beta = 0.1$; $\alpha = 1.5$; $c = 0.2$; $k = 10$.

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).

In addition $G'(d) > 0$.

Further, for $q > \dfrac{e^{1.05}}{2.1}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

3. NUMERICAL EXAMPLES

The model (2)–(3) for $\beta = 0.1$; $\alpha = 1.5$; $c = 0.2$; $k = 10$, $t_0 = 0.557721$ is visualized on Fig. 2.

From the nonlinear equation (4) and inequalities (6) we have: $d = 0.173242$, $d_l = 0.141208$, $d_r = 0.276418$.

The model (2)–(3)) for $\beta = 0.01$; $\alpha = 1.4$; $c = 0.1$; $k = 30$, $t_0 = 0.133295$ is visualized on Fig. 3.

From the nonlinear equation (4) and inequalities (6) we have: $d = 0.0578306$, $d_l = 0.032805$, $d_r = 0.1121$.

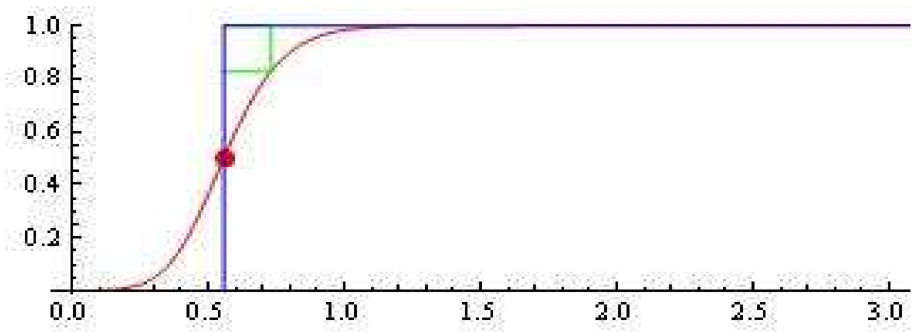

Figure 2: The model (2)–(3)) for $\beta = 0.1$; $\alpha = 1.5$; $c = 0.2$; $k = 10$, $t_0 = 0.557721$; H–distance $d = 0.173242$, $d_l = 0.141208$, $d_r = 0.276418$.

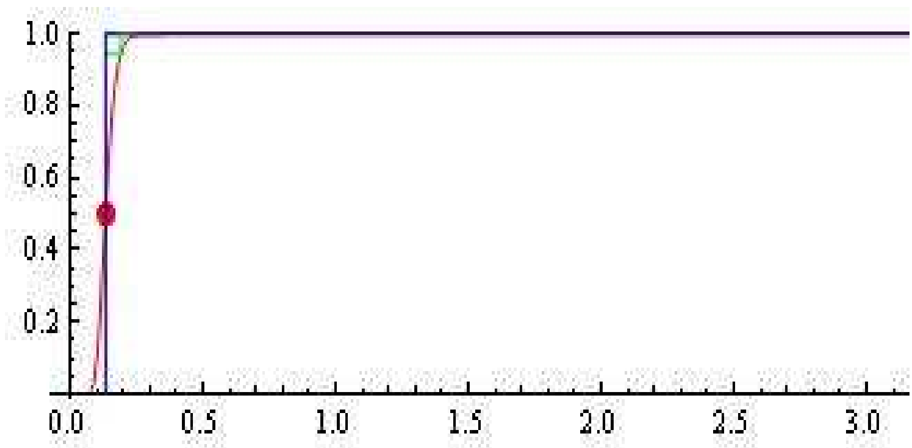

Figure 3: The model (2)–(3)) for $\beta = 0.01$; $\alpha = 1.4$; $c = 0.1$; $k = 30$, $t_0 = 0.133295$; H–distance $d = 0.0578306$, $d_l = 0.032805$, $d_r = 0.1121$.

From the above examples, it can be seen that the proven estimates (see Theorem) for the value of the Hausdorff approximation is reliable when assessing the important characteristic - "saturation".

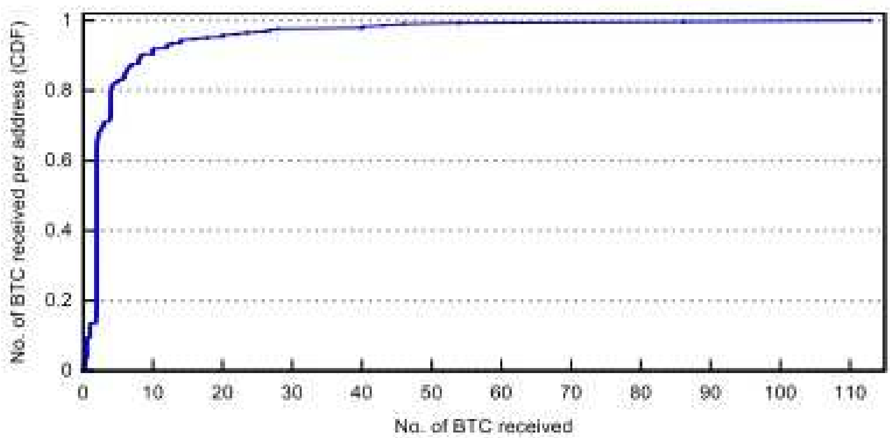

Figure 4: CDF of Bitcoin received (in ransoms) per address in C_{CL} [17].

4. APPROXIMATING CDF OF THE NUMBER OF BITCOIN RECEIVED PER ADDRESS

We consider the following data (see, [17]:

$$data\_CDF\_of\_Bitcoin\_received\_(inransoms)\_per\_address\_in\_C_{CL}$$

$$:= \{\{1, 0.0857\}, \{2, 0.1238\}, \{3, 0.6571\}, \{4, 0.6854\}, \{5, 0.8381\},$$

$$\{6, 0.8476\}, \{7, 0.8810\}, \{8, 0.9095\}, \{9, 0.9143\}, \{10, 0.9333\},$$

$$\{12, 0.9429\}, \{14, 0.9571\}, \{18, 0.9667\}, \{20, 0.9762\}, \{23, 0.9810\},$$

$$\{27, 0.9857\}, \{40, 0.9905\}, \{46, 0.9952\}, \{59, 0.9981\}\}.$$

Fig. 4 show cdf of the number of Bitcoin received per address respectively [17]. After that using the model $M(t)$ for $\beta = 0.12134$, $\alpha = 1.14562$, $c = 0.0479418$ and $k = 5.16358$ we obtain the fitted model (see, Fig. 5).

The proposed model can be successfully used to approximating data from Population Dynamic, Biostatistics, Debugging Theory and Computer Viruses Propagation Theory.

For some approximation, computational and modelling aspects, see [18]–[44].

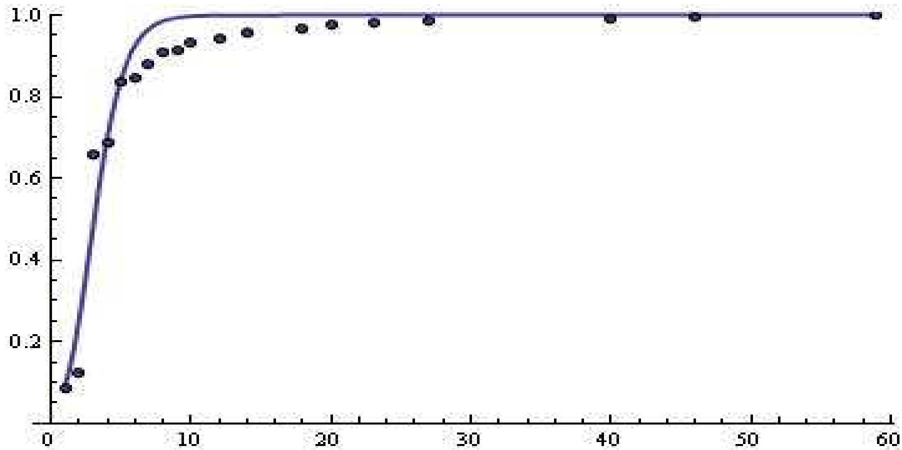

Figure 5: The fitted model $M(t)$.

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