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INVESTIGATIONS ON THE ZUBAIR–FAMILY WITH BASELINE GHOSH–BOURGUIGNON'S EXTENDED BURR XII CUMULATIVE SIGMOID. SOME APPLICATIONS

OLGA RAHNEVA[1], TODORKA TERZIEVA[2], AND ANGEL GOLEV[3]

[1]Faculty of Economy and Social Sciences
University of Plovdiv Paisii Hilendarski
24, Tzar Asen Str., 4000 Plovdiv, BULGARIA
[2,3]Faculty of Mathematics and Informatics
University of Plovdiv Paisii Hilendarski
24, Tzar Asen Str., 4000 Plovdiv, BULGARIA

ABSTRACT: In this paper we study the one–sided Hausdorff approximation of the shifted Heaviside step function by a class of the Zubair–family of cumulative distribution with baseline Ghosh–Bourguignon's extended Burr XII c.d.f. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study.

We will illustrate the advances of this new model for approximation and modelling of data for Witty worm for entire world and for USA [1] (see, also [16]) and "cancer data" (for some details see, [19], [20].

Numerical examples, illustrating our results are presented using programming environment *CAS Mathematica*.

AMS Subject Classification: 68N30, 41A46

Key Words: Zubair–G family of cumulative distribution, Zubair–G Family with baseline Ghosh–Bourguignon's extended Burr XII (cdf), Heaviside function, Hausdorff approximation, upper and lower bounds

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1. INTRODUCTION

In [2], a new family of lifetime distributions, called the Zubair–G family of distributions is introduces.

The new family is defined by the following cumulative distribution function (cdf)

$$F(t;\lambda) = \frac{e^{\lambda G^2(t)} - 1}{e^{\lambda} - 1}, \tag{1}$$

where $\lambda > 0$ and $G(t)$ is the (cdf) of the baseline model.

Some comments on a Zubair-G Family of cumulative lifetime distributions with baseline Weibull, Lomax and Burr (cdf) can be found in [3], [17]–[18].

In [4] Ghosh and Bourguignon proposed the following c.d.f. (so–called extended Bur XII c.d.f.):

$$G(t) = 1 - \frac{2}{1 + (1 + t^c)^{\mu}}; \quad c > 0,\ \mu > 0. \tag{2}$$

We consider the following class of the Zubair–family with baseline Ghosh–Bourguignon's extended Burr XII cumulative sigmoid:

$$M(t) = \frac{e^{\lambda\left(1 - \frac{2}{1+(1+t^c)^{\mu}}\right)^2} - 1}{e^{\lambda} - 1}, \tag{3}$$

with

$$t_0 = \left(\left(\frac{1 + \sqrt{\frac{1}{\lambda} \ln \frac{e^{\lambda}+1}{2}}}{1 - \sqrt{\frac{1}{\lambda} \ln \frac{e^{\lambda}+1}{2}}} \right)^{\frac{1}{\mu}} - 1 \right)^{\frac{1}{c}}; \qquad M(t_0) = \frac{1}{2}. \tag{4}$$

In this note we study the Hausdorff approximation of the *shifted Heaviside step function*

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases}$$

by this family.

Definition 1. [5] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

As an illustrative example we consider the fitting the new model against data for Witty worm for entire world and for USA [1].

2. MAIN RESULTS

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid - ((3)–(4)) satisfies the relation

$$M(t_0 + d) = 1 - d. \tag{5}$$

The following theorem gives upper and lower bounds for d

Theorem. *Let*

$$\begin{aligned} &p = -\tfrac{1}{2},\\ &q = 1 + \frac{c\mu\lambda(1+e^{\lambda})\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}\left(1-\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}\right)^2}{2(e^{\lambda}-1)}\times\\ &\left(\left(\frac{1+\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}}{1-\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}}\right)^{\frac{1}{\mu}} - 1\right)^{\frac{c-1}{c}}\left(\frac{1+\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}}{1-\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}}\right)^{\frac{\mu-1}{\mu}},\\ &r = 2.1q. \end{aligned} \tag{6}$$

For the one–sided Hausdorff distance d between $h_{t_0}(t)$ and the sigmoid ((3)–(4)) the following inequalities hold for: $q > \frac{e^{1.05}}{2.1}$

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{7}$$

Proof. Let us examine the function:

$$F(d) = M(t_0 + d) - 1 + d. \tag{8}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{9}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).

In addition $G'(d) > 0$.

Further, for $q > \frac{e^{1.05}}{2.1}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

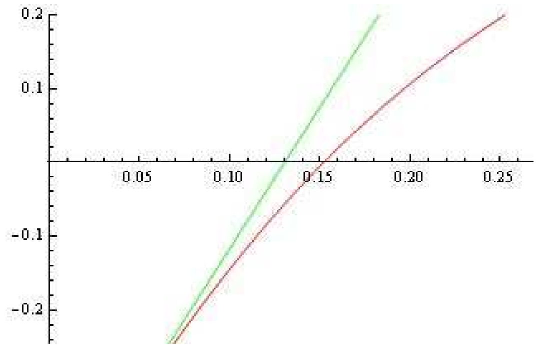

Figure 1: The functions $F(d)$ and $G(d)$ for $\mu = 4$; $c = 5$; $\lambda = 0.05$.

3. NUMERICAL EXAMPLES

The model ((3)–(4)) for $\mu = 4$; $c = 5$; $\lambda = 0.05$, $t_0 = 0.890723$ is visualized on Fig. 2.

From the nonlinear equation (5) and inequalities (7) we have: $d = 0.151979$, $d_l = 0.124188$, $d_r = 0.259051$.

The model ((3)–(4)) for $\mu = 8$; $c = 10$; $\lambda = 0.02$, $t_0 = 0.869714$ is visualized on Fig. 3.

From the nonlinear equation (5) and inequalities (7) we have: $d = 0.0830442$, $d_l = 0.0640181$, $d_r = 0.175959$.

The model ((3)–(4)) for $\mu = 10$; $c = 20$; $\lambda = 0.01$, $t_0 = 0.921083$ is visualized on Fig. 4.

From the nonlinear equation (5) and inequalities (7) we have: $d = 0.0493593$, $d_l = 0.0354992$, $d_r = 0.118505$.

From the above examples, it can be seen that the proven estimates (see Theorem) for the value of the Hausdorff approximation is reliable when assessing the important characteristic - "saturation".

4. SOME APPLICATIONS

1. Here we will give an application of the new cumulative sigmoid when provide analysis of the following data which we photographed for the situation that was happen on

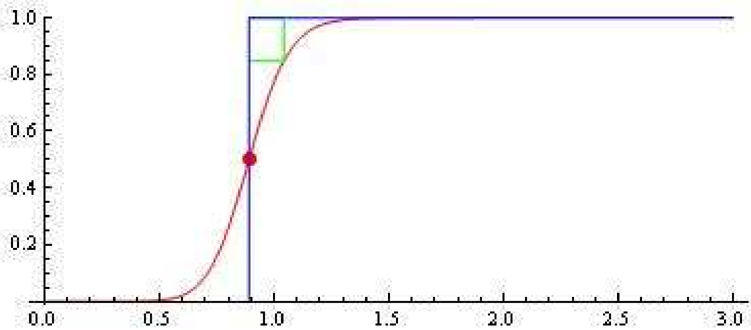

Figure 2: The model ((3)–(4)) for $\mu = 4$; $c = 5$; $\lambda = 0.05$, $t_0 = 0.890723$; H–distance $d = 0.151979$, $d_l = 0.124188$, $d_r = 0.259051$.

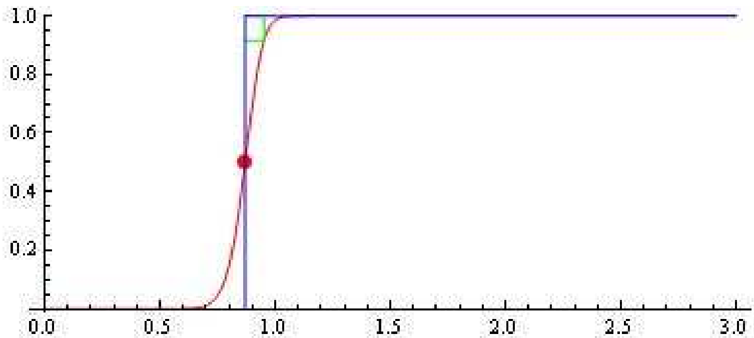

Figure 3: The model ((3)–(4)) for $\mu = 8$; $c = 10$; $\lambda = 0.02$, $t_0 = 0.869714$; H–distance $d = 0.0830442$, $d_l = 0.0640181$, $d_r = 0.175959$.

March 19, 2004, at approximately 8:45 p.m. Pacific Standard Time (PST), [1] where the first coordinate is time with step 5 minutes and second coordinate is number of infected hosts

data_World = {{0.1,150},{5,869},{10,2141},{15,3637},{20,5312},
{26,6602},{31,7562},{36,8340},{41,8941},{46,9389},{51,9734},
{56,10060},{61,10349},{66,10586},{71,10800},{76,11169},
{86,11362},{91,11532},{96,11684},{101,11823},{106,11972},
{111,12118},{116,12256},{121,12372}}

and

data_USA = {{0.1,150},{5,576},{10,1236},{15,1963},{20,2973},
{26,3488},{31,3953},{36,4343},{41,4630},{46,4825},{51,4986},

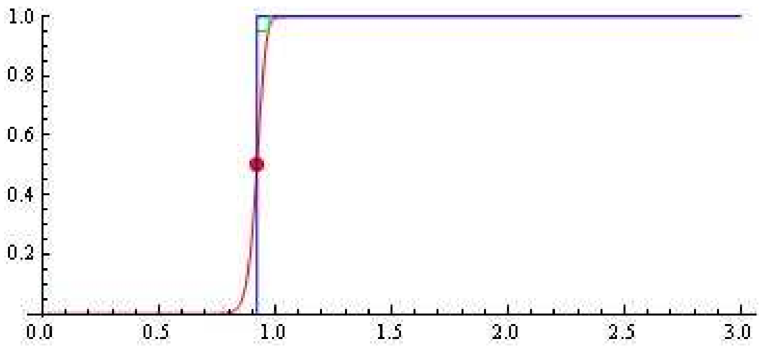

Figure 4: The model ((3)–(4)) for $\mu = 10$; $c = 20$; $\lambda = 0.01$, $t_0 = 0.921083$; H–distance $d = 0.0493593$, $d_l = 0.0354992$, $d_r = 0.118505$.

{56,5153},{61,5280},{66,5380},{71,5468},{76,5590},{86,5706},
{91,5769},{96,5831},{101,5877},{106,5939},{111,5989},
{116,6033},{121,6063}}.

From data analysis it follows that it is worthy to use the following model:

$$M(t) = \omega \left(\frac{e^{\lambda\left(1-\frac{2}{1+(1+t^c)^\mu}\right)^2} - 1}{e^\lambda - 1} \right). \tag{10}$$

When solving this task in programming environment Mathematica we receive the results (see, Fig. 5 and Fig. 6 respectively for entire World and USA spreading of Witty worm):

- for entire World spreading parameters are

$$\omega = 12372; c = 0.425265; \mu = 4.15722; \lambda = 116.886,$$

and

- for USA spreading parameters are

$$\omega = 6063; c = 0.450205; \mu = 3.96908; \lambda = 95.4303.$$

2. Here we an application of the new cumulative sigmoid for analysis of the following ”cancer data” (for some details see, [19], [20].

Consider the model:

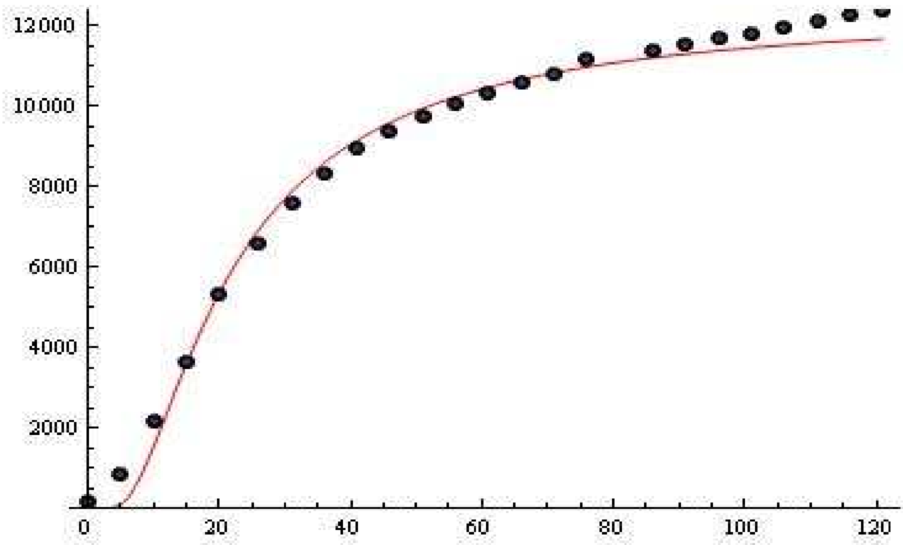

Figure 5: Epidemic data for entire world of Witty worm fitted with model (10)

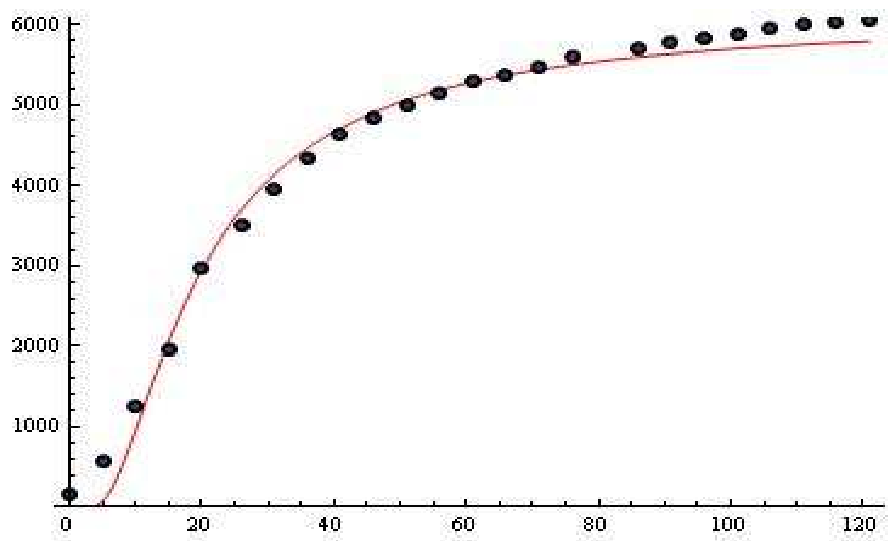

Figure 6: Epidemic data for USA of Witty worm fitted with with model (10)

$$M(t) = \omega \left(\frac{e^{\lambda\left(1-\frac{2}{1+(1+t^c)^\mu}\right)^2} - 1}{e^\lambda - 1} \right). \tag{11}$$

days	4	7	10	12	14	17	19	21
$R(t)$	0.415	0.794	1.001	1.102	1.192	1.22	1.241	1.3

Table 1: The ”cancer data” [19], [20]

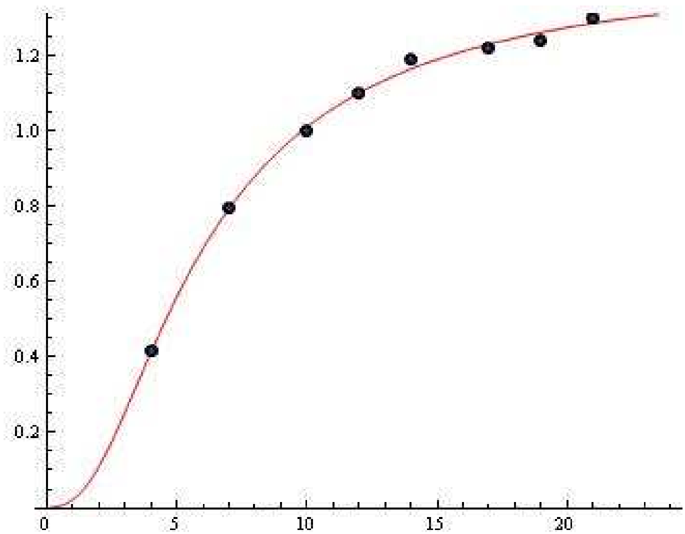

Figure 7: The model $M(t)$ based on the ”cancer data”.

The model (11) based on the data from Table 1 for the estimated parameters:

$$\omega = 1.43404;\ c = 1.10616;\ \mu = 1.52777;\ \lambda = 4.93163$$

is plotted on Fig. 7.

For the predictive power (PP) criterion:

$$PP = \sum_i \left(\frac{M(t_i) - y_i}{y_i} \right)^2$$

we find $PP = 0.00110303$.

From the conducted experiments (see, also Fig. 7 and Fig. 8) it can be concluded that the examined model can be successfully used in the field of Population dynamics.

For some approximation, computational and modelling aspects, see [6]–[12].

Some software reliability models, can be found in [13]–[15], [23]–[33].

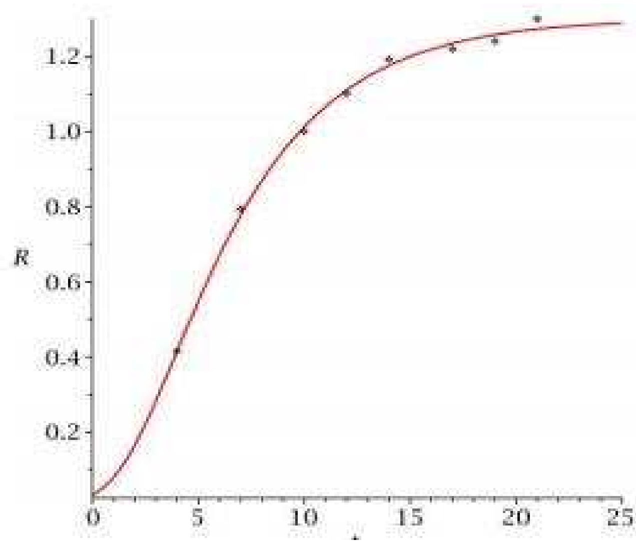

Figure 8: Numerical solution of the inhibition initial value problem and initial data–points (see Antonov, Nenov and Tsvetkov [20]).

For some specific properties of the solutions of impulse differential equations with applications to tumor growth theory, see [21], [22].

5. ACKNOWLEDGMENTS

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