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STABILITY AND F-ASYMPTOTIC STABILITY ANALYSIS OF NONLINEAR FRACTIONAL ORDER SYSTEMS WITH RIEMANN-LIOUVILLE DERIVATIVE

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ABSTRACT: In this paper we present analytical approaches for stability analysis of fractional orders systems (FDEs) involving Riemann-Liouville derivatives. Particularly, we derive sufficient conditions for stability and **F**-asymptotic stability for a class of nonlinear FDEs. We further apply numerical simulations on a test case to support our theoretical analysis.

Key Words: Riemann-Liouville derivative, fractional differential system, **F**-asymptotic stability, uniformly essentially bounded

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1. INTRODUCTION

Fractional calculus is a generalization of classical differentiation and integration to arbitrary order. In recent years, fractional calculus has been a fruitful field of research in science and engineering. Meanwhile, applications of fractional differential equations (FDEs) to physics, biology and engineering are a recent focus of interests [9, 10]. Recently, the theory of FDEs has been studied and some basic results are obtained including stability theory [3, 22, 23, 27]. The question of stability is of main interest in physical and biological systems. The analysis on stability of FDEs is more complex than that of classical differential equations, since fractional derivatives are nonlocal and have weakly singular kernels.

The earliest study on stability of FDEs started in [15] where the author studied the case of linear FDEs with Caputo derivative and the same fractional order α, where $0 < \alpha < 1$. The stability problem reduces to the eigenvalue problem of system matrix. Corresponding to the stability result in [15], Qian et al. [20] recently studied the case of linear FDEs with Riemann-Liouville derivative and the same fractional order α, for $0 < \alpha < 1$. Then, in [14, 17, 18] the same conclusions as [15] have been derived for the case $1 < \alpha < 2$.

A sufficient condition on Lyapunov global asymptotical stability for the linear systems with multi-order Caputo derivative was presented in [6] . Many researchers have shown interests in the stability of linear systems and various methods which emerged in succession. For example, frequency domain methods [2, 8, 21, 24, 25], Linear Matrix Inequalities (LMI) methods were presented in [17, 18]. By contrast, the development of stability of nonlinear FDEs is a bit slow. The structural stability of the system with Riemann-Liouville derivative has been presented in [7]. In [4] authors investigated the system of nonautonomous FDEs involving Caputo derivative and derived the result on continuous dependence of solution on initial conditions. The stability in the sense of Lyapunov has also been studied [16] by using Gronwall lemma and Schwartz inequality. Some researchers weakened the criterion of stability, such as [11] where the L^P-stability properties of nonlinear FDEs were investigated. In [12, 13], the Mittag-Leffler stability and the fractional Lyapunov of the second method were proposed. Deng [5] derived a sufficient stability condition of nonlinear FDEs.

The paper is organized as follows. In Section 2, we present some basic materials on fractional calculus. Some stability results of the system ${}_{RL}D^{\alpha}_{0,t}x(t) = Ax(t) + b(t)$ are presented in Section 3. In Section 4, the stability of fractional differential systems ${}_{RL}D^{\alpha}_{0,t}x(t) = Ax(t) + f(t, x(t))$ are analyzed. $\mathbf{F}$-asymptotic stability of the system ${}_{RL}D^{\alpha}_{0,t}x(t) = Ax(t) + B(t)x(t)$ and a note on the stability theorem given in [20] is presented in Section 5. In Section 6, we present a numerical example, in which we compute different orbits of the given systems by means of numerical simulations, to reveal validity of our analytical results. In Section 7, we conclude the paper.

2. PRELIMINARIES

Two types of fractional derivatives of Riemann-Liouville and Caputo derivatives, have been often used in fractional differential systems. We briefly introduce these two definitions.

Definition 2.1. The Riemann-Liouville integral $J^{\alpha}_{t_0,t}$ with fractional order $\alpha \in \mathbb{R}_+$

of function $x(t)$ is defined as:

$$J_{t_0,t}^{\alpha}x(t) := D_{t_0,t}^{-\alpha}x(t) := \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}x(\tau)d\tau$$

where $\Gamma(.)$ is the Eulers gamma function, for $\alpha = 0$ we set $J_{t_0,t}^{0} := Id$, the identity operator.

Definition 2.2. The Riemann-Liouville derivative with fractional order $\alpha \in \mathbb{R}_+$ of function $x(t)$ is defined by:

$$_{RL}D_{t_0,t}^{\alpha}x(t) := \frac{d^m}{dt^m}J_{t_0,t}^{(m-\alpha)}x(t)$$

where $m-1 < \alpha \leq m \in \mathbb{Z}_+$.

The Laplace transform of the Riemann-Liouville fractional derivative $_{RL}D_{0,t}^{\alpha}x(t)$ for $0 < \alpha \leq 1$ is

$$L\{_{RL}D_{0,t}^{\alpha}x(t)\} = s^{\alpha}X(s) - (D_0^{\alpha-1}x(t))_{t=0}$$

Here $X(s)$ is the Laplace transform of $x(t)$.

Definition 2.3. The Caputo derivative with fractional order $\alpha \in \mathbb{R}_+$ of function $x(t)$ is defined by:

$$_{C}D_{t_0,t}^{\alpha}x(t) := J_{t_0,t}^{(m-\alpha)}\frac{d^m}{dt^m}x(t)$$

where $m-1 < \alpha \leq m \in \mathbb{Z}_+$.

The Laplace transform of the Caputo fractional derivative $_{C}D_{a,t}^{\alpha}x(t)$ is

$$L\{_{C}D_{a,t}^{\alpha}x(t)\} = s^{\alpha}X(s) - \sum_{k=1}^{m}s^{\alpha-k}x^{(k-1)}(a), \qquad (m-1 < \alpha \leq m).$$

If $0 < \alpha \leq 1$ we have

$$L\{_{C}D_{a,t}^{\alpha}x(t)\} = s^{\alpha}X(s) - s^{\alpha-1}x(a).$$

Definition 2.4. [19] The Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$$

where $\alpha > 0, z \in \mathbb{C}$. The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

where $\alpha, \beta > 0, z \in \mathbb{C}$. It can be see easily that $E_{\alpha}(z) = E_{\alpha,1}(z)$.

For $j \in \mathbb{N}_0$, $\lambda \in \mathbb{R}$ and $\alpha, \beta > 0$ the Laplace transform of the function $f(t) = t^{j\alpha+\beta-1} E^{(j)}_{\alpha,\beta}(\pm\lambda t^{\alpha})$ can be easily found to be

$$L\{f(t)\} = \frac{j! s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)^{j+1}},$$

If $\beta = \alpha$ and $j = 0$ we have:

$$L\{t^{\alpha-1} E_{\alpha,\alpha}(\pm\lambda t^{\alpha})\} = \frac{1}{s^{\alpha} \mp \lambda},$$

and if $\beta = 1, j = 0$ we have:

$$L\{E_{\alpha}(\pm\lambda t^{\alpha})\} = \frac{s^{\alpha-1}}{s^{\alpha} \mp \lambda}.$$

The Mittag-Leffler function has the following asymptotic expression.

Lemma 2.5. *[19] If $0 < \alpha < 2$ and β is an arbitrary complex number, then for an arbitrary integer $p \geq 1$ the following expansions hold:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{(1-\beta)}{\alpha}} exp(z^{\frac{1}{\alpha}}) - \sum_{k=1}^{p} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O\Big(\frac{1}{|z|^{p+1}}\Big)$$

with $|z| \to \infty, |arg(z)| \leq \frac{\alpha\pi}{2}$, and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O\Big(\frac{1}{|z|^{p+1}}\Big)$$

with $|z| \to \infty, |arg(z)| > \frac{\alpha\pi}{2}$.

We consider the following general type of fractional differential equations involving Riemann-Liouville derivative

$${}_{RL}D^{\alpha}_{t_0,t} x(t) = f(t, x(t)) \tag{2.1}$$

with suitable initial values ${}_{RL}D^{\alpha-k}_{t_0,t} x(t)|_{t=t_0} = x_k = (x_{k1}, x_{k2}, ..., x_{kn})^T \in \mathbb{R}^n$ $(k = 1, ..., m)$ where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $m - 1 < \alpha \leq m \in \mathbb{Z}_+$ and $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.6. The system (2.1) is said to be stable if, for any initial values $x_k = (x_{k1}, x_{k2}, ..., x_{kn})^T \in \mathbb{R}^n$ $(k = 1, ..., m)$, there exists an $\varepsilon > 0$ such that any solution $x(t)$ of (2.1) satisfies $\|x(t)\| < \varepsilon$ for all $t > t_0$. The system (2.1) is said to be asymptotically stable if $\|x(t)\| \to 0$ as $t \to \infty$.

Recently D.Qian, et all; [20] studied the case of the following linear system of FDEs with Riemann-Liouville derivative by using the asymptotic expansions of Mittag-Leffler function, $0 < \alpha < 1$,

$${}_{RL}D^{\alpha}_{t_0,t} x(t) = Ax(t) \tag{2.2}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. We recall the following theorem from [20].

Theorem 2.7. *The system (2.2) with initial value* ${}_{RL}D^{\alpha-1}_{t_0,t}x(t)|_{t=t_0}$*, where* $0 < \alpha < 1$ *and* $t_0 = 0$*, is*
i) *asymptotically stable if all the non-zero eigenvalues of A satisfy* $|arg(spec(A))| > \frac{\alpha\pi}{2}$*, or A has k-multiple zero eigenvalues corresponding to a Jordan block* $diag(J_1, J_2, ..., J_i)$*, where* J_l *is a Jordan canonical form with order* $n_l \times n_l$*,* $\sum_{l=1}^{i} n_l = k$*, and* $n_l\alpha < 1$ *for each* $1 \le l \le i$*.*

ii) *stable if all the non-zero eigenvalues of A satisfy* $|arg(spec(A))| \ge \frac{\alpha\pi}{2}$ *and the critical eigenvalues satisfying* $|arg(spec(A))| = \frac{\alpha\pi}{2}$ *have the same algebraic and geometric multiplicities, or A has k-multiple zero eigenvalues corresponding to a Jordan block matrix* $diag(J_1, J_2, ..., J_i)$*, where* J_l *is a Jordan canonical form with order* $n_l \times n_l, \sum_{l=1}^{i} n_l = k$*, and* $n_l\alpha \le 1$ *for each* $1 \le l \le i$*.*

We derive the following theorem from the Theorem 2.7.

Theorem 2.8. *The following statement hold for the Mittag-Leffler function* $E_{\alpha,\alpha}(At^\alpha)$*:*
a) *If all the non-zero eigenvalues of A satisfy* $|arg(spec(A))| > \frac{\alpha\pi}{2}$ *then*
i) $E_{\alpha,\alpha}(At^\alpha)$ *remains bounded for* $t \to \infty$ *if zero eigenvalues of A have the same algebraic and geometric multiplicities.*
ii) $t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$ *remains bounded for* $t \to \infty$ *if A has k-multiple zero eigenvalues corresponding to a Jordan block* $diag(J_1, J_2, ..., J_i)$*, where* J_l *is a Jordan canonical form with order* $n_l \times n_l, \sum_{l=1}^{i} n_l = k$*, and* $n_l\alpha \le 1$ *for* $1 \le l \le i$*.*

b) *If all the non-zero eigenvalues of A satisfy* $|arg(spec(A))| \ge \frac{\alpha\pi}{2}$ *and those critical eigenvalues which satisfy* $|arg(spec(A))| = \frac{\alpha\pi}{2}$*, have the same algebraic and geometric multiplicities, then* $t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$ *remains bounded for* $t \to \infty$*.*

3. STABILITY OF $\boldsymbol{{}_{RL}D^{\alpha}_{0,T}X(T) = AX(T) + B(T)}$

In this section, we consider the nonlinear fractional differential system with Riemann-Liouville derivative

$$ {}_{RL}D^{\alpha}_{0,t}x(t) = Ax(t) + b(t), \quad (0 < \alpha < 1) \tag{3.1} $$

under the initial condition $x_0 = {}_{RL}D^{\alpha-1}_{0,t}x(t)|_{t=0}$, where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and $b(t) : [0, \infty) \to \mathbb{R}^n$ is a continuous vector function. We can get the solution of (3.1), by using the Laplace and inverse Laplace transforms, as

$$x(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})x_0 + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})b(\theta)d\theta. \tag{3.2}$$

We present the following stability results.

Theorem 3.1. *Suppose that all the non-zero eigenvalues of A satisfy*

$$|arg(spec(A))| > \frac{\alpha\pi}{2}$$

and the zero eigenvalues of A have the same algebraic and geometric multiplicities. Let $b(t) : [0,\infty) \to \mathbb{R}^n$ be a continuous vector function, and there exist $\beta > \alpha$ and $M, h > 0$ such that

$$\|b(t)\| \le \frac{M}{t^{\beta}} \qquad (t \ge h).$$

Then the system (3.1) is asymptotically stable.

Proof. By using the continuity of $b(t)$ we can assume that $\alpha < \beta < 1$, and by using the assumptions $N = \sup\limits_{\theta \ge 0} \|b(\theta)\|$ is finite, also by using Theorem 2.8, $E = \sup\limits_{0 \le t < \infty} \|E_{\alpha,\alpha}(At^{\alpha})\|$ is finite. So, from (3.2) it suffices to show that

$$\lim_{t\to\infty} \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})b(\theta)d\theta = 0. \tag{3.3}$$

Now, we have

$$\int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})b(\theta)d\theta =$$
$$\int_0^h (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})b(\theta)d\theta + \int_h^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})b(\theta)d\theta.$$

We observe that

$$\begin{aligned}\Big\| \int_0^h (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})b(\theta)d\theta \Big\| &\le \int_0^h (t-\theta)^{\alpha-1}EN d\theta \\ &= \frac{ENh}{\alpha}[\frac{t^{\alpha}-(t-h)^{\alpha}}{h}] \longrightarrow 0\end{aligned}$$

as $t \to \infty$ and

$$\begin{aligned}\Big\| \int_h^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})b(\theta)d\theta \Big\| &\le E\int_h^t (t-\theta)^{\alpha-1}\|b(\theta)\|d\theta \\ &\le EM\int_h^t (t-\theta)^{\alpha-1}\theta^{-\beta}d\theta\end{aligned}$$

$$\begin{aligned}
&\leq EM\int_0^t (t-\theta)^{\alpha-1}\theta^{-\beta}d\theta \\
&\leq EM\int_0^1 (t-ty)^{\alpha-1}(ty)^{-\beta}tdy \\
&= EMt^{\alpha-\beta}B(\alpha,1-\beta) \longrightarrow 0
\end{aligned}$$

as $t \to \infty$, where $B(.,.)$ stands for the Beta function. This completes the proof. □

We present the following definitions which are needed for the stability of the non-linear system to be discussed in the next section.

Definition 3.2. The system (2.1) with initial condition $x_0 = {}_{RL}D_{0,t}^{\alpha-1}x(t)|_{t=0}$ is said to be essentially bounded (ess-bdd) if, for any $\varepsilon > 0$ there exist the constants $T, M > 0$ such that $\|x_0\| < \varepsilon$ implies $\|x(t)\| < M$, for all $t > T$.

Definition 3.3. The solution $x(t) : (0, +\infty) \to \mathbb{R}^n$ of the system (2.1) is said to be ess-bdd if, there exists $T > 0$ such that the restriction of $x(t)$ on $[T, +\infty)$ is bounded.

Definition 3.4. The system (2.1) with initial condition $x_0 = {}_{RL}D_{0,t}^{\alpha-1}x(t)|_{t=0}$ is said to be uniformly essentially bounded (u-ess-bdd) if, there exist some $T > 0$ for which $\forall\varepsilon > 0,\ \exists M > 0$ such that $\|x_0\| < \varepsilon \Rightarrow \|x(t)\| < M$, $(t > T)$.

4. STABILITY OF ${}_{RL}D_{0,T}^{\alpha}X(T) = AX(T) + F(T, X(T))$

In this section, we study the following fractional differential system with Riemann-Liouville derivative

$${}_{RL}D_{0,t}^{\alpha}x(t) = Ax(t) + f(t, x(t)), \quad (0 < \alpha < 1) \tag{4.1}$$

under the initial condition $x_0 = {}_{RL}D_{0,t}^{\alpha-1}x(t)|_{t=0}$, where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. The system (4.1) can be solved analytically as

$$x(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})x_0 + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^{\alpha})f(\theta, x(\theta))d\theta. \tag{4.2}$$

We establish the following stability results.

Theorem 4.1. *Suppose f be a continuous vector function, and there exist constants $M, h > 0$ and $\beta > \alpha$ such that for all $t \geq h$,*

$$\|f(t, x(t))\| \leq \frac{M}{t^{\beta}}.$$

Then the system (4.1) *is asymptotically stable if all the non-zero eigenvalues of A satisfy*

$$|arg(spec(A))| > \frac{\alpha\pi}{2},$$

and the zero eigenvalues of A have the same algebraic and geometric multiplicities.

Proof. Following the proof of Theorem 3.1, we suppose that $\alpha < \beta < 1$ and there exists $E > 0$ such that $\|E_{\alpha,\alpha}(At^{\alpha})\| \leq E$. Thus

$$\|x(t)\| \leq \|t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})x_0\| + \int_0^t (t-\theta)^{\alpha-1}\|E_{\alpha,\alpha}(A(t-\theta)^{\alpha})\| \; \|f(\theta,x(\theta))\|d\theta$$

we set

$$I = \int_0^t (t-\theta)^{\alpha-1}\|E_{\alpha,\alpha}(A(t-\theta)^{\alpha})\| \; \|f(\theta,x(\theta))\|d\theta$$

so

$$I \leq E\Big[\int_0^h (t-\theta)^{\alpha-1}\|f(\theta,x(\theta))\|d\theta + \int_h^t (t-\theta)^{\alpha-1}\|f(\theta,x(\theta))\|d\theta\Big]. \tag{4.3}$$

By using the assumptions $F = \sup\limits_{0\leq t\leq h} \|f(t,x(t)\|$ is finite. then

$$\int_0^h (t-\theta)^{\alpha-1}\|f(\theta,x(\theta))\|d\theta \leq F\int_0^h (t-\theta)^{\alpha-1}d\theta \longrightarrow 0$$

as $t \to \infty$, and

$$\begin{aligned}\int_h^t (t-\theta)^{\alpha-1}\|f(\theta,x(\theta))\|d\theta \leq \int_h^t (t-\theta)^{\alpha-1}\frac{M}{\theta^{\beta}}d\theta &\leq M\int_0^t (t-\theta)^{\alpha-1}\theta^{-\beta}d\theta \\ &= M\int_0^1 (t-ty)^{\alpha-1}(ty)^{-\beta}tdy \\ &= Mt^{\alpha-\beta}B(\alpha,1-\beta) \longrightarrow 0,\end{aligned}$$

as $t \to \infty$. So the system (4.1) is asymptotically stable. □

Theorem 4.2. *Under the assumption of Theorem 4.1 the system* (4.1) *is u-ess-bdd if $\beta \geq \alpha$ and for all non-zero eigenvalues of A, we have*

$$|arg(spec(A))| > \frac{\alpha\pi}{2},$$

and the zero eigenvalues of A have the same algebraic and geometric multiplicities.

Proof. Since

$$\int_0^h (t-\theta)^{\alpha-1}\|f(\theta,x(\theta))\|d\theta \longrightarrow 0$$

as $t \to \infty$, there exists a $t_1 > 0$, such that for all $t > t_1$

$$\Big|\int_0^h (t-\theta)^{\alpha-1}\|f(\theta,x(\theta))\|d\theta\Big| < 1,$$

and, by

$$\int_h^t (t-\theta)^{\alpha-1}\|f(\theta,x(\theta))\|d\theta \leq Mt^{\alpha-\beta}B(\alpha,1-\beta),$$

we deduce that

$$\forall\ t \geq 1,\ \Big|\int_h^t (t-\theta)^{\alpha-1}\|f(\theta,x(\theta))\|d\theta\Big| \leq MB(\alpha,1-\beta).$$

Now by the assumption, there exists a $t_2 > 0$ such that $\|t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)\| < 1$ for all $t > t_2$. We set, $T = \max\{1, t_1, t_2\}$, and for $\varepsilon > 0$ set $M'(\varepsilon) := \varepsilon + E + EMB(\alpha, 1-\beta)$. If $\|x_0\| < \varepsilon$ then using the inequality (4.3) yields to

$$\|x(t)\| \leq \varepsilon + E\big(1 + MB(\alpha,1-\beta)\big) = M'(\varepsilon).$$

This completes the proof. □

Remark 1. Under the assumption of Theorem 4.2 any solution of the system (4.1) is ess-bdd if $\beta \geq \alpha$.

Remark 2. Suppose that $f(t,x(t)) = b(t)$ in which $b(t) : (0,+\infty) \to \mathbb{R}^n$ is a continuous function. Suppose for all non-zero eigenvalues of A, $|arg(spec(A))| > \frac{\alpha\pi}{2}$ holds, and the zero eigenvalues of A have the same algebraic and geometric multiplicities, and there exists a constant $M > 0$ such that $\|b(t)\| < \frac{M}{t^\beta}$. Then the system (4.1) is asymptotically stable for $\beta > \alpha$ and u-ess-bdd for $\beta \geq \alpha$.

5. F-STABILITY OF $\,_{RL}D^{\alpha}_{0,T}X(T) = AX(T) + B(T)X(T)$

In this section, we study the following fractional differential system with Riemann-Liouville derivative

$$\,_{RL}D^{\alpha}_{0,t}x(t) = Ax(t) + B(t)x(t), \quad (0 < \alpha < 1) \tag{5.1}$$

under the initial condition $x_0 = \,_{RL}D^{\alpha-1}_{0,t}x(t)|_{t=0}$, where
$x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. We can get the solution of (5.1), by using the Laplace transform and inverse Laplace transform, as

$$x(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)x_0 + \int_0^t (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(A(t-\theta)^\alpha)B(\theta)x(\theta)d\theta. \tag{5.2}$$

Now we give the following definition:

Definition 5.1. Let $\mathbf{F}$ be a family of vector functions defined on $[t_0,\infty)$. The fractional differential system (2.1) is said to be $\mathbf{F}$-asymptotically stable if every solution which belongs to $\mathbf{F}$, is asymptotically stable.

We derive the following stability results.

Theorem 5.2. *Suppose all the non-zero eigenvalues of A satisfy*

$$|arg(spec(A))| > \frac{\alpha\pi}{2},$$

and the zero eigenvalues of A have the same algebraic and geometric multiplicities. If $B(t)$ be an $n \times n$ matrix continuously depending on time t and there exist $M, h > 0$ and $\beta > \alpha$ such that

$$\|B(t)\| \leq \frac{M}{t^\beta}, \qquad (t \geq h)$$

then the system (5.1) is $\mathbf{F}$-asymptotically stable, where

$$\mathbf{F} := \left\{x(t) : [0,\infty) \to \mathbb{R}^n, \exists M' > 0, \|x(t)\| \leq M'\right\}.$$

Proof. We have

$$\Big\| \int_0^t (t-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(t-\theta)^\alpha) B(\theta) x(\theta) d\theta \Big\| \leq \tag{5.3}$$

$$\leq M' \int_0^t (t-\theta)^{\alpha-1} \|E_{\alpha,\alpha}(A(t-\theta)^\alpha)\| \|B(\theta)\| d\theta$$

By (3.3), we can conclude immediately that (5.3) tends to zero as $t \to \infty$. So, the system (5.1) is $\mathbf{F}$-asymptotically stable. □

Theorem 5.3. *Suppose the conditions of Theorem 5.2 hold and $\mathbf{F} := L^p(\mathbb{R}^+)$, then the system (5.1) is $\mathbf{F}$-asymptotically stable for $p > \frac{1}{\alpha}$.*

Proof. Without loss of generality, we suppose that $\beta < 1$. So there exists a constant $E > 0$ such that $\|E_{\alpha,\alpha}(At^\alpha)\| \leq E$. Thus

$$\|x(t)\| \leq \|t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) x_0\| + \int_0^t (t-\theta)^{\alpha-1} \|E_{\alpha,\alpha}(A(t-\theta)^\alpha)\| \|B(\theta)\| \|x(\theta)\| d\theta.$$

Hence

$$\|x(t)\| \leq \|t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) x_0\| + EM \int_0^t (t-\theta)^{\alpha-1} \theta^{-\beta} \|x(\theta)\| d\theta.$$

Applying the Holder inequality yields to

$$\|x(t)\| \leq \|t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) x_0\| + EM \left[\Big(\int_0^t ((t-\theta)^{\alpha-1}\theta^{-\beta})^q d\theta \Big)^{\frac{1}{q}} \Big(\int_0^t \|x(\theta)\|^p d\theta \Big)^{\frac{1}{p}} \right]$$

$$\leq \|t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})x_0\| + EM\|x(\theta)\|_p\left[\left(\int_0^t (t-\theta)^{(\alpha-1)q}\theta^{-\beta q}d\theta\right)^{\frac{1}{q}}\right].$$

Where $\frac{1}{p}+\frac{1}{q}=1$. The right hand side of the above inequality tends to zero as $t \to \infty$. Thus (5.1) is **F**-asymptotically stable. □

Corollary 1. Suppose that all the non-zero eigenvalues of A satisfy

$$|arg(spec(A))| > \frac{\alpha\pi}{2},$$

and the zero eigenvalues of A have the same algebraic and geometric multiplicities. The fractional differential system

$$_{RL}D^{\alpha}_{0,t}x(t) = Ax(t) + f(t, x(t)), \quad (0 < \alpha < 1)$$

is **F**-asymptotically stable for either:

- $\mathbf{F} = L^p(\mathbb{R}^+)$, for $p > \frac{1}{\alpha}$
- $\mathbf{F} = \{x(t) : [0,\infty) \to \mathbb{R}^n, \exists M' > 0, \|x(t)\| \leq M'\}$,

if there exist $M, h > 0$ and $\beta > \alpha$, for which

$$\|f(t, x(t))\| \leq \frac{M\|x(t)\|}{t^{\beta}}, \qquad (t \geq h).$$

5.1. NOTE

We show that Theorem (4.1) in [20] is incomplete and even wrong, we consider the system

$$_{RL}D^{\alpha}_{t_0,t}x(t) = Ax(t) + B(t)x(t) \tag{5.4}$$

in which A is an arbitrary $n \times n$ matrix and $B(t) : [0,\infty) \to \mathbb{R}^{n\times n}$ is a continuous bounded function. We can rewrite the equation (5.4) as

$$_{RL}D^{\alpha}_{t_0,t}x(t) = Ax(t) + B(t)x(t) = -Id\ x(t) + (Id + A + B(t))x(t) \tag{5.5}$$

where Id is the identity matrix. This system now satisfies in the hypothesis of Theorem (4.1) in [20] since $|arg(spec(-Id))| > \frac{\alpha\pi}{2}$ and $Id+A+B(t)$ is always a continuous bounded function. So the system (5.5) is always asymptotically stable according to the Theorem (4.1) in [20] and equivalently (5.4) is asymptotically stable independent of the matrix A, which is clearly a wrong conclusion.

6. NUMERICAL APPROACH AND EXAMPLE

Example 6.1. Consider the periodically forced linear oscillators model [26] presented by

$$\ddot{x} + \delta\dot{x} + {\omega_0}^2 x = \gamma\cos\omega t$$

which in turns can be written as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -{\omega_0}^2 x - \delta y + \gamma\cos\omega t \end{cases} \tag{6.1}$$

where $\omega_0, \delta, \omega, \delta$ are constants. We rewrite (6.1) as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -{\omega_0}^2 & -\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma\cos\omega t \end{pmatrix}. \tag{6.2}$$

We now introduce the fractional order derivatives into the above system, and then modify the above system to obtain the following modified fractional order oscillator model:

$$\begin{pmatrix} x^\alpha \\ y^\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -{\omega_0}^2 & -\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\gamma\cos\omega t}{s+t^\beta} \end{pmatrix}. \tag{6.3}$$

If we set

$$A = \begin{pmatrix} 0 & 1 \\ -{\omega_0}^2 & -\delta \end{pmatrix}, \quad b(t) = \begin{pmatrix} 0 \\ \frac{\gamma\cos\omega t}{s+t^\beta} \end{pmatrix},$$

then $\|b(t)\| \le \frac{|\gamma|}{s+t^\beta}$. For choosing appropriate parameter values, we get $|arg(spec(A))| > \frac{\alpha\pi}{2}$, so according to the Theorem 4.1, for $s > 0$ and $\beta > \alpha$ the system (6.3) becomes asymptotically stable.

To verify the stability results of this example numerically, we perform numerical simulation by means of the method by Atanackovic and Stankovic [1]. In [1] it was shown that for a function $f(t)$, the Riemann-Liouville derivative of order α with $0 < \alpha < 1$ may be expressed as

$$\begin{aligned} {}_{RL}D^\alpha_{0,t} f(t) &= \frac{1}{\Gamma(2-\alpha)} \times \\ &\left[\frac{f'(t)}{t^{\alpha-1}} \Big(1 + \sum_{p=1}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\Big) - \Big(\frac{\alpha-1}{t^\alpha} f(t) + \sum_{p=2}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!} \Big(\frac{f(t)}{t^\alpha} + \frac{v_p(f)(t)}{t^{p-1+\alpha}}\Big)\Big) \right], \end{aligned} \tag{6.4}$$

where

$$v_p(f)(t) = -(p-1)\int_0^t \tau^{p-2} f(\tau)d\tau, \quad p = 2, 3, \cdots$$

For the sake of simplicity, we proceed the computations as follows.
First we approximate ${}_{RL}D^{\alpha}_{0,t}f(t)$ by using the first M terms in the sum appearing in Eq. (6.4) by

$$
{}_{RL}D^{\alpha}_{0,t}f(t) \simeq \frac{1}{\Gamma(2-\alpha)}\times
\left[\frac{f'(t)}{t^{\alpha-1}}\Big(1+\sum_{p=1}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\Big)-\Big(\frac{\alpha-1}{t^{\alpha}}f(t)+\sum_{p=2}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}\big(\frac{f(t)}{t^{\alpha}}+\frac{v_p(f)(t)}{t^{p-1+\alpha}}\big)\Big)\right]. \tag{6.5}
$$

We can rewrite Eq. (6.5) as follows

$$
{}_{RL}D^{\alpha}_{0,t}f(t) \simeq \Omega(\alpha,t,M)f'(t)+\Phi(\alpha,t,M)f(t)+\sum_{p=2}^{M}A(\alpha,t,p)\frac{v_p(f)(t)}{t^{p-1+\alpha}},
$$

where

$$
\Omega(\alpha,t,M)=\frac{1+\sum\limits_{p=1}^{M}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}}{\Gamma(2-\alpha)t^{\alpha-1}},\quad R(\alpha,t)=\frac{1-\alpha}{t^{\alpha}\Gamma(2-\alpha)},
$$

and

$$
\Phi(\alpha,t,M)=R(\alpha,t)+\sum_{p=2}^{M}\frac{A(\alpha,t,p)}{t^{\alpha}},\quad A(\alpha,t,p)=-\frac{\Gamma(p-1+\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)(p-1)!}.
$$

We set

$$
v_p(x)(t)=w_p(t),\ v_p(y)(t)=u_p(t),\quad p=2,3,\cdots,M.
$$

For the system (6.3) , we have

$$
\Omega(\alpha,t,M)x'(t)+\Phi(\alpha,t,M)x(t)+\sum_{p=2}^{M}A(\alpha,t,p)\frac{w_p(t)}{t^{p-1+\alpha}}={}_{RL}D^{\alpha}_{0,t}x(t)=y(t),
$$

where

$$
w_p(t)=-(p-1)\int_0^t\tau^{p-2}x(\tau)d\tau,\quad p=2,3,\cdots,M.
$$

Also, we have

$$
\Omega(\alpha,t,M)y'(t)+\Phi(\alpha,t,M)y(t)+\sum_{p=2}^{M}A(\alpha,t,p)\frac{u_p(t)}{t^{p-1+\alpha}}={}_{RL}D^{\alpha}_{0,t}y(t)=-{\omega_0}^2x(t)-\delta y(t)+\frac{\gamma cos\omega t}{s+t^{\beta}},
$$

where

$$
u_p(t)=-(p-1)\int_0^t\tau^{p-2}y(\tau)d\tau,\quad p=2,3,\cdots,M.
$$

Now we can rewrite the above equations as the following forms

$$
x'(t)=\frac{1}{\Omega(\alpha,t,M)}\Big[y(t)-\Phi(\alpha,t,M)x(t)-\sum_{p=2}^{M}A(\alpha,t,p)\frac{w_p(t)}{t^{p-1+\alpha}}\Big], \tag{6.6}
$$

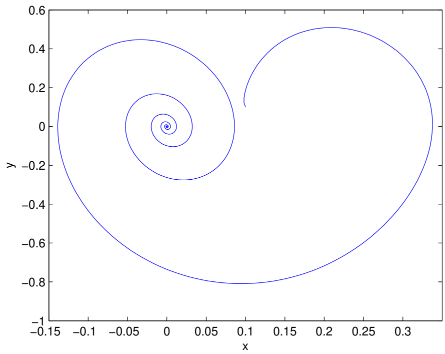

Figure 1: Phase portrait of (6.3), for $\delta = 1$.

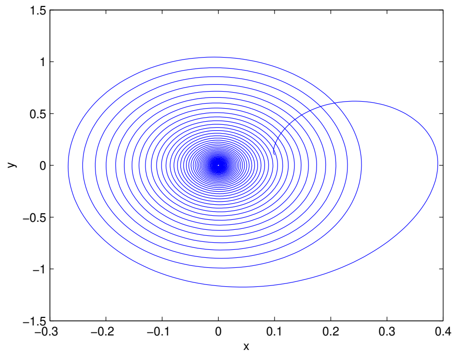

Figure 2: Phase portrait of (6.3), for $\delta = .1$.

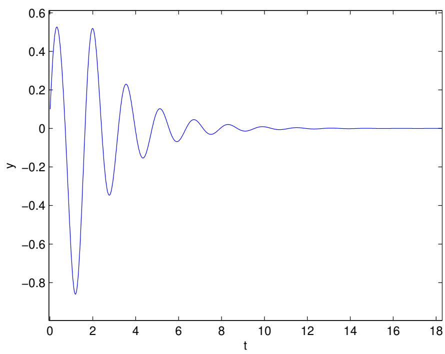

Figure 3: Numerical value of (6.3), for $\delta = 1$.

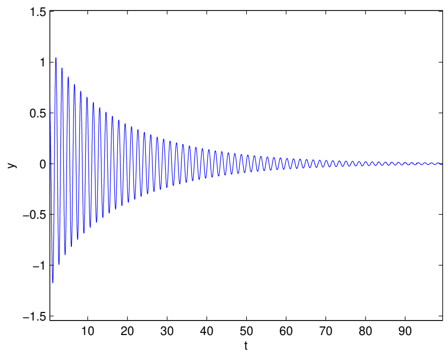

Figure 4: Numerical value of (6.3), for $\delta = .1$.

where

$$w_p'(t) = -(p-1)t^{p-2}x(t), \quad p = 2, 3, \cdots, M,$$

and

$$y'(t) = \frac{1}{\Omega(\alpha, t, M)}\Big[-{\omega_0}^2 x(t) - \delta y(t) + \frac{\gamma cos\omega t}{s + t^{\beta}} - \Phi(\alpha, t, M)y(t) - \sum_{p=2}^{M} A(\alpha, t, p)\frac{u_p(t)}{t^{p-1+\alpha}}\Big], \tag{6.7}$$

in which

$$u_p'(t) = -(p-1)t^{p-2}y(t), \quad p = 2, 3, \cdots, M,$$

along with the following initial conditions

$$x(\xi) = x_0, \; w_p(\xi) = 0, \; p = 2, 3, \cdots, M,$$

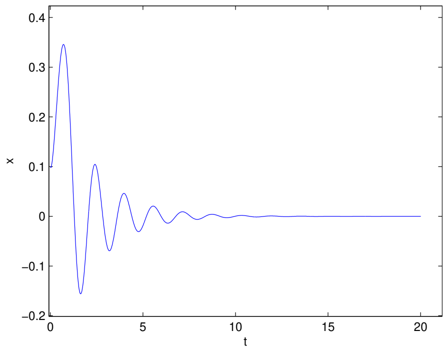

Figure 5: Numerical value of (6.3), for $\delta = 1$.

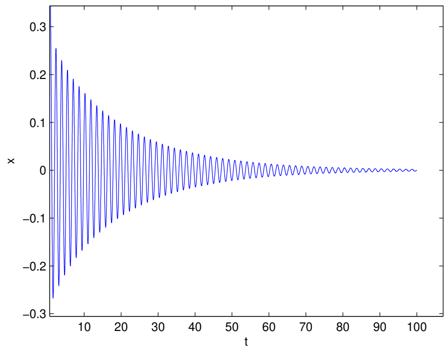

Figure 6: Numerical value of (6.3), for $\delta = .1$.

$$y(\xi) = y_0, \;\; u_p(\xi) = 0, \;\; p = 2, 3, \cdots, M, \tag{6.8}$$

where ξ is a positive constant. Now we consider the numerical solution of system of ordinary differential Eqs. (6.6), (6.7), with the initial conditions (6.8) by using the well known Runge-Kutta method of the fourth order and depict orbits of the system (6.3) for different set of parameters.

Phase portrait and numerical value of system (6.3) for the fixed parameter values $\alpha = .98, \omega_0 = 4, \gamma = 2, s = .5, \beta = 7, \omega = .1, x_0 = .1, y_0 = .1$, are depicted in Figs. 1, 2, 3, 4, 5, 6.

7. CONCLUSION

In the present paper, we provide analytical methods to examine the asymptotic stability for a class of nonlinear differential systems with Riemann-Liouville fractional derivative for the commensurate order $0 < \alpha < 1$. We also establish **F**-asymptotic stability theorems of the nonlinear differential system. To reveal validity of the obtained analytical results we examine a test example in which we compute different orbits of the given system by means of numerical simulations.

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