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SOME APPROXIMATION ASPECTS FOR A NEW CLASS CUMULATIVE DISTRIBUTION FUNCTIONS

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ABSTRACT: In this paper we introduce and consider a new class of cumulative distribution functions. This class belongs to the important class of functions arising from the theory of impulse techniques, neural networks and debugging theory.

By this family we study the Hausdorff approximation of the impulse function $\sigma^{**}(t)$.

Numerical examples, illustrating our results using the programming environment CAS MATHEMATICA are presented.

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Key Words: impulse function $\sigma^{**}(t)$, cumulative distribution function, emitting chart, Hausdorff approximation, upper and lower bounds

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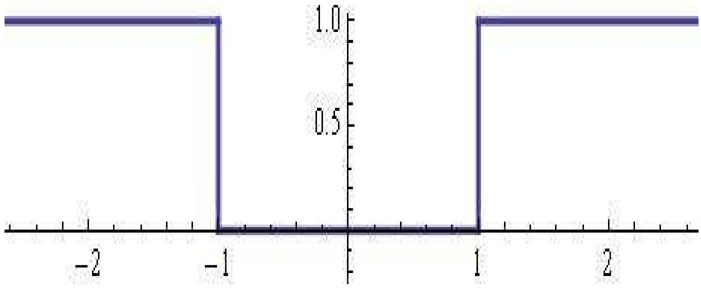

Figure 1: The signal of $\sigma^{**}(t)$ – type.

1. INTRODUCTION

The extended Chen [1] software reliability model is given as follows (see Chaubey and Zhang [2]):

$$M(t) = \omega\left(1 - e^{\lambda(1-e^{t^\beta})}\right)^\alpha \tag{1}$$

where $\lambda > 0$, $\alpha > 0$, $\beta > 0$.

For other extensions of the Chen distribution, see [3] – [7].

Another model that uses the "Gompertz–type correction" is the Poisson–exponential cumulative distribution function (Pcdf). The (Pcdf) is given by (see for instance [8]):

$$M_1(t;\lambda;\beta) = \frac{e^{\lambda e^{-\beta t}} - e^\lambda}{1 - e^\lambda} \tag{2}$$

where $\beta > 0$; $\lambda > 0$.

For other extensions and estimations, see [9] – [10].

Some applications of the (Pcdf) to rainfall and aircraft data with zero occurrence can be found in [10].

We consider the following new cumulative function which belongs to the important class of functions arising from the theory of impulse techniques, debugging theory, population dynamics and cell growth models:

$$M_2(t) = \frac{e^{\lambda e^{-\beta t^m}} - e^\lambda}{1 - e^\lambda}, \tag{3}$$

with

$$t_0 = \left(-\frac{1}{\beta}\ln\left(\frac{1}{\lambda}\ln\left(\frac{1+e^\lambda}{2}\right)\right)\right)^{\frac{1}{m}}; \quad M_2(t_0) = \frac{1}{2}, \tag{4}$$

where β, $\lambda > 0$ and m is an even number.

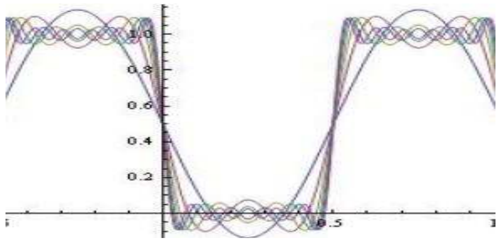

Figure 2: The polynomial approximation of the $\sigma^{**}(t)$ – type function.

The typical example of an impulse function from antenna feeder technique has the following shape (see, Fig. 1):

$$\sigma^{**}(t) = \begin{cases} 1, \ t \in [-\infty, -1) \cup (1, +\infty) \\ 0, \ t \in [-1, 1]. \end{cases} \tag{5}$$

The polynomial Hausdorff approximations of a signal of type (5) is visualized on Fig. 2.

In this paper we study the Hausdorff approximation of the impulse function $\sigma^{**}(t)$ (see Definition 1) by the family (3)–(4).

Furthermore, we propose a software module (intellectual property) within the programming environment CAS Mathematica for the analysis. The models have been tested with real-world data.

2. HAUSDORFF APPROXIMATION OF THE IMPULSE FUNCTION $\sigma^{**}(T)$

Definition 1. [11] The Hausdorff distance (the H–distance) $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f, g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t, x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

The one–sided Hausdorff distance d between the impulse function $\sigma^{**}(t)$ and the function ((3)–(4)) satisfies the relation

$$M_2(t_0 - d) = d. \tag{6}$$

The following theorem gives upper and lower bounds for d

Theorem 2. *Let*

$$p = \frac{1}{2},$$

$$q = -1 + \frac{m\beta(1+e^{\lambda})\ln\frac{1+e^{\lambda}}{2}}{2(1-e^{\lambda})}\left(-\frac{1}{\beta}\ln\left(\frac{1}{\lambda}\ln\left(\frac{1+e^{\lambda}}{2}\right)\right)\right)^{\frac{m-1}{m}},$$

$$r = -2.1q > 0$$

For the one–sided Hausdorff distance d between $\sigma^{**}(t)$ and the function ((3)–(4)) for

$$r > e^{1.05} \approx 1.36079$$

the following inequalities hold:

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{7}$$

Proof. Let us examine the function:

$$F(d) = M_2(t_0 - d) - d. \tag{8}$$

From $F'(d) < 0$ we conclude that function F is decreasing.

Consider the function

$$G(d) = p + qd. \tag{9}$$

From the Taylor expansion we obtain $G(d) - F(d) = O(d^2)$. Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 3).

In addition $G'(d) < 0$.

Further, for $r > e^{1.05}$ we have $G(d_l) > 0$ and $G(d_r) < 0$.

This completes the proof of the theorem. □

The family of functions $M_2(t)$ is visualized on Fig. 4

The model ((3)–(4)) for $\beta = 20$, $\lambda = 0.7$, $m = 12$, $t_0 = 0.73949$ is visualized on Fig. 5. From the nonlinear equation (6) and inequalities (7) we have: $d = 0.106819$, $d_l = 0.0757091$, $d_r = 0.195394$.

The model ((3)–(4)) for $\beta = 30$, $\lambda = 0.5$, $m = 30$, $t_0 = 0.876576$ is visualized on Fig. 6. From the nonlinear equation (6) and inequalities (7) we have: $d = 0.0663519$, $d_l = 0.0386655$, $d_r = 0.125771$.

From the above examples, it can be seen that the proven estimates (see Theorem 2) for the value of the Hausdorff approximation are reliable when assessing the important characteristic - "saturation".

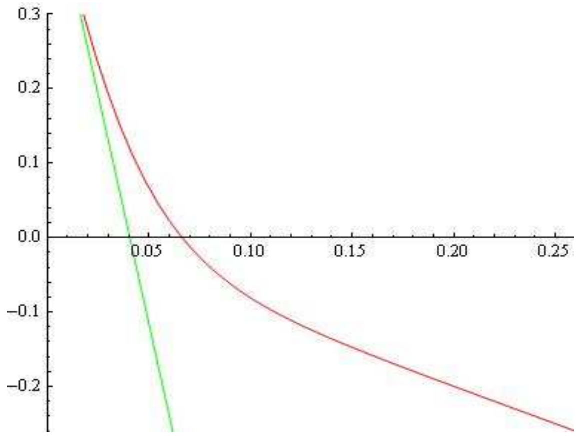

Figure 3: The functions $F(d)$ and $G(d)$ for $\beta = 30; \lambda = 0.5; m = 30$.

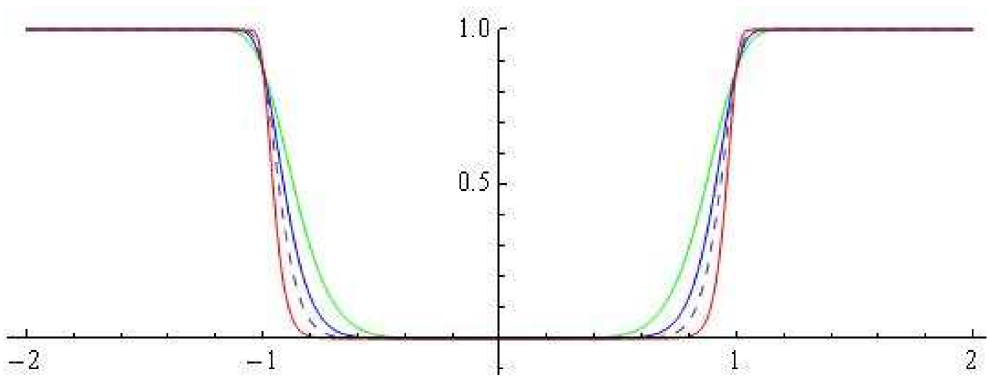

Figure 4: The family of functions $M_2(t)$ for $m = 8$ (green); $m = 12$ (blue); $m = 16$ (dashed); $m = 26$ (red); ($\beta = 2$ and $\lambda = 0.1$ are fixed).

3. SOFTWARE MODULE FOR ANALYSIS WITHIN CAS MATHEMATICA

We propose a software module (intellectual property) within the programming environment *CAS Mathematica* for the analysis of the considered family $M_2(t)$ of cumulative functions.

The module offers the following possibilities:

- generation of the impulse function under user defined values of the parameters λ, m and β;

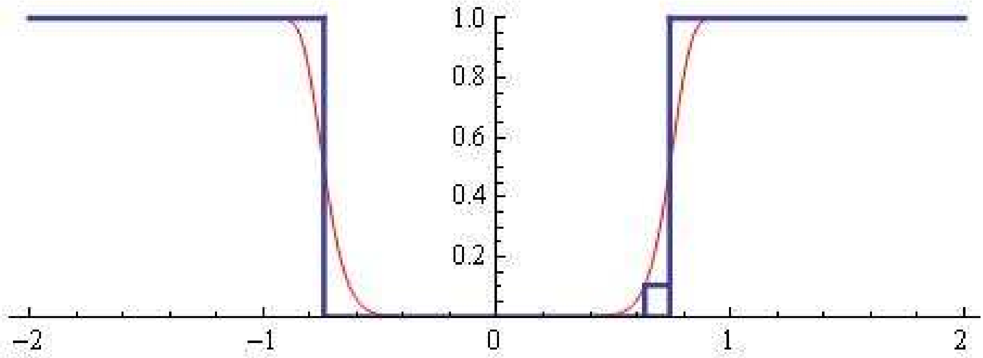

Figure 5: The model ((3)–(4)) for $\beta = 20$, $\lambda = 0.7$, $m = 12$, $t_0 = 0.73949$; H–distance $d = 0.106819$, $d_l = 0.0757091$, $d_r = 0.195394$.

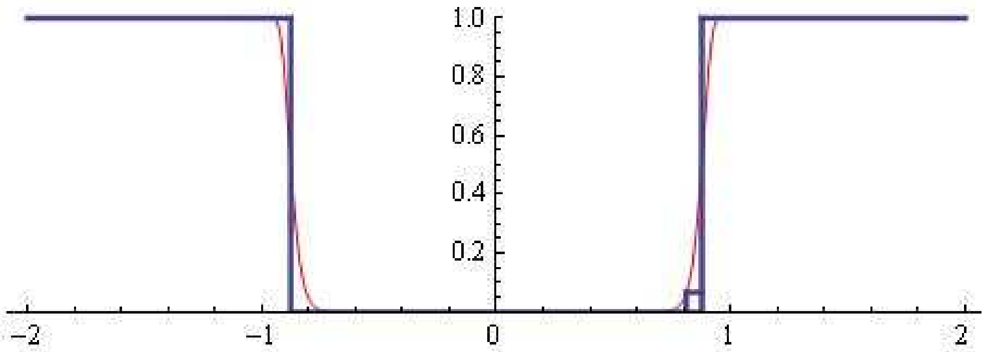

Figure 6: The model ((3)–(4)) for $\beta = 30$, $\lambda = 0.5$, $m = 30$, $t_0 = 0.876576$; H–distance $d = 0.0663519$, $d_l = 0.0386655$, $d_r = 0.125771$.

- calculation of the H-distance d between the function $\sigma^{**}(t)$ and the function $M_2(t)$;
- generation of the emitting chart of antenna factor;
- software tools for animation and visualization.

After the substitution

$$t = kl\cos\theta + a,$$

where $k = \frac{2\pi}{\lambda}$, λ is the wave length; a is the phase difference; θ is the azimuthal angle and l is the distance between the emitters ($l = \frac{\lambda}{2}$ is fixed), the function $M_2(t)$ (or

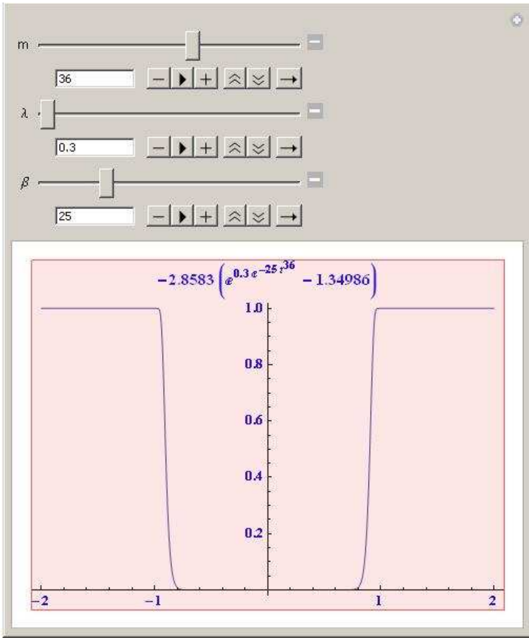

Figure 7: An example of the usage of dynamical and graphical representation for the family $M_2(t)$. For example $\lambda = 0.3$, $\beta = 25$, $m = 36$; H–distance $d = 0.0408078$. The plots are prepared using CAS Mathematica.

emitting chart of an antenna factor) can be rewritten in the form

$$M_2(\theta) = \frac{e^{\lambda e^{-\beta(\pi \cos\theta + a)^m}} - e^{\lambda}}{1 - e^{\lambda}}. \quad (10)$$

Typical emitting chart is visualized on Fig. 8.

If $l = \lambda$ we have the chart - Fig. 9

4. OPEN PROBLEMS AND CONCLUSIONS

Of course, the question of the practical realization of the activation functions which are generated as emitting charts remains open.

The mathematical apparatus proposed in the article can be successfully used for imitation and simulation of such charts.

We will explicitly say that the results have independent significance in the study of issues related to impulse techniques [12].

Hausdorff approximation of some impulse functions and some modeling aspects in the field of antenna–feeder technique can be found in [13]–[27].

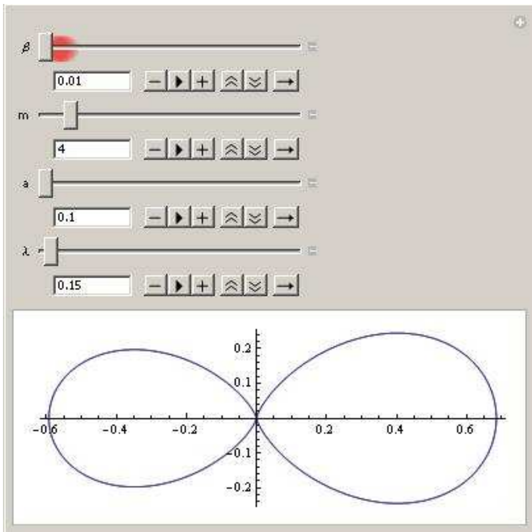

Figure 8: Typical emitting chart ($M_2(\theta)$) for $\beta = 0.015$; $a = 0.1$; $\lambda = 0.15$; $m = 4$.

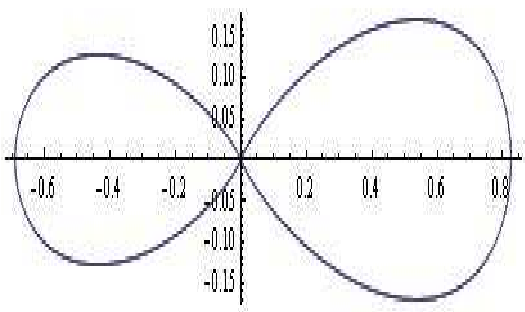

Figure 9: Typical emitting chart for $\beta = 0.001$; $a = 0.02$; $\lambda = l = 0.15$; $m = 16$.

We hope that the results will be useful for a lot of specialists in this scientific area.

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