Neural, Parallel, and Scientific Computations, **26**, No. 2 (2018), 119-141 ISSN: 1056-2176

CONTINUOUS TIME INTERPOLATION OF MONOTONE MARKED RANDOM MEASURES WITH APPLICATIONS

JEWGENI H. DSHALALOW[1] AND KIZZA M. NANDYOSE

Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901, USA

ABSTRACT: We study a class of monotone delayed marked point processes that model stochastic networks (under attacks), status of queueing systems during vacation modes, responses to cancer treatments (such as chemotherapy and radiation), hostile ambushes in economics and warfare. We are interested in the behavior of such a process about a fixed threshold. It presents an analytic challenge, because of the arbitrary nature of random marks. We target the first passage time, pre-first passage time, the status of the associated continuous time parameter process between these two epochs, and the status of the process upon these two epochs. A joint functional of these stochastic quantities is investigated in the transient mode. Analytically tractable formulas are obtained and demonstrated on special cases of marked Poisson processes.

AMS Subject Classification: 60G50, 60G51, 60G52, 60G55, 60G57, 60K05, 60K35, 60K40, 60G25, 90B18, 90B10, 90B15, 90B25

Key Words: random walk, independent and stationary increments processes, fluctuations of stochastic processes, marked point processes, Poisson process, exit time, first passage time, renewal theory, time sensitive analysis

Received: January 11, 2018 ; **Accepted:** May 8, 2018 ;
Published: May 14, 2018 **doi:** 10.12732/npsc.v26i2.1
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/npsc

1. INTRODUCTION

Consider a piecewise constant process N_t (valued in $\mathbb{R}$) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with independent and stationary increments such that $\{t_n\}$ is a point process on $\mathbb{R}_+$ of stopping times relative to $(\mathcal{F}_t)$ and N_t is constant between t_{n-1} and

$t_n, n = 1, 2, \ldots$. If N_t is monotone nondecreasing, often of interest is to determine (probabilistically) the crossing a fixed threshold M by N_t at some time t_n and the value N_{t_n} at the crossing. To continue, let us introduce some more notation. Define

$$\nu = \min\{n = 0, 1, \ldots : N_{t_n} \geq M\}.$$

Then, N_t crosses M at some moment t_ν, referred to as the *first passage time*. Note that because the paths of N_t are not continuous, the value N_{t_ν} is likely to exceed M rather than take M at the crossing, even if N_t and M are integer-valued. Consequently, N_{t_ν} is called the *first excess of* M.

In the past work, Dshalalow et. al [1-3, 10-24] studied transforms of various classes of such processes in connection with stochastic games, queueing, stochastic networks, and finance, targeting functionals like $Ee^{i\phi N_{t_\nu}} e^{-\theta t_\nu}$ and their embellishments. They were called ıtime insensitive functionals, because the associated reference values were not related to real time parameter t. In some way, this is a common shortcoming of embedded processes compared to those with continuous time parameter. There were efforts made to revive lost information on the behavior of N_t around the first passage time. In all of them, Dshalalow and his co-authors [15,16,24] studied N_t observed over a sequence $\{\tau_m\}$ (being independent of filtration $(\mathcal{F}_t)$). For that matter, the interpolation of N_t (pertaining to *time sensitivity*) was referred to the interval $(\tau_{\rho-1}, \tau_\rho]$, where

$$\rho = \min\{m = 0, 1, \ldots : N_{\tau_m} \geq M\}.$$

It is understood that the real crossing of M at t_ν takes place at an earlier time than τ_ρ, but in various applications, data collection is impossible in real time. This class of problems makes associated modeling more realistic, but there are still many applications where real time information is possible or when changes of N_t between t_{n-1} and t_n can be neglected. Note that even if there are no changes between t_{n-1} and t_n, it is still of great importance to find the distribution of N_t when t is from $(t_{\nu-1}, t_\nu]$. We thus are interested in the following *time sensitive functional*.

$$\Phi := Ee^{i(\delta N_t + \phi N_{t_{\nu-1}} + \xi N_{t_\nu})} e^{-\vartheta t_{\nu-1} - \theta t_\nu} \mathbf{1}_{(t_{\nu-1}, t_\nu]}(t).$$

This is a joint transform of *pre-first passage time* $t_{\nu-1}$, the *first passage time* t_ν, *pre-excess* value $N_{t_{\nu-1}}$, the *excess value* N_{t_ν}, and the value of N_t continuously observed between $t_{\nu-1}$ and t_ν, the most significant reference points. Again even though N_t is not supposed to alter between these two moments, the time t-sensitivity would be a significant refinement of Φ compared to a more limited $Ee^{i\phi N_{t_\nu}} e^{-\theta t_\nu}$. (We will refer it to *real time sensitivity* as opposed to a *delayed time sensitivity* of associated functionals related to the interval $(\tau_{\rho-1}, \tau_\rho]$.)

While the process N_t can be real- or integer-valued, in the present paper, we focus on the latter. Firstly, we are eager to explore and demonstrate benefits of *discrete* *operational calculus* related to fluctuations of N_t. Secondly, the integer-valued nature of N_t can be easily refined with an arbitrarily small multiple factor coming close to the usual continuous topology. (Yet a real-valued version of N_t may still be worth considering.)

Motivation. The stochastic modeling considered in this work is driven by problems arising from real-life events such as cyber security, cancer treatment, stock markets, finance, and queuing systems. Below, we expand more on these applications.

(i) Cyberattacks

High-profile cyberattacks all over the world have amplified fears and led to heavy monetary, computer system, and information losses. For instance, in 2004, a German College student Sven Jaschan, released a computer virus that disabled Delta Airline's computer system, resulting in many flight cancellations and over $500 million dollars in losses. In yet another high-profile attack, during the 2008 presidential election, Chinese an Russian Hackers hacked into Barack Obama and John McCain's campaign computers systems gaining access to sensitive data. The computers were subsequently confiscated by the FBI.

Cyberattacks have become capable of far more than stealing consumer information or embarrassing politicians and business executives. Whether conducted by lone intruders or nation-states, they can compromise the safety of medical food and water systems, disrupt transportation, and destabilize nuclear power plants. Such attacks can undermine democratic institutions or encourage violence by spreading false information. The cyber threat has become existential. (Cf., The Wall Street Journal, July 12, 2017.)

Very recently we witnessed an escapade of cyberattacks on multiple infrastructure and industry throughout the world, such as Faux Ransomware (cf. The Wall Street Journal, June 30, 2017) and the infamous breach in Equifax credit institution compromising more than 145 millions personal files. Among many other places affected by Fauz Ransomwarethe attack, was Princeton Community Hospital in West Virginia, USA. Here the attack froze the hospital's electronic medical record system leaving doctors unable to review patient's medical history or transmit laboratory and pharmacy orders. Officials were unable to restore services, and found there was no way to pay a ransom for the return of their system. The cyberattack almost left Princeton Community Hospital without even paper templates, which were stored on a computer file, to be printed. Surgeons at the Heritage Valley Hospital in Beaver, Pennsylvania, canceled elective surgeries for two days.

These kinds of economic, social, and privacy violations raise important stochastic analysis questions concerning cyberattacks proofing and worst case scenarios.

Once some systems fail, how long would it take before the attack spreads to the entire network or the attack spreads to a point of no return? What level of risk is posed by failures of individual components based on network connectivity? These and many other questions can be answered by modeling an underlying computer network as a marked point process where the points are the attack times and the marks the number of downed computers, whereas a given threshold is a minimal number of nodes by whose crossing the network becomes totally compromised. Whereas our modeling does not prevent those attacks, nor is it a firewall in any sense, it aims at containing damages by predicting the time and caliber of casualties and thereby allowing a surviving part of the network to be separated from an infected subnetwork that is to be quarantined.

(ii) Cancer Treatment

When diagnosed with cancer, a patient is often prescribed an aggressive treatment such as radiation or chemo therapy. In quite a few cases, an underlying cancer does not respond well to either chemo or radiation, and this is a bad news for a patient, not only because he may run out of options, but also because much time is wasted that could have been used for alternative treatments. Knowing this, one key challenge is to decide ahead of the time whether the cancer is going to be treatable shortly after the therapy starts. One approach is to model the response to the therapy by a marked point process. The idea is to predict the cancer progression (or regression) ahead of the time and if needed give the patient another treatment before patient's condition deteriorates.

(iii) Finance

In option trading, it is of interest to predict the time to sell an underlying stock before the corresponding call option expires. The latter makes sense when the stock continues to appreciate, so that predicting the time and the stock price upon crossing a fixed threshold would be a good reason to use fluctuation analysis.

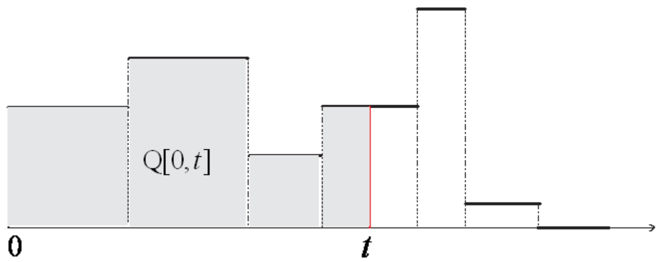
(iv) Queueing

In queuing theory an often studied-to scenario is when a server waits or vacates until the queue accumulates to a certain level (cf., N-Policy), and until then, service is suspended. Once threshold N is crossed by the contents of the waiting room, the server resumes his service. This situation can be treated using fluctuation theory and further interpolated if there is a need to work on a continuous time parameter process. (Cf. Al-Matar and Dshalalow [3].)

Significance of Time Sensitive Analysis. In some of the above cases we can use time insensitive analysis such as [10,11]. However, the real time interpolation allows one to employ stochastic control making it a very useful embellishment. For example, suppose the main process N_t gives the status of a patient measured in integer units and $h(k)$ is a weight function of k units of such measurement. If the underlying indicator is of the white blood cell count, say k, $h(k)$ is the hemoglobin level (that can be determined by using regression analysis). Then,

$$Q[0,t] = E\int_{y=0}^{t}(N_y)dy$$

gives the mean hemoglobin level in interval $[0,t]$.



We can represent Q$[0,t]$ as follows, by using Fubini's theorem:

$$Q[0,t] = \sum_{k\geq 0} E\left[\int_{y=0}^{t}\mathbf{1}_{\{k\}}(N_y)h(N_y)dy\right] = \sum_{k\geq 0} h(k)\int_{y=0}^{t} P\{N_y = k\}dy. \quad (1)$$

Now we assume that we know the stationary distribution of N_t

$$\pi_k = \lim_{t\to\infty} P\{N_t = k\} = \lim_{t\to\infty}\frac{1}{t}\int_{y=0}^{t} P\{N_y = k\}dy, k = 0,1,\ldots \quad (2)$$

Applying (1.1) to (1.2) and using the monotone convergence theorem we have

$$Q = \lim_{t\to\infty}\frac{\mathrm{Q}[0,t]}{t} = \sum_{k\geq 0} h(k)\pi_k$$

the mean hemoglobin level per unit time calculated over the time $[0,\infty)$. □

Related Literature. The main contents of the present article falls into the area of fluctuation theory of stochastic processes originally stemming from random walk analysis. The literature on fluctuations is very rich. A very interesting survey paper is by Bingham [6] and a seminal work on fluctuations belongs to Takács [34] and it relates to fluctuations of recurrent and semi-Markov processes. The main keywords (some synonymous) associated with fluctuations are first passage time, first exit time, first excess level, and level crossing and are examined in [4,5,7-11,14,21,26,27,31-33,35,36]. These papers focus on determining probability distributions of the first passage (exit/crossing) time and the process value upon crossing a critical threshold or manifold or departing from a compact set. As previously mentioned, just a handful of work (to the best of our knowledge, mainly by the first co-author), belongs to time sensitive analysis [1-3,12,15,16,24]. Fluctuation analysis is a powerful method that finds applications in stochastic games [2,15,16,18-20], stochastic networks [22,23], finance [13,14,28-30,35], queueing [2,3], physics and astronomy [25,32], earthquakes [33], and general stochastic processes [1], in particular, with independent and stationary increments [5,7,8,10,24,26,36]. As far as analytical tools, the Laplace-Carson transform is being used for real-valued processes [15,16,24] and discrete operational calculus for integer-valued processes [10,13,14,17,18, 22,23] (developed by the first co-author).

Brief Overview. The paper is laid out as follows. Section 2 deals with a more rigorous description of the underlying marked random process, reference stopping times, and a related functional. Section 3 introduces the notion of piecewise constant interpolation of a marked random measure and proves several preliminary lemmas and a proposition. Section 4 deals with a stochastic expansion of an underlying joined functional in series that allows us to obtain a fully tractable expression of that functional. Section 5 aims to demonstrate analytic tractability by an important special case of a marked Poisson process.

2. FORMALISM

Let $(\Omega, \mathcal{F}, P)$ be a probability space and

$$\mathcal{A} = \sum_{k=0}^{\infty} X_k \varepsilon_{t_k} \ (\varepsilon_a \text{ is the point mass at } a) \tag{3}$$

a marked random measure, with position dependent marking, that is, $X_k \otimes (t_k - t_{k-1}) : \Omega \to \mathbb{N} \times \mathbb{R}_+$ being independent and all but $X_0 \otimes t_0$ identically distributed. The underlying support counting measure $\sum_{k=0}^{\infty} \varepsilon_{t_k}$ is a delayed renewal process. One of the key questions that arise in applications for processes like $\mathcal{A}$ is the behavior of $\mathcal{A}$ around some threshold, say M. We assume that the marks X_k's are nonnegative, that $t_n \to \infty$, and, without loss of generality, the sequence of sums A_n defined as

$$A_n = X_0 + \ldots + X_n, n = 0, 1, \ldots,$$

runs to ∞ a.s. as $n \to \infty$. Thus the total of the marks A_n will a.s. cross M at some point t_n. Obviously, such an A_n will be equal to or greater than M. The integer-valued r.v.

$$\nu = \inf\{n = 0, 1, \ldots : A_n \geq M\} \tag{4}$$

is called the *exit index*. The r.v. A_ν is the *excess level* over M and t_ν is the *first passage time* (a standard terminology from fluctuation theory).

In various past work (cf. Dshalalow [10,11]), the fluctuations of $\mathcal{A}$ around M were thoroughly investigated and the joint distribution of the key components A_ν, t_ν,and ν, along with $A_{\nu-1}, t_{\nu-1}$, were found. Many other tools were implemented to refine the results. In some of them the authors observed $\mathcal{A}$ over a third-party point process $\mathcal{T} = \{\tau_0, \tau_1, \ldots\}$ [22,23] to get an additional information on $\mathcal{A}$ and include some auxiliary thresholds lower than M that $\mathcal{A}$ was to cross prior to crossing M [20].

Here we attempt to refine the results by introducing the continuous time parameter process

$$N_t = \mathcal{A}\left[0,t\right], t \geq 0, \tag{5}$$

that gives us more information about $\mathcal{A}$ which we plan to obtain around the key reference points $t_{\nu-1}$ and t_ν. More significantly, as we will see it, the presence of N_t makes $\mathcal{A}$ *time sensitive* (as it relates to real time t) allowing us to implement control. We want to focus on interval $(t_{\nu-1}, t_\nu]$ just before the crossing of M at t_ν takes place. It thus stands for reason to investigate the joint functional

$$Ez^{N_t}u^{A_{\nu-1}}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}\left(t\right), \\ \|z\| \leq 1,\ \|u\| \leq 1,\ \|v\| \leq 1,\ \mathrm{Re}\vartheta_0 \geq 0, \mathrm{Re}\vartheta \geq 0, \tag{6}$$

of process N_t observed in interval $(t_{\nu-1}, t_\nu]$. In this particular case, we assume that the marks are also integer-valued that as mentioned, works in various applications and can accurately approximate a continuous topology when using a small multiplier. Whereas the tools we use here do not pertain to discrete-valued r.v.'s X_k only, there are benefits of discrete operational calculus (that goes with the discrete marks) we are going to employ that yield analytically more tractable formulas compared to their continuous counterparts.

3. TIME-SENSITIVE ANALYSIS PRELIMINARIES

To work on functional (1.4) we begin with the following assertions.

Lemma 1. Suppose (A,T) and (U,Δ) are random vectors on probability space $(\Omega, \mathcal{F}, P)$ each valued in $(\mathbb{N}_0, \mathbb{R}_+)$ and with a joint probability distribution $P_{A\otimes V\otimes T\otimes\Delta}$ on the product space

$$\left(\mathbb{N}_0 \times \mathbb{R}_+ \times \mathbb{N}_0 \times \mathbb{R}_+, \mathcal{P}\left(\mathbb{N}_0\right) \otimes \mathrm{B}_+ \otimes \mathcal{P}\left(\mathbb{N}_0\right) \otimes \mathrm{B}_+\right),$$

where $\mathrm{B}_+ = \mathrm{B}(\mathbb{R}_+)$ is the Borel σ-algebra. Then the following formula holds.

$$\Gamma(u,\xi,v,\vartheta;\theta) := \int_{t\ge 0} e^{-\theta t} E u^{A} e^{-\xi T} v^{U} e^{-\vartheta\Delta} \mathbf{1}_{t\in(T,T+\Delta]} dt$$

$$= \frac{1}{\theta}\left[Eu^{A}v^{U}e^{-(\xi+\theta)T}e^{-\vartheta\Delta} - Eu^{A}v^{U}e^{-(\xi+\theta)T}e^{-(\vartheta+\theta)\Delta}\right],$$

$$|u|\le 1, |v|\le 1, \operatorname{Re}\xi \ge 0, \operatorname{Re}\vartheta \ge 0, \operatorname{Re}\theta \ge 0. \tag{L1}$$

Corollary 2. In the event that (A,T) are independent of (U,Δ), the functional Γ of Lemma 1 reduces to

$$\Gamma(u,\xi,v,\vartheta;\theta) = \frac{1}{\theta} Eu^{A}e^{-(\xi+\theta)T}\left[Ev^{U}e^{-\vartheta\Delta} - Ev^{U}e^{-(\vartheta+\theta)\Delta}\right]. \tag{C2}$$

Proof. Unfolding the expectation we have the following chain of equations.

$$\int_{t\ge 0} e^{-\theta t} E u^{A} e^{-\xi T} v^{U} e^{-\vartheta\Delta} \mathbf{1}_{t\in(T,T+\Delta]} dt$$

$$= \sum_{k=0}^{\infty} u^{k} \sum_{j=0}^{\infty} v^{j} \int_{t=0}^{\infty} e^{-\theta t} \int_{s\ge 0} e^{-\xi s} \int_{\delta\ge 0} e^{-\vartheta\delta} \mathbf{1}_{t\in(s,s+\delta]} dP_{A\otimes U\otimes T\otimes\Delta}(k,j,s,\delta)\, dt$$

$$= \sum_{k=0}^{\infty} u^{k} \sum_{j=0}^{\infty} v^{j} \int_{s\ge 0} e^{-(\xi+\theta)s} \int_{\delta\ge 0} e^{-\vartheta\delta} \int_{t=s}^{s+\delta} e^{-\theta(t-s)} dt dP_{A\otimes U\otimes T\otimes\Delta}(k,j,s,\delta)$$

due to the translation invariance of the Borel-Lebesgue measure and by the change of variables

$$= \frac{1}{\theta}\sum_{k=0}^{\infty} u^{k} \sum_{j=0}^{\infty} v^{j} \int_{s\ge 0} e^{-(\xi+\theta)s} \int_{\delta\ge 0} \left[e^{-\vartheta\delta} - e^{-(\vartheta+\theta)\delta}\right] dP_{A\otimes U\otimes T\otimes\Delta}(k,j,s,\delta)$$

$$= \frac{1}{\theta}\left[Eu^{A}v^{U}e^{-(\xi+\theta)T}e^{-\vartheta\Delta} - Eu^{A}v^{U}e^{-(\xi+\theta)T}e^{-(\vartheta+\theta)\Delta}\right]$$

that proves formula (L1). If (A,T) and (U,Δ) are independent, we easily arrive at formula (C2) stated in Corollary 1. □

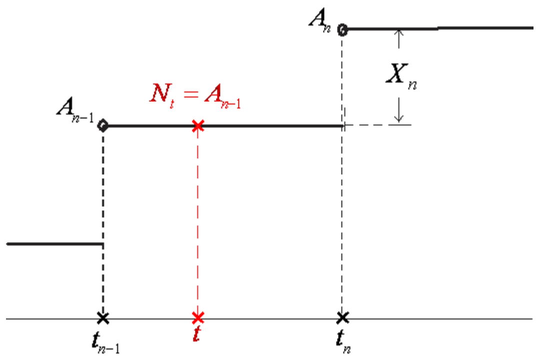
In the context of section 2, with $\mathcal{A}$ being a delayed marked renewal process, we introduce the following notation.

$$A_n = X_0 + \ldots + X_n, \Delta_n = t_n - t_{n-1}, n = 0,1,\ldots, t_{-1} = 0 \tag{7}$$

$$\gamma_0(u,\vartheta) = Eu^{X_0}e^{-\vartheta t_0},\ \gamma(u,\vartheta) = Eu^{X_i}e^{-\vartheta\Delta_i}, i = 1,2,\dots \tag{8}$$

As per (2.3), $\mathcal{A}[0,t] = N_t = \sum_{k=0}^{\infty}\mathbf{1}_{[0,t]}(t_k)$ is the counting process associated with the point process $\sum_{k=0}^{\infty}\varepsilon_{t_k}$.

In light of the figure below,



consider the functional

$$F_n(t) = Ez^{N_t}u^{A_{n-1}}v^{A_n}e^{-\vartheta_0 t_{n-1}-\vartheta t_n}\mathbf{1}_{\{t_{n-1}\le t<t_n\}}, n = 1,2,\dots \tag{9}$$

unfolded as

$$\begin{aligned} F_n(z) &= E(zu)^{A_{n-1}}v^{A_{n-1}+X_n}e^{-(\vartheta_0+\vartheta)t_{n-1}-\vartheta(t_n-t_{n-1})}\mathbf{1}_{\{t_{n-1}\le t<t_n\}} \\ &= E(uvz)^{A_{n-1}}e^{-(\vartheta_0+\vartheta)t_{n-1}}v^{X_n}e^{-\vartheta\Delta_n}\mathbf{1}_{\{t_{n-1}\le t<t_n\}}, n = 1,2\dots \end{aligned} \tag{10}$$

Due to the independence of (A_{n-1}, t_{n-1}) and (X_n, Δ_n) [(A,T) and (U,Δ) in the context of Lemma 1], applying Corollary 2 to the Laplace transform of F_n we have

$$\begin{aligned} F_n^*(\theta) &= \int_{t=0}^{\infty} e^{-\theta t}F_n(t)\,dt \\ &= \int_{t\ge 0} e^{-\theta t}E(uvz)^{A_{n-1}}e^{-(\vartheta_0+\vartheta)t_{n-1}}v^{X_n}e^{-\vartheta\Delta_n}\mathbf{1}_{t\in(t_{n-1},t_n]}dt \\ &= \frac{1}{\theta}\mathit{\Gamma}_{n-1}(uvz,\vartheta_0+\vartheta+\theta)[\gamma(v,\vartheta)-\gamma(v,\vartheta+\theta)] \end{aligned}$$

where

$$\mathit{\Gamma}_{n-1}(uvz,\vartheta_0+\vartheta+\theta) = \gamma_0(uvz,\vartheta_0+\vartheta+\theta)\,\gamma^{n-1}(uvz,\vartheta_0+\vartheta+\theta)\ \text{ for } n\ge 1. \tag{11}$$

Summing up F_n for all $n = 1,2,\dots$, with (3.5) in mind, we formally arrive at the expression

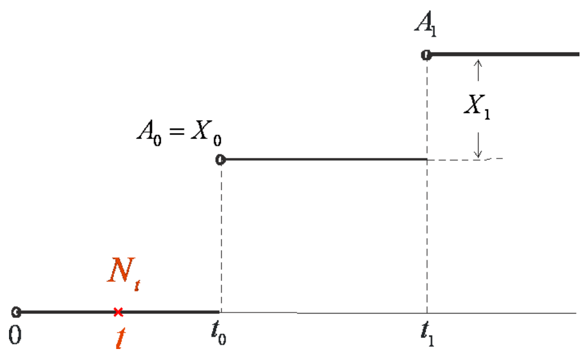
$$\sum_{n=1}^{\infty} F_n^*(\theta)$$
$$= \frac{1}{\theta}\gamma_0(uvz, \vartheta_0 + \vartheta + \theta)[\gamma(v,\vartheta) - \gamma(v, \vartheta+\theta)]\frac{1}{1-\gamma(uvz, \vartheta_0+\vartheta+\theta)}. \quad (12)$$

In Proposition A.1 (see the Appendix), we show that $\|\gamma(uvz, \vartheta_0 + \vartheta + \theta)\| < 1$.

Proposition 3. Let $F_0(t) = Ez^{N_t}v^{A_0}e^{-\vartheta t_0}\mathbf{1}_{[0,t_0]}(t)$. Then

$$F_0^*(\theta) = \frac{1}{\theta}[\gamma_0(v,\vartheta) - \gamma_0(v,\vartheta+\theta)].$$

Proof. From the figure below



we readily deduce that

$$F_0(t) = Ev^{u_0}e^{-\vartheta t_0}\mathbf{1}_{[0,t_0]}(t).$$

The statement follows from Corollary 2 with $T = 0$, $U = u_0$, and $\Delta = \Delta_0 = t_0$. □

With Proposition 3, we can adhere F_0^* to the series $\sum_{n=1}^{\infty} F_n^*$ of formula (3.6):

$$\sum_{n=0}^{\infty} F_n^*(\theta) = \frac{1}{\theta}[\gamma_0(v,\vartheta) - \gamma_0(v,\vartheta+\theta)]$$
$$+ \frac{1}{\theta}\gamma_0(uvz, \vartheta_0 + \vartheta + \theta)[\gamma(v,\vartheta) - \gamma(v,\vartheta+\theta)]\frac{1}{1-\gamma(uvz,\vartheta_0+\vartheta+\theta)}. \quad (13)$$

Formula (3.7) will be used in section 4.

4. FIRST PASSAGE TIME OF $\mathcal{A}$ AND ITS RAMIFICATIONS

Now we return to the formalism of section 2 about random measure $\mathcal{A}$, the associated continuous time parameter jump process $N_t = \mathcal{A}[0,t]$, the exit index ν, and the first passage time t_ν. Of equal interest are A_ν, the first excess value of $\{A_n\}$ of threshold

M upon t_ν. We also target $t_{\nu-1}$ (pre-first passage time) and $A_{\nu-1}$ (pre-first excess value). Because X_k's are nonnegative, $A_{\nu-1} < M$.

As previously noted, not only do we want to screen N_t from $t_{\nu-1}$ to t_ν (providing us with a refined information about $\mathcal{A}$ between the two key reference points), but even more importantly, we want to connect underlying fluctuation parameters with real time. So, we target the joint distribution of the introduced r.v.'s under the following transform

$$\varPhi_\nu(t) = Ez^{N_t} u^{A_{\nu-1}} v^{A_\nu} e^{-\vartheta_0 t_{\nu-1} - \vartheta t_\nu} \mathbf{1}_{(t_{\nu-1}, t_\nu]}(t),$$
$$\|z\| \le 1, \|u\| \le 1, \|v\| \le 1, \mathrm{Re}\vartheta_0 \ge 0, \mathrm{Re}\vartheta \ge 0. \quad (14)$$

We notice that $\varPhi_\nu$ cannot be treated directly by applying Lemma 1 or Corollary 2, simply because we do not know the distribution of $(A_\nu - A_{\nu-1}, t_\nu - t_{\nu-1})$, nor is the latter independent of $(A_{\nu-1}, t_{\nu-1})$. The method of dealing with functional $\varPhi_\nu$ will include several steps. In step 1, we introduce the auxiliary sequence $\{\nu(p)\}$ of exit indices relative to the sequence $\{0, 1, \ldots\}$ of thresholds to be crossed by A_n, of which $\nu = \nu(M-1)$. Namely, let

$$\nu(p) = \inf\{n = 0, 1, \ldots : A_n > p\}, p = 0, 1, \ldots$$

Given a fixed p, we have

$$\varPhi_{\nu(p)}(t) = Ez^{N_t} u^{A_{\nu(p)-1}} v^{A_{\nu(p)}} e^{-\vartheta_0 t_{\nu(p)-1} - \vartheta t_{\nu(p)}} \mathbf{1}_{(t_{\nu(p)-1}, t_{\nu(p)}]}(t).$$

In step 2, we apply to $\varPhi_{\nu(p)}$ transformation D_p defined as

$$D_p\{f(p)\}(x) := \sum_{p=0}^{\infty} x^p f(p)(1-x), \ ||x|| < 1,$$

where f is a real-valued function with the domain $\mathbb{N}_0 = \{0, 1, \ldots\}$. The inverse of D_p is the so-called $\mathcal{D}$-operator defined in Dshalalow [17] as

$$\mathcal{D}_x^k \varphi(x, y) = \begin{cases} \lim_{x \to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left[\frac{1}{1-x} \varphi(x, y) \right], & k \ge 0 \\ 0, & k < 0 \end{cases}$$

(φ is analytic at zero in variable x).

From $\varPhi_{\nu(p)}(t) = \sum_{n=0}^{\infty} \varPhi_{\nu(p)}(t) \mathbf{1}_{\{v(p)=n\}}$, we have

$$\varPhi(t, x) := D_p\left[\varPhi_{\nu(p)}(t)\right](x) = \sum_{n=0}^{\infty} \varPhi_{\nu(p)}(t) D_p \mathbf{1}_{\{\nu(p)=n\}}(x)$$
$$= \sum_{n=0}^{\infty} \varPhi_{\nu(p)=n}(t) D_p \mathbf{1}_{\{\nu(p)=n\}}(x),$$

with $\Phi_{\nu(p)=n}(t) = Ez^{N_t}u^{A_{n-1}}v^{A_n}e^{-\vartheta_0 t_{n-1}-\vartheta t_n}\mathbf{1}_{\{t_{n-1}\leq t<t_n\}} = F_n(t)$. From $1_{\{v(p)=n\}} = \mathbf{1}_{\{A_{n-1}\leq p\}}\mathbf{1}_{\{A_n>p\}}$,

$$D_p\mathbf{1}_{\{v(p)=n\}}(x) = (1-x)\sum_{p=0}^{\infty}x^p\mathbf{1}_{\{A_{n-1}\leq p\}}\mathbf{1}_{\{A_n>p\}}$$
$$= (1-x)\sum_{p=A_{n-1}}^{A_n-1}x^p = x^{A_{n-1}} - x^{A_n}$$

that yields

$$\Phi(t,x) = \sum_{n=0}^{\infty}F_n(t)\left(x^{A_{n-1}} - x^{A_n}\right)$$
$$= \sum_{n=0}^{\infty}F_n(ux,v,z,\vartheta_0,\vartheta,t) - F_n(u,vx,z,\vartheta_0,\vartheta,t)\,, \text{ where } A_{-1}=0.$$

Finally, applying the Laplace transform to $\Phi(t,x)$, in notation

$$\Phi^*(\theta,x) = \int_{t=0}^{\infty}e^{-\theta t}\Phi(t,x)\,dt,$$

we have

$$\theta\Phi^*(\theta,x) = \sum_{n=0}^{\infty}[F_n^*(ux,v,z,\vartheta_0,\vartheta,t) - F_n^*(u,vx,z,\vartheta_0,\vartheta,t)].$$

We can copy the results for $\sum_{n=1}^{\infty}F_n^*$ from (3.6). As far as F_0^*, we have to proceed with caution, since $x^{A_{n-1}} - x^{A_n} = 1 - x^{X_0}$, if $n=0$. Thus in this case we have,

$$F_0^*(v,\vartheta,t) - F_0^*(vx,\vartheta,t)$$
$$= \frac{1}{\theta}\left[\gamma_0(v,\vartheta) - \gamma_0(v,\vartheta+\theta)\right] - \frac{1}{\theta}\left[\gamma_0(vx,\vartheta) - \gamma_0(vx,\vartheta+\theta)\right]. \quad (15)$$

Furthermore,

$$\sum_{n=1}^{\infty}F_n^*(ux,v,z,\vartheta_0,\vartheta,t) - F_n^*(u,vx,z,\vartheta_0,\vartheta,t)$$
$$= \frac{1}{\theta}\gamma_0(uvzx,\vartheta_0+\vartheta+\theta)\left[\gamma(v,\vartheta) - \gamma(v,\vartheta+\theta)\right]\frac{1}{1-\gamma(uvzx,\vartheta_0+\vartheta+\theta)}$$
$$- \frac{1}{\theta}\gamma_0(uvzx,\vartheta_0+\vartheta+\theta)\left[\gamma(vx,\vartheta) - \gamma(vx,\vartheta+\theta)\right]\frac{1}{1-\gamma(uvzx,\vartheta_0+\vartheta+\theta)}$$
$$= \frac{1}{\theta}\gamma_0(uvxz,\vartheta_0+\vartheta+\theta)\frac{1}{1-\gamma(uvzx,\vartheta_0+\vartheta+\theta)}\left[\gamma(v,\vartheta)\right.$$
$$\left.-\gamma(vx,\vartheta) + \gamma(vx,\vartheta+\theta) - \gamma(v,\vartheta+\theta)\right]. \quad (16)$$

Hence combining (4.2) and (4.3), summing up over all $n \geq 0$ yields:

$$\theta \Phi^* (\theta, x) = \gamma_0 (v, \vartheta) - \gamma_0 (v, \vartheta + \theta) - \gamma_0 (vx, \vartheta) + \gamma_0 (vx, \vartheta + \theta)$$
$$+\gamma_0 (uvxz, \vartheta_0 + \vartheta + \theta) \frac{1}{1 - \gamma (uvzx, \vartheta_0 + \vartheta + \theta)} [\gamma (v, \vartheta)$$
$$-\gamma (vx, \vartheta) + \gamma (vx, \vartheta + \theta) - \gamma (v, \vartheta + \theta)] . \quad (17)$$

Applying the $\mathcal{D}$-operator to Φ^* of (4.4) we get

$$\theta \Phi^* (\theta) = \mathcal{D}_x^{M-1} \theta \Phi^* (\theta, x) . \quad (18)$$

$\Phi_\nu (t) = E z^{N_t} u^{A_{\nu-1}} v^{A_\nu} e^{-\vartheta_0 t_{\nu-1} - \vartheta t_\nu} \mathbf{1}_{(t_{\nu-1}, t_\nu]} (t)$ can be extracted from (4.4-4.5) by applying the inverse Laplace transform (subject to our discussion in section 5).

Without a “delay” of $\mathcal{A}$, $\gamma_0 = 1$ and thus (4.4) is simplified to

$$\theta \Phi^* (\theta, x) =$$
$$\frac{1}{1 - \gamma (uvzx, \vartheta_0 + \vartheta + \theta)} [\gamma (v, \vartheta) - \gamma (vx, \vartheta) + \gamma (vx, \vartheta + \theta) - \gamma (v, \vartheta + \theta)] . \quad (19)$$

Finally, applying the $\mathcal{D}$-operator to Φ^* version (4.6) we get

$$\theta \Phi_\nu^* (\theta) = \mathcal{D}_x^{M-1} \frac{1}{1 - \gamma (uvzx, \vartheta_0 + \vartheta + \theta)} [\gamma (v, \vartheta)$$
$$-\gamma (vx, \vartheta) + \gamma (vx, \vartheta + \theta) - \gamma (v, \vartheta + \theta)] . \quad (20)$$

5. SPECIAL CASE: MARKED POISSON PROCESS WITH POSITION INDEPENDENT MARKING

To illustrate tractability of the results obtained in (4.6-4.7), let $\mathcal{A} = \sum_{n=1}^{\infty} X_n \varepsilon_{t_n}$ be a marked Poisson measure with position independent marking and support counting measure $\sum_{n=1}^{\infty} \varepsilon_{t_n}$ of intensity λ. We assume that the marks $X_1, X_2, \ldots \in [\mathrm{Geo}_1 (p)]$ are independent and identically distributed (iid), thus with the common pgf

$$a (z) = \frac{pz}{1 - qz}, \quad (21)$$

and that interrenewal times $\Delta_n = t_n - t_{n-1}$, $n = 1, 2, 3, \ldots$ (assuming no delay and $t_0 = 0$) are iid with the common LST

$$\gamma (\theta) = E e^{-\Delta_1 \theta} = \frac{\lambda}{\lambda + \theta}, \ \mathrm{Re}\theta \geq 0. \quad (22)$$

So, due to position independent marking,

$$\gamma (u, \theta) = E u^{X_1} e^{-\Delta_1 \vartheta} = a (u) \gamma (\vartheta) = \frac{pu}{1 - qu} \frac{\lambda}{\lambda + \theta}. \quad (23)$$

We will work on formula (4.7) substituting there (5.3) for γ. Furthermore, from

$$1-\gamma\left(uvzx,\vartheta_0+\vartheta+\theta\right)=1-\frac{\lambda}{\lambda+\vartheta_0+\vartheta+\theta}\frac{puvzx}{1-quvzx},$$

after some algebra,

$$\frac{1}{1-\gamma\left(uvzx,\vartheta_0+\vartheta+\theta\right)}=\frac{A-Bx}{A-Cx}=\left[\frac{B}{C}+\frac{A\left(C-B\right)}{C\left(A-Cx\right)}\right]$$

$$A=\lambda+\vartheta_0+\vartheta+\theta$$

$$B=(qvuz)(\lambda+\vartheta_0+\vartheta+\theta)$$

$$C=(qvuz)(\lambda+\vartheta_0+\vartheta+\theta)+\lambda pvuz=B+\lambda pvuz$$

Then we have

$$\gamma(v,\vartheta)-\gamma(vx,\vartheta)+\gamma(vx,\vartheta+\theta)-\gamma(v,\vartheta+\theta)$$
$$=G\left(\frac{1}{1-qv}-\frac{1}{1-qvx}\right),\ qv\neq 1,\ qvx\neq 1,$$

where

$$G=\frac{\lambda p\theta}{q\left(\vartheta+\lambda\right)\left(\vartheta+\lambda+\theta\right)}$$

Then, using Dshalalow [17],

$$\theta\Phi_\nu^*\left(\theta\right)=D_x^{M-1}\left[G\left(\frac{B}{C}+\frac{A\left(C-B\right)}{C\left(A-Cx\right)}\right)\left(\frac{1}{1-qv}-\frac{1}{1-qvx}\right)\right]$$

$$=\frac{BG}{C\left(1-qv\right)}-\frac{BG}{C}\frac{1-\left(qv\right)^M}{1-qv}+\frac{G\left(C-B\right)}{C\left(1-qv\right)}\frac{1-\left(\frac{C}{A}\right)^M}{1-\frac{C}{A}}-\frac{G\left(C-B\right)}{C\left(1-qv\right)}$$
$$\left(\frac{1-\left(\frac{C}{A}\right)^M}{1-\frac{C}{A}}-\left(qv\right)^M\frac{1-\left(\frac{C}{Aqv}\right)^M}{1-\frac{C}{Aqv}}\right)$$

$$=\frac{G}{C\left(1-qv\right)}\left[\left(qv\right)^M B+\left(C-B\right)\left(qv\right)^M\frac{1-\left(\frac{C}{Aqv}\right)^M}{1-\frac{C}{Aqv}}\right].$$

or returning to the original expressions

$$\Phi_\nu^*\left(\theta\right)=\frac{\lambda p}{q\left(\vartheta+\lambda\right)\left(\vartheta+\lambda+\theta\right)\left(1-qv\right)\left[q\left(\lambda+\vartheta_0+\vartheta+\theta\right)+\lambda p\right]}$$
$$\times\left[q\left(qv\right)^M\left(\lambda+\vartheta_0+\vartheta+\theta\right)+\lambda p\left(qv\right)^M\frac{1-\left(\frac{qvuz(\lambda+\vartheta_0+\vartheta+\theta)+\lambda pvuz}{(\lambda+\vartheta_0+\vartheta+\theta)qv}\right)^M}{1-\frac{qvuz(\lambda+\vartheta_0+\vartheta+\theta)+\lambda pvuz}{(\lambda+\vartheta_0+\vartheta+\theta)qv}}\right]. \tag{24}$$

We are interested in various marginal versions of functional

$$\Phi_\nu(t) = Ez^{N_t} u^{A_{\nu-1}} v^{A_\nu} e^{-\vartheta_0 t_{\nu-1} - \vartheta t_\nu} \mathbf{1}_{(t_{\nu-1}, t_\nu]}(t).$$

For brevity we will write

$$\Phi_\nu(t) = \Phi_\nu(t; z, u, v, \vartheta_0, \vartheta),$$

which can be derived by taking the Laplace inverse of $\Phi_\nu^*(\theta) = \Phi_\nu^*(\theta; z, u, v, \vartheta_0, \vartheta)$.

(i) First, consider the marginal functional $\Phi_\nu(t; z, 1, 1, 0, 0) = Ez^{N_t} \mathbf{1}_{(t_{\nu-1}, t_\nu]}(t)$ of the Poisson counting process N_t. Here,

$$\Phi_\nu^*(\theta; z, 1, 1, 0, 0)$$
$$= \frac{\lambda}{q(\lambda)(\lambda+\theta)[q(\lambda+\theta)+\lambda p]} \left[q^{M+1}(\lambda+\theta) + \lambda p q^M \frac{1 - \left(\frac{qz(\lambda+\theta)+\lambda pz}{(\lambda+\theta)q}\right)^M}{1 - \frac{qz(\lambda+\theta)+\lambda pz}{(\lambda+\theta)q}} \right]$$

that can be reduced to

$$\Phi_\nu^*(\theta; z, 1, 1, 0, 0) = \frac{((\lambda+\theta)q)^{M-1} + \lambda p \sum_{k=1}^{M-1} ((\lambda+\theta)q)^{k-1} z^{M-k} (\lambda + q\theta)^{M-1-k}}{(\lambda+\theta)^M},$$

after some algebra, and then further to

$$\Phi_\nu^*(\theta; z, 1, 1, 0, 0) = \frac{q^{M-1}}{\lambda+\theta} + \lambda p \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} z^{M-k}$$
$$q^{k+i-1} (\lambda p)^{M-1-k-i} \left(\frac{1}{\lambda+\theta}\right)^{M+1-k-i}.$$

Thus, denoting $\mathcal{L}_\theta^{-1}$ for the Laplace inverse operator in variable θ we have

$$\Phi_\nu(t; z, 1, 1, 0, 0) = Ez^{N_t} \mathbf{1}_{(t_{\nu-1}, t_\nu]}(t)(t) = \mathcal{L}_\theta^{-1} \{\Phi_\nu^*(\theta; z, 1, 1, 0, 0)\}(t)$$
$$= \mathcal{L}_\theta^{-1} \left\{ \frac{q^{M-1}}{\lambda+\theta} \right\}(t) + \lambda p \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} z^{M-k}$$
$$q^{k+i-1} (\lambda p)^{M-1-k-i} \mathcal{L}_\theta^{-1} \left\{ \left(\frac{1}{\lambda+\theta}\right)^{M+1-k-i} \right\}(t)$$
$$= q^{M-1} e^{-\lambda t} + \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} z^{M-k} q^{k+i-1} (\lambda p)^{M-k-i}$$
$$e^{-\lambda t} \frac{t^{M-k-i}}{(M-k-i)!}. \tag{25}$$

In particular, for $M = 1$,$M = 2$,and $M = 3$ we get:

$$M = 1 : \Phi_\nu(t; z, 1, 1, 0, 0) = e^{-\lambda t} \tag{26}$$

$$M=2:\varPhi_{\nu}\left(t;z,1,1,0,0\right)\ =\ qe^{-\lambda t}\ +\ \lambda pzte^{-\lambda t}\ =(q+\lambda pzt)e^{-\lambda t} \tag{27}$$

$$M=3:\ \varPhi_{\nu}\left(t;z,1,1,0,0\right)\ =\ q^{2}e^{-\lambda t}\ +\lambda pqzt(1+z)e^{-\lambda t}\ +\ \frac{(\lambda pzt)^{2}}{2}e^{-\lambda t}. \tag{28}$$

We confirm the results for $Ez^{N_t}\mathbf{1}_{(t_{\nu-1},t_{\nu}]}\left(t\right)$ obtained from (5.5) for $M=1,2,$ and 3 using direct probability arguments.

When $M=1$, the first passage time occurs in the event that $X_1\geq 1$. Thus for $M=1$ and $z=1$,

$$\begin{aligned}\varPhi_{\nu}\left(t;1,1,1,0,0\right)=E\mathbf{1}_{(t_{\nu-1},t_{\nu}]}\left(t\right)&=P\{t_{\nu-1}\leq t\leq t_{\nu}\}\,P\left\{X_1\geq 1\right\}\\ &=P\{0\leq t\leq t_1\}\cdot 1=e^{-\lambda t}\end{aligned} \tag{29}$$

which agrees with (5.6).

When $M=2$, the first passage time could occur in the event that $X_1\geq 2$ (interval $[0,t_1]$) or in the event that $X_1=1$ and $X_2\geq 1$ (interval $[t_1,t_2]$). Thus, when $M=2$ and $z=1$,

$$\begin{aligned}\varPhi_{\nu}\left(t;1,1,1,0,0\right)&=E\mathbf{1}_{(t_{\nu-1},t_{\nu}]}\left(t\right)\ =P\{t_{\nu-1}\leq t\leq t_{\nu}\}\\ &=P\{0\leq t\leq t_1\}P\left\{X_1\geq 2\right\}+\ P\{t_1\leq t\leq t_2\}P\left\{X_1=1\right\}P\left\{X_2\geq 1\right\}\\ &=P\{N_t=0\}P\left\{X_1\geq 2\right\}+\ P\{N_t=1\}P\left\{X_1=1\right\}P\left\{X_2\geq 1\right\}\\ &=qe^{-\lambda t}\ +\ \lambda pte^{-\lambda t},\end{aligned} \tag{30}$$

which agrees with (5.7).

Similarly, when $M=3$, the first passage time occurs when $X_1\geq 3$, or $X_1=1$ and $X_2>1$, or when $X_1=1$, $X_2=1$, and $X_3\geq 1$ in their respective intervals. Therefore, when $M=3$ and $z=1$,

$$\begin{aligned}\varPhi_{\nu}\left(t;1,1,1,0,0\right)=&E\mathbf{1}_{(t_{\nu-1},t_{\nu}]}\left(t\right)\ =P\{t_{\nu-1}\leq t\leq t_{\nu}\}\\ =&P\{0\leq t\leq t_1\}P\left\{X_1\geq 3\right\}\\ &+\ P\{t_1\leq t\leq t_2\}\left[P\left\{X_1=1\right\}P\left\{X_2\geq 2\right\}+P\left\{X_1=2\right\}P\left\{X_2\geq 1\right\}\right]\\ &+P\{t_2\leq t\leq t_3\}P\left\{X_1=1\right\}P\left\{X_2=1\right\}P\left\{X_3\geq 1\right\}\\ =&P\{N_t=0\}P\left\{X_1\geq 3\right\}\\ &+P\{N_t=1\}\left[P\left\{X_1=1\right\}P\left\{X_2\geq 2\right\}+P\left\{X_1=2\right\}P\left\{X_2\geq 1\right\}\right]\\ &+\ P\{N_t=2\}P\left\{X_1=1\right\}P\left\{X_2=1\right\}P\left\{X_3\geq 1\right\}\\ =&q^{2}e^{-\lambda t}\ +\ 2pq\lambda te^{-\lambda t}\ +\ \frac{\left(\lambda pt\right)^{2}}{2}\ e^{-\lambda t}\end{aligned} \tag{31}$$

which agrees with (5.8).

(*ii*) Now, considering the marginal functional $\varPhi_{\nu}\left(t;1,1,1,0,\vartheta\right)=Ee^{-\vartheta t_{\nu}}\mathbf{1}_{(t_{\nu-1},t_{\nu}]}\left(t\right)$ of t_{ν}, the first passage time (i.e., when the first threshold crossing occurs), we have

$$\Phi_\nu^*(\theta;1,1,1,0,\vartheta) = \frac{\lambda}{q(\vartheta+\lambda)(\vartheta+\lambda+\theta)[q(\vartheta+\lambda+\theta)+\lambda p]} \left[q^{M+1}(\vartheta+\lambda+\theta) + \lambda p q^M \frac{1-\left(\frac{q(\vartheta+\lambda+\theta)+\lambda p}{(\vartheta+\lambda+\theta)q}\right)^M}{1-\frac{q(\vartheta+\lambda+\theta)+\lambda p}{(\vartheta+\lambda+\theta)q}} \right]$$

reducing it to

$$\Phi_\nu^*(\theta;1,1,1,0,\vartheta) = \frac{\lambda q^{M-1}}{(\vartheta+\lambda)(\vartheta+\lambda+\theta)} \left[1 + \lambda p \sum_{k=1}^{M-1} \frac{q^{k-M}}{(\vartheta+\lambda+\theta)^{M-k}} \left(q(\vartheta+\lambda+\theta)+\lambda p\right)^{M-k-1} \right]$$

and then to

$$\Phi_\nu^*(\theta;1,1,1,0,\vartheta) = \frac{\lambda q^{M-1}}{(\vartheta+\lambda)(\vartheta+\lambda+\theta)} + \lambda^2 p \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} \frac{q^{k+i-1}}{\vartheta+\lambda} (\lambda p)^{M-k-1-i} \left(\frac{1}{\vartheta+\lambda+\theta}\right)^{M-k+1-i}.$$

Thus,

$$\begin{aligned}\Phi_\nu(t;1,1,1,0,\vartheta) &= \mathcal{L}_\theta^{-1}\left\{\Phi_\nu^*(\theta;1,1,1,0,\vartheta)\right\}(t) \\ &= \mathcal{L}_\theta^{-1}\left\{\frac{\lambda q^{M-1}}{(\vartheta+\lambda)(\vartheta+\lambda+\theta)}\right\}(t) \\ &\quad + \lambda^2 p \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} \frac{q^{k+i-1}}{(\vartheta+\lambda)} (\lambda p)^{M-k-1-i} \\ &\quad \mathcal{L}_\theta^{-1}\left\{\left(\frac{1}{\vartheta+\lambda+\theta}\right)^{M-k+1-i}\right\}(t).\end{aligned}$$

Hence,

$$\begin{aligned}\Phi_\nu(t;1,1,1,0,\vartheta) &= Ee^{-\vartheta t_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t). \\ &= \frac{\lambda q^{M-1}}{\vartheta+\lambda} e^{-(\vartheta+\lambda)t} + \lambda^2 p \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} \frac{q^{k+i-1}}{\vartheta+\lambda} \\ &\quad (\lambda p)^{M-k-1-i} e^{-(\vartheta+\lambda)t} \frac{t^{M-k-i}}{(M-k-i)!}.\end{aligned}$$

Specifically for

$$M = 1 :, \Phi_\nu(t;1,1,1,0,\vartheta) = \frac{\lambda}{\vartheta+\lambda} e^{-(\vartheta+\lambda)t}$$

$$M=2:\Phi_{\nu}\left(t;1,1,1,0,\vartheta\right) = \frac{\lambda q}{\vartheta+\lambda}e^{-(\vartheta+\lambda)t}+\lambda^{2}pt\frac{1}{\vartheta+\lambda}e^{-(\vartheta+\lambda)t}$$
$$= \frac{\lambda}{\vartheta+\lambda}\left(q+\lambda pt\right)e^{-(\vartheta+\lambda)t},$$

which agree with (5.9) and (5.10), respectively, when $\vartheta = 1$.

Furthermore, for general M, applying the inverse Laplace transform in variable ϑ

$$\frac{d}{dx}\left[P\left\{t_{\nu}\leq x,t\in\left(t_{\nu-1},t_{\nu}\right]\right\}\right]$$
$$=\mathcal{L}_{\vartheta}^{-1}\left\{\Phi_{\nu}\left(t;1,1,1,0,\vartheta\right)\right\}(x)$$
$$=\lambda q^{M-1}e^{-\lambda x}\mathbf{1}_{(t,\infty)}(x)+\lambda^{2}p\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}q^{k+i-1}\left(\lambda p\right)^{M-k-1-i}$$
$$\frac{t^{M-k-i}}{(M-k-i)!}e^{-\lambda x}\mathbf{1}_{(t,\infty)}(x).$$

(iii) Thirdly, for the marginal functional $\Phi_{\nu}\left(\theta;1,1,v,0,0\right)=Ev^{A_{\nu}}\mathbf{1}_{(t_{\nu-1},t_{\nu}]}(t)$ of the first excess level A_{ν}, setting $z=1$, $u=1$, $\vartheta_{0}=0$, $\vartheta=0$ in $\Phi_{\nu}^{*}\left(\theta;z,u,v,\vartheta_{0},\vartheta\right)$ we arrive at:

$$\Phi_{\nu}^{*}\left(\theta;1,1,v,0,0\right)=\frac{pv^{M}q^{M-1}}{(\lambda+\theta)(1-qv)}+v^{M}\lambda p^{2}\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}\frac{q^{k+i-1}}{1-qv}$$
$$\left(\lambda p\right)^{M-k-1-i}\left(\frac{1}{\lambda+\theta}\right)^{M-k+1-i}$$

after a similar algebraic routine. Thus,

$$\mathcal{L}_{\theta}^{-1}\left\{\Phi_{\nu}^{*}\left(\theta;1,1,v,0,0\right)\right\}(t)$$
$$=\mathcal{L}_{\theta}^{-1}\left\{\frac{pv^{M}q^{M-1}}{(\lambda+\theta)(1-qv)}\right\}(t)$$
$$+v^{M}\lambda p^{2}\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}\frac{q^{k+i-1}}{(1-qv)}\left(\lambda p\right)^{M-k-1-i}$$
$$\mathcal{L}_{\theta}^{-1}\left\{\left(\frac{1}{\lambda+\theta}\right)^{M-k+1-i}\right\}(t).$$

Hence,

$$\Phi_{\nu}\left(t;1,1,v,0,0\right)=Ev^{A_{\nu}}\mathbf{1}_{(t_{\nu-1},t_{\nu}]}(t).$$
$$\frac{pv^{M}q^{M-1}}{1-qv}e^{-\lambda t}+v^{M}\lambda p^{2}\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}\frac{q^{k+i-1}}{(1-qv)}\left(\lambda p\right)^{M-k-1-i}$$
$$e^{-\lambda t}\frac{t^{M-k-i}}{(M-k-i)!}.$$

Notice that expanding $Ev^{A_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)$ in Taylor series in v we have

$$Ev^{A_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)$$
$$=\sum_{n=0}^{\infty}\left(pv^M q^{M-1}e^{-\lambda t}+v^M\lambda p^2\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}q^{k+i-1}(\lambda p)^{M-k-1-i}\right.$$
$$\left.e^{-\lambda t}\frac{t^{M-k-i}}{(M-k-i)!}\right)(qv)^n$$
$$=\sum_{n=M}^{\infty}\left(pq^{n+M-1}e^{-\lambda t}+\lambda p^2\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}q^{n+k+i-1}(\lambda p)^{M-k-1-i}\right.$$
$$\left.e^{-\lambda t}\frac{t^{M-k-i}}{(M-k-i)!}\right)v^n.$$

Therefore,

$$P\{A_\nu=n,t\in(t_{\nu-1},t_\nu]\}$$
$$=\left[\lambda p^2\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}q^{n+k+i-1}(\lambda p)^{M-k-1-i}e^{-\lambda t}\frac{t^{M-k-i}}{(M-k-i)!}\right.$$
$$\left.+pq^{n+M-1}e^{-\lambda t}\right]\mathbf{1}_{\{M,M+1,\dots\}}(n),n=0,1,\dots$$

which agrees with the fact that $A_\nu\geq M$ a.s.
Specifically, for $M=1$,

$$\Phi_\nu(t;1,1,v,0,0)=\frac{pv}{1-qv}e^{-\lambda t}$$

and for $M=2$,

$$\Phi_\nu(t;1,1,v,0,0)=\frac{pqv^2}{1-qv}e^{-\lambda t}+\lambda p^2v^2t\frac{1}{1-qv}e^{-\lambda t}=(q+\lambda pt)\,e^{-\lambda t}\frac{v^2p}{1-qv}$$

which agree with (5.9) and (5.10) respectively when $v=1$.

REFERENCES

[1] Agarwal, R.P., Dshalalow, J.H., and O'Regan, D., Time sensitive functionals of marked Cox processes, Journ. of Math. Analysis and Appl., 293 (2004), 14-27.

[2] Al-Matar, N. and Dshalalow, J.H., A game-theoretic approach in single-server queues with maintenance. Time sensitive analysis, Commun. in Appl. Nonlin. Analysis, 17, No. 1 (2010), 65-92.

[3] Al-Matar, N. and Dshalalow, J.H., Time sensitive functionals in classes of queues with sequential maintenance, Stochastic Models, 27 (2011), 687–704.

[4] Atiya, A.F. and Metwally, S.A., Efficient estimation of first passage time density function for jump-diffusion processes, SIAM J. Sci. Comput., 26(5), 1760–1775.

[5] Aurzada, F., Iksanov, A., and Meiners, M., Exponential moments of first passage times and related quantities for Lévy processes, Mathematische Nachrichten (2015), (DOI: 10.1002/mana.201400289).

[6] Bingham, N.H., Random walk and fluctuation theory, in Handbook of Statistics (Eds. D.N. Shanbhag and C.R. Rao), Volume 19, 2001, Elsevier Science, 171-213.

[7] Borokov, A.A., On the first passage time for one class of processes with independent increments. Theor. Probab. Appl., 10 (1964), 331–334.

[8] Coutin, L. and Dorobantu, D., First passage time law for some Levy processes with compound Poisson: Existence of a density, Bernoulli, 17:4 (2011), 1127-1135.

[9] Crescenzo, A.D. and Martinucci, B., On a first passage-time problem for the compound power-law process, Stochastic Models, 25 (2009), 420-435.

[10] Dshalalow, J.H., First excess level of vector processes, J. Appl. Math. Stoch. Anal., 7:3, 457-464, 1994.

[11] Dshalalow, J.H., On the level crossing of multidimensional delayed renewal processes, in: Special Issue STOCHASTIC SYSTEMS, Journ. Appl. Math. Stoch. Anal., 10:4 (1997), 355-361.

[12] Dshalalow, J.H., Time dependent analysis of multivariate marked renewal processes, Journ. Appl. Prob., 38, 2001.

[13] Dshalalow, J.H., Fluctuations of recurrent processes and their application to the stock market, Stoch. Anal. Appl., 22:1, 2004, 67-79.

[14] Dshalalow, J.H., On exit times of a multivariate random walk with some applications to finance, Nonlinear Analysis, 63 (2005), 569-577.

[15] Dshalalow, J.H., Random walk analysis in antagonistic stochastic games, Stochastic Analysis and Applications, **26** (2008), 738–783.

[16] Dshalalow, J.H., On multivariate antagonistic marked point processes, Math. and Comp. Modeling, 49 (2009) 432-452.

[17] Dshalalow, J.H., Stochastic Processes, Lecture Notes, FIT, Melbourne, FL, 2015.

[18] Dshalalow, J.H. and Iwezulu, K., Discrete versus continuous operational calculus in antagonistic stochastic games, São Paulo Journal of Math. Sci., Springer, NY, 11 (2017), 471-489.

[19] Dshalalow, J.H., Iwezulu, K., and White, R.T., Discrete operational calculus in delayed stochastic games, Neural, Parallel, and Scientific Computations, 24 (2016), 55-64.

[20] Dshalalow, J.H. and Ke, H.-J., Multilayers in a modulated stochastic game, Journ. of Math. Analysis and Applications, 353 (2009), 553-565.

[21] Dshalalow, J.H. and Liew, A., On exit time of a multivariate random walk and its embedding in a quasi-Poisson process, Stochastic Analysis and Applications, 24 (2006), 451-474.

[22] Dshalalow, J.H. and White, R.T., On reliability of stochastic networks, Neural, Parallel, and Scientific Computations, 21 (2013) 141-160.

[23] Dshalalow, J.H. and White, R.T., On strategic defense in stochastic networks, Stochastic Analysis and Applications, 32 (2014), 365–396.

[24] Dshalalow, J.H. and White, R.T., Time sensitive analysis of independent and stationary increment processes, Journal of Mathematical Analysis and Applications 443 (2016), 817-833.

[25] Hida, T. (Editor), Mathematical Approach to Fluctuations: Astronomy, Biology and Quantum Dynamics: Proceedings of the Iias Workshop: Kyoto, Japan, May 18-21, 1992, World Scientific Publishers, 1995).

[26] Fishburn, P.C., Non-cooperative stochastic dominance games, International Journal of Game Theory, 7:1 (1978), 51-61.

[27] Kadankova, T.V., Exit, passage, and crossing times and overshoots for a Poisson compound process with an exponential component, Theor. Probability and Math. Statist., 75 (2007), 23-29.

[28] Kyprianou, A.E. and Pistorius, M.R., Perpetual options and Canadization through fluctuation theory, Ann. Appl. Prob., 13:3 (2003), 1077-1098.

[29] Mellander, E., Vredin, A, and Warne, A., Stochastic trends and economic fluctuations in a small open economy, J. Applied Econom., 7:4 (1992), 369-94.

[30] Muzy, J., Delour1, J., and Bacry, E., Modelling fluctuations of financial time series: from cascade process to stochastic volatility model, Eur. Phys. J. B 17 (2000), 537-548.

[31] Novikov, A, Melchers, R.E., Shinjikashvili, E., and Kordzakhia, N., First passage time of filtered Poisson process with exponential shape function, Probabilistic Engineering Mechanics, 20:1 (2005), 57-65

[32] Redner, S., A Guide to First-Passage Processes, Cambridge University Press, Cambridge, 2001.

[33] Shinozuka, M. and Wu, W-F., On the first passage problem and its application to earthquake engineering, Proceedings of Ninth World Conference on Earthquake Engineering, August 2-9, (VIII) 1988, Tokyo-Kyoto, Japan.

[34] Takács, L., On fluctuations of sums of random variables, in Studies in Probability and Ergodic Theory. Advances in Mathematics; Supplementary Studies, Volume 2, ed. by G.-C. Rota, (1978), 45-93.

[35] Yin, C., Wen, Y., Zong, Z., and Shen, Y., The first passage time problem for mixed-exponential jump processes with applications in insurance and finance, Abstract and Applied Analysis (2014), 9 pages.

[36] Zolotarev, V.M. The first passage time of a level and the behavior at infnity for a class of processes with independent increments, Theor. Probab. Appl, 9 (1964) 653–664.

APPENDIX

Proposition A.1. *The series*

$$\sum_{n=1}^{\infty} F_n^*(\theta) = \sum_{n=1}^{\infty} \int_{t=0}^{\infty} e^{-\theta t} E z^{N_t} u^{A_{n-1}} v^{A_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{t_{n-1} \le t < t_n\}} dt$$

converges to

$$\sum_{n=1}^{\infty} F_n^*(\theta) = \frac{1}{\theta} \gamma_0(uvz, \vartheta_0 + \vartheta + \theta)\left[\gamma(v, \vartheta) - \gamma(v, \vartheta + \theta)\right] \frac{1}{1 - \gamma(uvz, \vartheta_0 + \vartheta + \theta)}$$

and $\|\gamma(uvz, \vartheta_0 + \vartheta + \theta)\| < 1.$ (A.1.1)

Proof. The first part of the proposition is due to the above steps that ended in formula (3.6). The inequality (A.1.1) is due to the following arguments.

$$\left\|\gamma(uvz, \vartheta_0 + \vartheta + \theta)\right\| = \left\|E\,(uvz)^{X_1} e^{-(\vartheta_0+\vartheta+\theta)\Delta_1}\right\|$$

$$= \Big\| \sum_{k=0}^{\infty} (uvz)^k \int_{t=0}^{\infty} e^{-(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(k, dt) \Big\|$$

$$\le \sum_{k=0}^{\infty} \|(uvz)\|^k \int_{t=0}^{\infty} \left\|e^{-(\vartheta_0+\vartheta+\theta)t}\right\| P_{X_1 \otimes \Delta_1}(k, dt)$$

$$= \sum_{k=0}^{\infty} \|(uvz)\|^k \int_{t=0}^{\infty} e^{-Re(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(k, dt)$$

$$= \int_{t=0}^{\infty} e^{-Re(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(0, dt) + \sum_{k=1}^{\infty} \|(uvz)\|^k \int_{t=0}^{\infty} e^{-Re(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(k, dt)$$

$$< \int_{t=0}^{1} P_{X_1 \otimes \Delta_1}(0, dt) + e^{-Re(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1 \otimes \Delta_1}(0, dt)$$
$$+ \left\| (uvz) \right\| \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_1 \otimes \Delta_1}(k, dt) + \left\| (uvz) \right\| \sum_{k=1}^{\infty} e^{-Re(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1 \otimes \Delta_1}(k, dt) ,$$

since $\left\| (uvz) \right\| > \left\| (uvz) \right\|^k$ for $\left\| (uvz) \right\| < 1$ and $k > 1$.
Let,

$$a := \int_{t=0}^{1} P_{X_i \otimes \Delta_i}(0, dt)$$
$$b := \int_{t=1}^{\infty} P_{X_i \otimes \Delta_i}(0, dt)$$
$$c := \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_i \otimes \Delta_i}(k, dt)$$
$$d := \sum_{k=1}^{\infty} e^{-Re(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_i \otimes \Delta_i}(k, dt) .$$

Then clearly, $a + b + c + d = 1$ and thus,

$$a + e^{-Re(\vartheta_0+\vartheta+\theta)} b + \left\| (uvz) \right\| c + \left\| (uvz) \right\| e^{-Re(\vartheta_0+\vartheta+\theta)} d < 1$$

since $\left\| (uvz) \right\| < 1$ and because $Re(\vartheta_0 + \vartheta + \theta) \geq 0$ is a requirement for the existence of the LST, $e^{-Re(\vartheta_0+\vartheta+\theta)} \leq 1$.

$\square$