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A COLLOCATION SCHEME FOR SINGULAR BOUNDARY VALUE PROBLEMS ARISING IN PHYSIOLOGY

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ABSTRACT: A new collocation method for the solution of a class of second-order two-point boundary value problems associated with physiology and other areas with a singular point at one endpoint is constructed. The singularity of the differential equation is modified by L'Hôpital's rule and the boundary condition $y'(0) = 0$. Quintic B-spline functions on equidistant collocation points are used to approximate the solution. The quasi-linearization technique is used to reduce a non-linear problem to a sequence of linear problems. The system obtained on discretization is transformed to the system of linear algebraic equations which is easy to be solved. It is proved that the proposed algorithm converges to a smooth approximate solution of the singular boundary value problems and the error estimates are given. To check the theory and to demonstrate the efficiency of the proposed method, several numerical illustrations from physical model problems have been carried out. To show the effectiveness of the proposed method comparisons with several existing methods has also been done.

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1. INTRODUCTION

We consider the numerical solution of linear two-point boundary value problems

(BVPs) with a singularity at one endpoint. In particular, the class of problems is formulated as:

$$y''(x) + \frac{k}{x}y'(x) + f(x)y(x) = g(x), \quad 0 < x < 1, \tag{1a}$$

subject to the boundary conditions

$$y'(0) = 0, \quad py(1) + qy'(1) = \beta, \tag{1b}$$

where $k \geq 0$ and $f(x)$, $g(x)$ are sufficiently smooth to ensure the existence and uniqueness of the solution of (1a)-(1b).

These singular boundary value problems (RSVPS) for ordinary differential equations frequently arise in a variety of applied mathematics, scientific, and engineering applications such as heat conduction, nuclear physics, gas dynamics, boundary value theory, flow networks of biology, atomic calculations, chemical reaction, electro-hydrodynamics, in the study of generalized axially symmetric potentials after separation of variables, thermal explosions, control and optimization theory and many more. For example, the BVP

$$\frac{d^2T}{dx^2} + \frac{k}{x}\frac{dT}{dx} + \phi(T) = 0, \quad \left.\frac{dT}{dx}\right|_{x=0} = 0, \; T(1) = 1,$$

results from an analysis of heat conduction through a solid with heat generation. The function $\phi(T)$ represents the heat generation within the solid, T is the temperature and the constant k is equal to 0, 1 or 2 depending on whether the solid is a plate, a cylinder or a sphere. Another example is the Thomas-Fermi model

$$y''(x) = \sqrt{\frac{y(x)}{x}}, \quad y(0) = 1, \; y(b) = 0,$$

in atomic physics that describes the charge concentration $y(x)$ of electrons in an ion. Such problems are also encountered in stochastic control problems when studying the steady-state properties of systems driven by noise which is proportional to the state or which are non-linear functions of the state. These also arise in physiology for the study of various tumor problems, the study of steady states oxygen diffusion in a spherical cell with michaelis-menten uptake kinetics and the study of heat sources distribution in the human head.

Because of such a wide scope and applications, this classical problem has attracted much attention [41, 28] and many researchers have focused on these type of problems. These problems have been investigated intensively by using a variety of numerical methods [36, 37, 38, 7, 26, 22, 15, 45, 11, 12, 13, 24, 44]. These methods include the finite difference method, spline methods, finite element methods and other important class of numerical methods that includes the Rayleigh-Ritz, Galerkin and collocation methods [13, 3, 4, 18].

Because of their simplicity and intuitive, the finite difference method is always a good choice for solving SBVPs. In 2003, Kanth and Reddy [36] proposed a fourth order finite difference method by re-approximating the central difference approximation to solve SBVPs. Because of their simplicity and piecewise polynomial characteristics of the spline functions, a number of methods based on splines for solving SBVPs have been extensively studied. In 2005, Kanth and Reddy [37] studied two-point SBVPs by applying cubic spline interpolation method and extended it to solve non-linear SBVPs [38]. After reducing the non-linear problem into a sequence of linear problems by using quasi-linearization techniques and then modifying the resulting sets of differential equations around the singular point, Kanth and Bhattacharya [39] employed B-spline functions to solve two-point BVPs with a singularity at $x = 0$. Based on cubic B-spline bases, in 2006, Caglar and Caglar [7] applied a direct method to find a solution of SBVPs.

To overcome the slow convergence of the Taylor series solution Cohen and Jones [14] have used an economized expansion for the problems and employed deferred correction outside the range of economized expansion. Reddien [40] has studied collocation method for the numerical solution of such problems. Kadalbajoo and Raman [25] have discussed the numerical solution of SBVPs using the invariant imbedding method.

For a homogenous and linear SBVPs, to remove the singularity, Kadalbajoo and Aggarwal [26] first used Chebyshev economization in the vicinity of the singular point and then they derived boundary condition at a point in the vicinity of the singularity. The resulting regular BVP is then efficiently treated by cubic B-spline for finding the numerical solution. Recently, Goh *et al.* [22] used quartic B-spline approximations where the values of coefficients are chosen via optimization. Cui and Geng [15] proposed a method for solving SBVPs where the exact solution is represented in the form of a series in reproducing kernel space. But the deduction of this method seems to be relatively complicated and not intuitive. In [1], Abukhaled *et al.* modified the singularity by L'Hôpital's rule and then the economized Chebyshev polynomial is implemented in the vicinity of the singular point. The readers may also see Chawla and Katti [10], and El-Gebeily and Abbu-Zaid [19] for extra readings. After modifying the singularity by L'Hôpital's rule an adaptive spline method is introduced by Khuri and Sayfy [29].

In the present study to obtain a continuous solution with a higher order approximation, a method of collocation using quintic B-spline functions is proposed for two-point BVPs with a singularity at $x = 0$. The new boundary condition is derived by using L'Hôpital's rule in the vicinity of the singularity. Quintic B-spline functions are then employed to solve the SBVP. The problem with a singularity at $x = 1$ or with the singularities at both ends can be treated in a similar way.

This paper is organized as follows. In Section 2, the quintic B-spline collocation method is applied to two-point SBVPs. For non-linear SBVPs the quasi-linearization approach is given in Section 3. The convergence of the proposed method is analyzed in Section 4 and in Section 5, some numerical examples from the literature are presented and the comparisons of the solutions obtained by different existing methods are made to show the better reliability of our method. Finally, at the end, some conclusions are given in Section 6.

2. PROPOSED METHOD

In this section, we introduce a spline collocation method for solving SBVP (1a)-(1b). To overcome the singularity at $x = 0$ in the coefficient of the convection term, we apply L'Hôpital's rule as x approaches zero to the term $\frac{y'(x)}{x}$ in (1a). Since $y'(0) = 0$, so $\frac{y'(x)}{x}$ is in indeterminate form at $x = 0$ and thus the use of L'Hôpital's rule gives $\lim_{x\to 0} \frac{y'(x)}{x} = y''(0)$. Thus, we obtain the BVP

$$Ly(x) \equiv y''(x) + u(x)y'(x) + v(x)y(x) = r(x), \quad 0 < x < 1, \tag{2a}$$

with the boundary conditions

$$y'(0) = 0, \quad py(1) + qy'(1) = \beta, \tag{2b}$$

where

$$u(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{k}{x}, & \text{if } x \neq 0, \end{cases}$$

$$v(x) = \begin{cases} \frac{f(0)}{k+1}, & \text{if } x = 0, \\ f(x), & \text{if } x \neq 0, \end{cases}$$

$$r(x) = \begin{cases} \frac{g(0)}{k+1}, & \text{if } x = 0, \\ g(x), & \text{if } x \neq 0. \end{cases}$$

Let $\pi :\equiv 0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 1$ be the partition of $[0, 1]$, with equidistant spacing $h = 1/n$.

We use quintic B-spline functions to approximate the solution $y(x)$ of (1a)-(1b). The quintic B-splines $B_i,\ i = -2, -1, \ldots, n+2$ at the nodes x_i are defined to form a basis over the interval $[0, 1]$ (see ref. [34]). A quintic B-spline $B_i,\ i = -2, -1, \ldots, n+2$, covers six elements and defines over the interval $[0, 1]$ as follows:

$B_i(x) =$

$$\frac{1}{120h^5}\begin{cases}(x-x_{i-3})^5, & x_{i-3}\le x\le x_{i-2},\\ (x-x_{i-3})^5-6(x-x_{i-2})^5, & x_{i-2}\le x\le x_{i-1},\\ (x-x_{i-3})^5-6(x-x_{i-2})^5+15(x-x_{i-1})^5, & x_{i-1}\le x\le x_i,\\ (x_{i+3}-x)^5-6(x_{i+2}-x)^5+15(x_{i+1}-x)^5, & x_i\le x\le x_{i+1},\\ (x_{i+3}-x)^5-6(x_{i+2}-x)^5, & x_{i+1}\le x\le x_{i+2},\\ (x_{i+3}-x)^5, & x_{i+2}\le x\le x_{i+3},\\ 0, & \text{elsewhere}.\end{cases} \tag{3}$$

The values of $B_i(x), B_i'(x), B_i''(x)$ and $B_i'''(x)$ at the nodal points x_j obtained from the definition are given in Table 1.

Table 1: The values of B-Splines and their derivatives at nodal points

	Nodal values					
	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	elsewhere
$B_i(x)$	$\frac{1}{120}$	$\frac{26}{120}$	$\frac{66}{120}$	$\frac{26}{120}$	$\frac{1}{120}$	0
$B_i'(x)$	$\frac{1}{24h}$	$\frac{10}{24h}$	0	$-\frac{10}{24h}$	$-\frac{1}{24h}$	0
$B_i''(x)$	$\frac{1}{6h^2}$	$\frac{2}{6h^2}$	$-\frac{6}{6h^2}$	$\frac{2}{6h^2}$	$\frac{1}{6h^2}$	0
$B_i'''(x)$	$\frac{1}{2h^3}$	$-\frac{2}{2h^3}$	0	$\frac{2}{2h^3}$	$-\frac{1}{2h^3}$	0

The quintic spline space is defined as follows:

$$S_5([0,1]) = \{s(x)\in \mathbb{C}^4([0,1]):\ s(x)|_{[x_i,x_{i+1}]}\in P_5,\ i=0,1,2,\dots,n-1\},$$

where $s(x)|_{[x_i,x_{i+1}]}$ is the restriction of $s(x)$ on subinterval $[x_i,x_{i+1}]$ and P_5 is set of all quintic polynomials. Note that each $s(x)\in S_5([0,1])$ can be written as $s(x)=\sum_{i=-2}^{n+2} c_iB_i(x),\ c_i\in\mathbb{R}$, where the functions $B_i(x),\ i=-2,-1,\dots,n+1,n+2$ are linearly independent B-spline functions on $[0,1]$ and so they are the basis splines of $S_5([0,1])$, the dimension of $S_5([0,1])$ is thus $n+5$ (see ref. [43]). Clearly each $B_i(x)$ is non-negative and is locally supported on $[x_{i-3},x_{i+3}]$. It is also easy to observe that $B_i(x_j)=B_{i+1}(x_{j+1})$ for all $i,j=-2,-1,\dots,n+1,n+2$ and $\sum_{i=-2}^{n+2}B_i(x)=1,\ x\in[0,1]$.

We include two artificial points on each side of the partition π and then partition π becomes $\pi :\equiv x_{-2}<x_{-1}<x_0<x_1<x_2<\dots<x_{n-1}<x_n<x_{n+1}<x_{n+2}$. We use the quintic B-spline basis functions $B_i(x)$ for $i=0,1,2,\dots,n$ (see ref. [34]). It is easy to see that each $B_i(x)$ is also a piecewise quintic with knots at π and $B_i(x)\in S_5([0,1])$. We seek a function $\phi(x)\in S_5([0,1])$ that approximates the solution of BVP (2a)-(2b),

represented as

$$\phi(x) = \sum_{i=-2}^{n+2} c_i B_i(x), \tag{4}$$

where c_i are unknown real coefficients to be determined from the boundary conditions and collocation form of the differential equation. Here we have introduced four extra cubic B-splines B_{-2}, B_{-1}, B_{n+1} and B_{n+2} to satisfy the boundary conditions. Each quintic B-spline covers six elements so that an element is covered by six quintic B-splines.

At nodal points, the values of ϕ and its derivatives ϕ', ϕ'' and ϕ''' can be determined in terms of the element parameters c_j as

$$\phi(x_j) = \frac{1}{120}(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}), \tag{5}$$

$$\phi'(x_j) = \frac{1}{24h}(-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}), \tag{6}$$

$$\phi''(x_j) = \frac{1}{6h^2}(c_{j-2} + 2c_{j-1} - 6c_j + 2c_{j+1} + c_{j+2}), \tag{7}$$

$$\phi'''(x_j) = \frac{1}{2h^3}(-c_{j-2} + 2c_{j-1} - 2c_{j+1} + c_{j+2}). \tag{8}$$

The order of these approximations are $y'(x_j) = \phi'(x_j) + O(h^6)$, $y''(x_j) = \phi''(x_j) + O(h^4)$ and $y'''(x_j) = \phi'''(x_j) + O(h^4)$. Thus, we obtain the following set of collocation equations

$$L\phi(x_j) \equiv \phi''(x_j) + u(x_j)\phi'(x_j) + v(x_j)\phi(x_j) = r(x_j), \quad j = 0, 1, \ldots, n, \tag{9}$$

with boundary conditions

$$\phi'(x_0) = 0, \quad p\phi(x_n) + q\phi'(x_n) = \beta. \tag{10}$$

Using equations (5), (6) and (7) in equation (9), we obtain

$$\frac{1}{6h^2}(c_{j-2} + 2c_{j-1} - 6c_j + 2c_{j+1} + c_{j+2}) + \frac{u_j}{24h}(-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}) + \frac{v_j}{120}(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}) = r_j, \; j = 0, 1, 2, \ldots, n,$$

where f_j stands for $f(x_j)$ etc.. On simplifying above equation, we get

$$\left(\frac{1}{6h^2} - \frac{u_j}{24h} + \frac{v_j}{120}\right) c_{j-2} + \left(\frac{2}{6h^2} - \frac{10u_j}{24h} + \frac{26v_j}{120}\right) c_{j-1} + \left(-\frac{6}{6h^2} + \frac{66v_j}{120}\right) c_j + \left(\frac{2}{6h^2} + \frac{10u_j}{24h} + \frac{26v_j}{120}\right) c_{j+1} + \left(\frac{1}{6h^2} + \frac{u_j}{24h} + \frac{v_j}{120}\right) c_{j+2} = r_j, \; j = 0, 1, 2, \ldots, n. \tag{11}$$

Now the use of equation (6) in first boundary condition gives

$$-c_{-2} - 10c_{-1} + 10c_1 + c_2 = 0. \tag{12}$$

Similarly, the use of equations (5) and (6) in second boundary condition turns into

$$\frac{p}{120}(c_{n-2}+26c_{n-1}+66c_n+26c_{n+1}+c_{n+2})+\frac{q}{24h}(-c_{n-2}-10c_{n-1}+10c_{n+1}+c_{n+2}) = \beta,$$

or

$$\left(\frac{p}{120} - \frac{q}{24h}\right) c_{n-2} + \left(\frac{26p}{120} - \frac{10q}{24h}\right) c_{n-1} + \left(\frac{66p}{120}\right) c_n + \left(\frac{26p}{120} + \frac{10q}{24h}\right) c_{n+1} + \left(\frac{p}{120} + \frac{q}{24h}\right) c_{n+2} = \beta. \tag{13}$$

We still need two equations in c_j's. On differentiating both sides, equation (1a) gives

$$y'''(x) + k\frac{xy''(x) - y'(x)}{x^2} + f(x)y'(x) + f'(x)y(x) = g'(x). \tag{14}$$

Like earlier, we again modify this equation at $x = 0$. As at $x = 0$ the factor $\frac{xy''(x)-y'(x)}{x^2}$ is in indeterminant form so by using L'Hôpital's rule, we have

$$\lim_{x\to 0} \frac{xy''(x) - y'(x)}{x^2} = \frac{y'''(0)}{2}$$

and so at $x = 0$, equation (14) can be approximated as

$$\left(1 + \frac{k}{2}\right) y'''(0) + f(0)y'(0) + f'(0)y(0) = g'(0).$$

Making use of Equations (5), (8) and the first boundary condition, it becomes

$$\frac{\left(1 + \frac{k}{2}\right)}{2h^3}(-c_{-2} + 2c_{-1} - 2c_1 + c_2) + \frac{f'(0)}{120}(c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2) = g'(0),$$

or

$$\left(-\frac{1}{2h^3}\left(1 + \frac{k}{2}\right) + \frac{f'(0)}{120}\right) c_{-2} + \left(\frac{2}{2h^3}\left(1 + \frac{k}{2}\right) + \frac{26f'(0)}{120}\right) c_{-1} + \frac{66f'(0)}{120}c_0 + \left(-\frac{2}{2h^3}\left(1 + \frac{k}{2}\right) + \frac{26f'(0)}{120}\right) c_1 + \left(\frac{1}{2h^3}\left(1 + \frac{k}{2}\right) + \frac{f'(0)}{120}\right) c_2 = g'(0). \tag{15}$$

Now to get one more equation, we differentiate (2a) to obtain

$$y''' + uy'' + u'y' + vy' + v'y = r',$$

or

$$y''' + u(g - \frac{k}{x}y' - fy) + u'y' + vy' + v'y = r',$$

or

$$y''' + \left(-\frac{ku}{x} + u' + v\right) y' + (v' - uf)y = r' - ug.$$

The use of equation (5), (6) and (8) at $x = 1$ gives the following collocation equation

$$\frac{1}{2h^3}(-c_{n-2} + 2c_{n-1} - 2c_{n+1} + c_{n+2}) + \frac{\alpha}{24h}(-c_{n-2} - 10c_{n-1} + 10c_{n+1} + c_{n+2})$$
$$+ \frac{\gamma}{120}(c_{n-2} + 26c_{n-1} + 66c_n + 26c_{n+1} + c_{n+2}) = \theta,$$

where $\alpha = -ku(1) + u'(1) + v(1),\ \gamma = v'(1) - u(1)f(1)$ and $\theta = r'(1) - u(1)g(1)$. The above equation can be written as

$$\left(-\frac{1}{2h^3} - \frac{\alpha}{24h} + \frac{\gamma}{120}\right)c_{n-2} + \left(\frac{2}{2h^3} - \frac{10\alpha}{24h} + \frac{26\gamma}{120}\right)c_{n-1} + \frac{66\gamma}{120}c_n$$
$$+ \left(\frac{-2}{2h^3} + \frac{10\alpha}{24h} + \frac{26\gamma}{120}\right)c_{n+1} + \left(\frac{1}{2h^3} + \frac{\alpha}{24h} + \frac{\gamma}{120}\right)c_{n+2} = \theta. \qquad (16)$$

Now eliminating c_{-2} and c_{-1} from equations (12), (15) and the first equation of (11) and eliminating c_{n+1} and c_{n+2} from equations (13), (16) and the last equation of (11), we obtain a linear system $AC = B$ with $n + 1$ linear equations in $n + 1$ unknowns $c_0, c_1, \ldots, c_{n-1}, c_n$ where matrix A is a pentadiagonal matrix. It is easy to check that for small enough step size h the matrix A is strictly diagonally dominant and hence invertible. Thus this $(n + 1) \times (n + 1)$ linear system can be solved by any Gauss eliminations or any iterative method. Hence, we obtain the quintic spline approximate solution given by (4).

3. NON-LINEAR PROBLEMS

The solution of non-linear SBVPs of the form

$$y'' + \frac{k}{x}y' = g(x, y), \quad 0 < x < 1, \qquad (17)$$

with the boundary conditions

$$y'(0) = 0, \quad py(1) + qy'(1) = \beta, \qquad (18)$$

arising in physiology is also considered. We assume that $g(x, y)$ is continuous, $\frac{\partial g}{\partial y}$ exists, continuous and non-negative for all $0 \le x \le 1$. For the case $k = 2,\ p = \beta$ and $q = 1$ the existence and uniqueness of the solution of (17)-(18) has been given in [23]. The SBVP (17)-(18) with $k = 0, 1, 2$ arise in the study of various problems [2, 5, 6].With linear $g(x, y)$ and non-linear $g(x, y)$ of the form $g(x, y) = \frac{\sigma y}{\mu + y},\ \sigma > 0, \mu > 0$ and $k = 2$ the problem arises in the study of steady state oxygen diffusion in a spherical cell with michaelis-menten uptake kinetics [31, 32]. A similar equation for $k = 2$ arise in the study of the distribution of heat sources in the human head [20, 21]. For $g(x, y) = -\delta e^{-\epsilon y},\ \delta > 0, \epsilon > 0$ the problem has been discussed in [17] and point-wise bounds and uniqueness results are given. To solve the non-linear problems, we use quasi-linearization technique. In the quasi-linearization technique, the non-linear differential equation is solved recursively by a sequence of linear differential equations. Using Taylor's series expansion the non-linear function $g(x, y)$ at $(m+1)$-th iteration can be expressed as $g(x, y^{m+1}) \approx g(x, y^m) + (y^{m+1} - y^m)\left(\frac{\partial g}{\partial y}\right)_{y=y^m}$, $m = 0, 1, 2\ldots$, where $y^m(x)$ is m-th iteration solution of the equation and $y^0(x)$ is a reasonable initial approximation for the function $y(x)$. The main advantage of this method is that if the procedure converges, it converges quadratically to the solution of the original problem that means that the error in the $(m+1)$-th iteration is proportional to the square of the error in the m-th iteration. Now at $(m+1)$-th iteration Eq. (17) can be written as

$$(y'')^{m+1} + \frac{k}{x}(y')^{m+1} = g(x, y^m) + (y^{m+1} - y^m)\left(\frac{\partial g}{\partial y}\right)_{y=y^m},$$

or

$$(y'')^{m+1} + \frac{k}{x}(y')^{m+1} + P^m(x)y^{m+1} = Q^m(x),$$

where $P^m(x) = -\left(\frac{\partial g}{\partial y}\right)_{y=y^m}$ and $Q^m(x) = g(x, y^m) - y^m\left(\frac{\partial g}{\partial y}\right)_{y=y^m}$. The last equation is linear in y^{m+1} and can be solved by the proposed method for linear problems. The boundary conditions become $(y')^{m+1}(0) = 0$ and $py^{m+1}(1) + q(y')^{m+1}(1) = \beta$.

The proposed method can also be applicable to the non-linear SBVPs of the form

$$y'' + \left(a + \frac{k}{x}\right)y' = h(x, y), \quad a > 0,\ 0 < x < 1, \tag{19}$$

with the boundary conditions

$$y'(0) = 0, \quad \xi_1 y(1) + \xi_2 y'(1) = \rho, \tag{20}$$

which arising in physiology. As above, using Taylor's expansion at $(m+1)$-th iteration equation (19) can be written as

$$(y'')^{m+1} + \left(a + \frac{k}{x}\right)(y')^{m+1} + R^m(x)y^{m+1} = S^m(x), \tag{21}$$

where $R^m(x) = -\left(\frac{\partial h}{\partial y}\right)_{y=y^m}$ and $S^m(x) = h(x, y^m) - y^m\left(\frac{\partial h}{\partial y}\right)_{y=y^m}$. The boundary conditions become $(y')^{m+1}(0) = 0$ and $\xi_1 y^{m+1}(1) + \xi_2(y')^{m+1}(1) = \rho$. As in the case of linear SBVPs, to overcome the singularity at $x = 0$, an application of L'Hôpital's rule on the first derivative term in (21) gives

$$(y'')^{m+1}(x) + s(x)(y')^{m+1}(x) + t(x)y^{m+1}(x) = w(x), \quad 0 < x < 1, \tag{22a}$$

with the boundary conditions

$$(y')^{m+1}(0) = 0, \quad \xi_1 y^{m+1}(1) + \xi_2(y')^{m+1}(1) = \rho, \tag{22b}$$

where

$$s(x) = \begin{cases} a, & \text{if } x = 0, \\ a + \frac{k}{x}, & \text{if } x \neq 0, \end{cases}$$

$$t(x) = \begin{cases} \frac{R^m(0)}{k+1}, & \text{if } x = 0, \\ R^m(x), & \text{if } x \neq 0, \end{cases}$$

$$w(x) = \begin{cases} \frac{S^m(0)}{k+1}, & \text{if } x = 0, \\ S^m(x), & \text{if } x \neq 0. \end{cases}$$

Equation (22a) is linear in y^{m+1} and can be solved by the proposed method for linear problems.

4. CONVERGENCE ANALYSIS

In this section, we shall show that the proposed collocation method described in the previous section is fourth order convergent. To prove the convergence of our method, we assume the following hypotheses

H_1 Problem (2) with homogeneous boundary conditions has a unique solution.

H_2 The boundary value problem $y'' = 0$ subject to the homogeneous boundary conditions is uniquely solvable.

The second hypothesis implies that there exists a Green's function $G(x, \xi)$ for this problem. Let $y(x)$ and $\phi(x)$ are the exact and the quintic spline solutions respectively of (1a)-(1b). If we denote $y''(x) = \eta(x)$ and $\phi''(x) = \psi(x)$, then for $m = 0, 1$, we have

$$y^{(m)}(x) = \int_0^1 \frac{\partial^m G(x,\xi)}{\partial x^m} \eta(\xi)\, d\xi, \quad \phi^{(m)}(x) = \int_0^1 \frac{\partial^m G(x,\xi)}{\partial x^m} \psi(\xi)\, d\xi.$$

Now, we define the operator $K : \mathbb{C}[0,1] \to \mathbb{C}[0,1]$ such that

$$K[g(x)] = u(x) \int_0^1 \frac{\partial G(x,\xi)}{\partial x} g(\xi)\, d\xi + v(x) \int_0^1 G(x,\xi) g(\xi)\, d\xi - r(x).$$

The following Lemma [34] will be used to show the convergence of the proposed method.

Lemma 4.1. *Let $P_n g$ be the piecewise Lagrange's polynomial of degree d on each of the intervals $[x_{j\kappa}, x_{(j+1)\kappa}]$, $0 \leq j \leq l-1$, where $ld = n$, interpolating g at knots of the partition $0 = x_0 < x_1 < \ldots < x_{ld} = 1$. Then the Tchebycheff error estimates for Lagrange's interpolates is given by*

$$\|g - P_n g\| \leq \frac{\|g^{(d+1)}\|}{2(d+1)} h^{d+1}.$$

Theorem 4.1. *Let $\phi(x)$ be the collocation approximation from $S_5([0,1])$ to the solution $y(x)$ of the boundary value problem (2a)-(2b). If $y \in \mathbb{C}^6[0,1]$, then the error estimate is given by*

$$\|y-\phi\| = \max_{i=0,1,\ldots,n} |y(x_i)-\phi(x_i)\,| \leq Ch^4,$$

where C is a positive constant independent of h.

Proof. To estimate the error $|y(x)-\phi(x)\,|$, let $\phi(x)$ be the unique spline interpolant from $S_5([0,1])$ to the solution $y(x)$ of boundary value problem (2a)-(2b) given by (4) between the knots $0 = x_0 < x_1 < \ldots < x_{3\kappa} = 1$. Then we have

$$L\phi(x_j) = r(x_j), \quad j = 0,1,\ldots,n,$$

iff $T_n(x) = \phi''(x)$ is the solution of integral equation

$$T_n + P_nKT_n = P_nr, \tag{23}$$

where P_n is an operator that maps $S_3([0,1])$ onto the cubic splines with the knots x_i and thus P_nw is the unique piecewise Lagrange polynomial of degree three on each of the intervals $[x_{3j}, x_{3(j+1)}], j = 0,1,\ldots,\kappa-1$ interpolating w at the knots of the partition π.

Let $T = y''$, where y is a true solution of (2a)-(2b), then T solves the integral equation

$$T + KT = r.$$

On applying P_n and then adding T, we obtain

$$T + P_nKT = P_nr + (T - P_nT). \tag{24}$$

On subtracting (23) from (24), we get

$$T - T_n = (I + P_nK)^{-1}(T - P_nT). \tag{25}$$

Now let $G(x,\xi)$ be the Green's function associated with (2a)-(2b). Then $y = GT$ and $\phi = GT_n$. Apply G on both sides of (25), to obtain $G(T-T_n) = G((I+P_nK)^{-1}(T - P_nT))$ or $y - \phi = G((I+P_nK)^{-1}(y'' - P_ny''))$. The L_∞ norm on both sides gives

$$\|y-\phi\| \leq \|G\|\|(I+P_nK)^{-1}\|\|y''-P_ny''\|. \tag{26}$$

From the hypotheses H_1 and H_2, we have that $(I+K)^{-1}$ exists and is a bounded linear operator. So using that $P_NK \to K$, we conclude that $(I+P_NK)^{-1}$ exists and it is bounded, therefore there exists $M > 0$ such that $\|(I+P_nK)^{-1}\| \leq M$, $\forall\, n \geq n_0$ for some n_0. Also if $y \in \mathbb{C}^6[0,1]$, then use of Lemma 4.1 in (26) gives

$$\|y-\phi\| \leq Ch^4,$$

where $C = M\|G\|\frac{\|y^4\|}{8}$ is a constant independent of h. $\square$

5. NUMERICAL EXPERIMENTS

In this section, the proposed method is implemented on linear as well as non-linear SBVPs. To demonstrate the effectiveness of this novel method several test examples have been carried out. Comparisons with exact solutions and existing numerical methods have also been done. The discrete L_∞-norm and L_2-norm are defined as follows

$$\|\tilde{y}-y\| = \max_{i=0,1,\ldots,n} |\tilde{y}(x_i)-y(x_i)|, \qquad \|\tilde{y}-y\|_2 = \sqrt{\sum_{i=0}^{n}(\tilde{y}(x_i)-y(x_i))^2},$$

where $\tilde{y}(x_i)$ and $y(x_i)$ denote the numerical and analytic solutions respectively at the knots x_i.

Example 1. We take $k=1, f(x)=1, g(x)=0, p=1, q=-1, \beta = 1+\frac{J_1(1)}{J_0(1)}$ in model problem (1a)-(1b). The associated equation is well known Bessel's equation of order zero. The analytic solution of this SBVP is given by $y(x)=\frac{J_0(x)}{J_0(1)}$.

To examine the error of the approximation, let $E^n = \max_{i=0,1,\ldots,n} |\tilde{y}(x_i)-y(x_i)|$. It is known that, if p is the order of convergence of the method, then for large n, $E^n \le Cn^{-p}$ where C is the error constant. Following the generalized algorithm for the order verification of numerical methods given in [42], we can find C and p from the line $y = px + \log C$ that best fits the equation $\log E^n = \log C - p\log n$. Applying this algorithm to Example 1, the equation of the best fit line whose graph is shown in Figure 1 is $y = 4.0007x - 3.8018$, where $x = -\log n$ and $y = \log E^n$. Thus, we conclude that the order of convergence $p \approx 4$ with error constant $C = 10^{-3.8018} \approx 1.5782\times10^{-4}$.

Table 2: Comparison of error norms for Example 1 for different values of n

	Quartic B-spline [22]		Proposed method	
n	L_∞-norm	L_2-norm	L_∞-norm	L_2-norm
10	1.67E-06	1.89E-06	1.49E-08	3.04E-08
20	2.04E-07	2.34E-07	9.85E-10	2.81E-09
50	1.42E-08	1.62E-08	2.58E-11	1.16E-10
100	1.44E-09	1.64E-09	1.39E-12	8.80E-12

We have used $n = 32$ to plot the graph between analytic and approximate solutions for Example 1 and the graph is presented in Figure 2. For Example 1, the maximum absolute errors in maximum norm (L_∞-norm) and Euclidean norm (L_2-norm) for different values of n are tabulated in Table 2 and compared with the results obtained

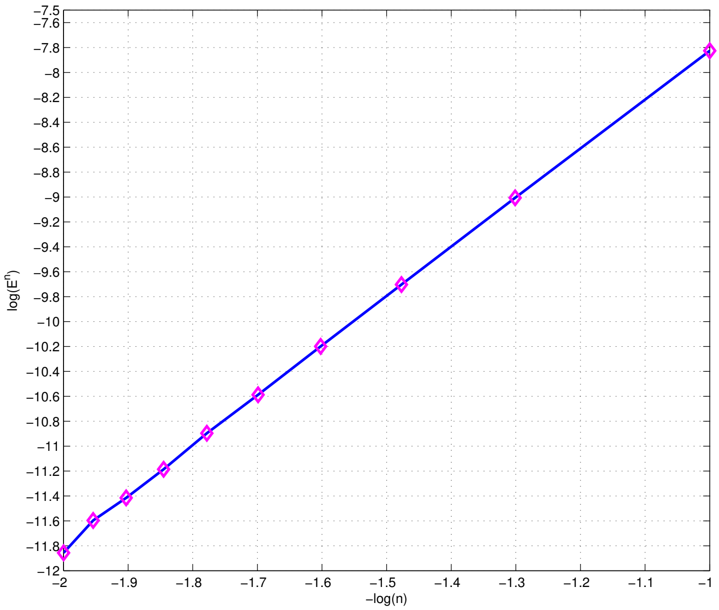

Figure 1: The best fit line for the order of convergence for Example 1

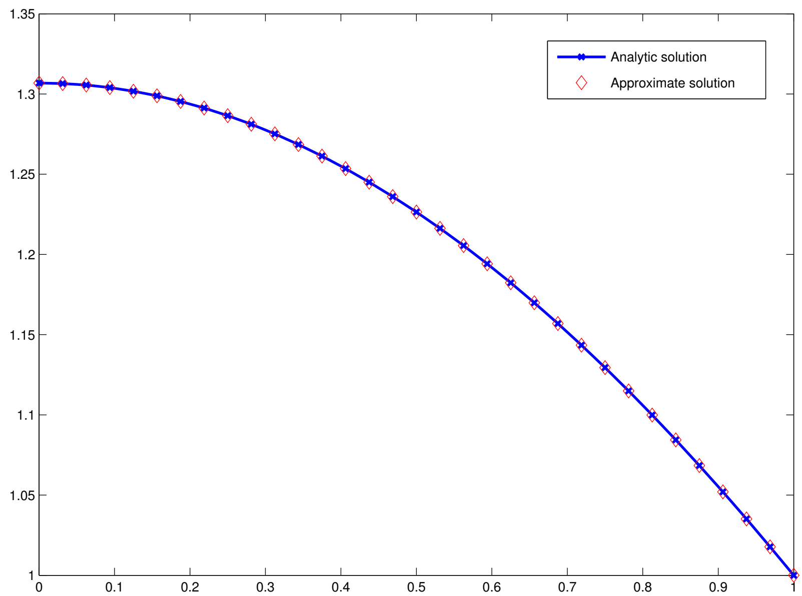

Figure 2: Analytic and approximate solutions for Example 1

Table 3: Comparison of numerical results for Example 1

n	Method in [9]	Proposed method
10	1.48E-02	1.49E-08
20	4.50E-03	9.85E-10
40	1.30E-03	6.32E-11
60	6.38E-04	1.27E-11
120	1.81E-04	2.30E-12
520	1.21E-05	1.84E-11

by Goh *et al.* [22]. The maximum absolute errors for different values of n are also compared in Table 3 with those given by Caglar *et al.* [9].

Example 2. We take $k = 2, f(x) = -4, g(x) = -2, p = 1/11, q = 1/10, \beta = \coth 2$ in model problem (1a)-(1b).

This problem has earlier been discussed by Russell and Shampine [41], which has the analytic solution

$$y(x) = \frac{1}{2} + \frac{5 \sinh 2x}{x \sinh 2}.$$

The numerical results of the proposed method for $n = 20$ at different nodal points are compared with those given in [38] and [45] and are presented in Table 4. As is evident from the table, our method is of high precision, and the results obtained by using our method are much better than those of [38] and [45]. For $n = 20$, a comparison of absolute errors of first and second order derivatives with those given in [47] at different nodal points are given in Table 5.

In Table 6, a comparison between the maximum absolute errors obtained by our proposed method and obtained by a finite difference approach and Patch bases suggested by Russell and Shampine [41] and obtained by optimal grid suggested by Kadalbajoo and Aggarwal [27] is given. In Tables 6 and 8 the entry * shows that the authors have not given the associated maximum absolute error. The graph between analytic and approximate solutions for Example 2 is presented in Figure 3 by taking $n = 32$.

Example 3. We take $k = 1, f(x) = 1, g(x) = 4 - 9x + x^2 - x^3, p = 1, q = -1, \beta = 1$ in model problem (1a)-(1b). The analytic solution of this SBVP is given by $y(x) = x^2 - x^3$.

The relative error at nodal point x_i is defined as $E_r = \frac{|\tilde{y}(x_i) - y(x_i)|}{|y(x_i)|}$. For Example 3, a comparison between the relative errors obtained by using the Fourier sine series

Table 4: Comparison of numerical results for Example 2 at different nodal points

x	Cubic spline [38]	Series solution [45]	Proposed method
0.05	2.92E-04	4.54E-05	2.54E-07
0.10	2.92E-04	5.92E-05	2.78E-07
0.20	2.88E-04	6.27E-05	4.08E-08
0.30	2.82E-04	6.50E-05	1.67E-08
0.40	2.71E-04	6.61E-05	3.88E-08
0.50	2.55E-04	6.35E-05	5.05E-08
0.60	2.33E-04	5.27E-05	5.73E-08
0.70	2.00E-04	2.64E-05	6.10E-08
0.80	1.55E-04	2.62E-05	6.23E-08
0.90	1.02E-04	1.21E-05	6.13E-08

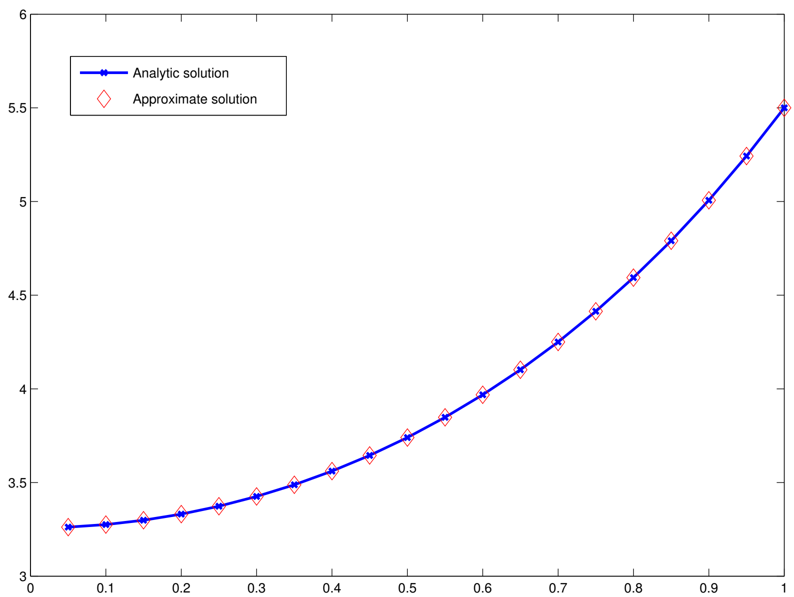

Figure 3: Analytic and approximate solutions for Example 2

[46] and obtained by our proposed method is given in Table 7. The graph between analytic and approximate solutions for Example 3 is presented in Figure 4 by taking $n = 32$.

Table 5: Comparison of absolute errors of first-order/second-order derivatives for Example 2 at different nodal points

	Absolute errors in first derivative		Absolute errors in second derivative	
x	Method in [47]	Proposed method	Method in [47]	Proposed method
0.10	4.21E-06	5.85E-07	8.58E-05	1.17E-05
0.30	3.89E-07	1.79E-08	3.01E-06	1.15E-07
0.50	5.96E-08	5.46E-09	4.48E-07	9.22E-09
0.70	5.58E-08	1.56E-09	4.63E-08	1.08E-08
0.90	1.37E-07	1.42E-09	2.66E-07	1.85E-08

Table 6: Comparison of numerical results for Example 2

n	FDM [41]	Patch bases [41]	Optimal grid [27]	Proposed method
4	2.12E-01	2.02E-01	2.12E-02	4.32E-05
9	5.05E-02	4.09E-02	4.19E-03	1.53E-06
16	1.80E-02	1.30E-02	1.33E-03	4.86E-07
25	8.02E-03	5.33E-03	5.44E-04	1.55E-07
36	4.13E-03	2.57E-03	*	5.68E-08
64	1.43E-03	8.14E-04	8.25E-05	1.10E-08

Example 4. Now we take $k = 1, g(x, y) = -\exp(y), p = 1, q = -1, \beta = \frac{4c}{c+1}$, where $c = 3 + 2\sqrt{2}$ in non-linear model problem (17)-(18). The analytic solution of this SBVP is given by $y(x) = 2\ln\frac{c+1}{cx^2+1}$.

This non-linear problem has earlier been discussed by many researchers like Russell and Shampine [41], kadalbajoo and Agarwal [27], Çağlar *et al.* [8]. The numerical results are obtained by reducing it into a linear problem by using quasi-linearization technique. To linearize the problem the initial approximation satisfying both the boundary conditions is taken as $y^0(x) = -\frac{4cx^2}{c+1}$ and five iterations are used. The numerical results of the proposed method for different values of n compared with those given in [8] and [30] are presented in Table 8. As is evident from the table, our method is of high precision, and the results obtained by using our method are better than that of [8] and [30]. The graphs for first two iterations and the analytic solution are drawn in Figure 5. The graphs show that how good the proposed method approximates the solution just in two iterations.

Table 7: Comparison of relative errors at different values of x for the Example 3

	Taking $n = 40$		Taking $n = 160$	
x	Method in [46]	Proposed method	Method in [46]	Proposed method
0.2	0.83E-02	0.77E-04	0.52E-03	0.12E-05
0.4	0.26E-02	0.27E-05	0.16E-03	0.42E-07
0.6	0.16E-02	0.72E-05	0.10E-03	0.11E-06
0.8	0.16E-02	0.15E-04	0.11E-03	0.24E-06

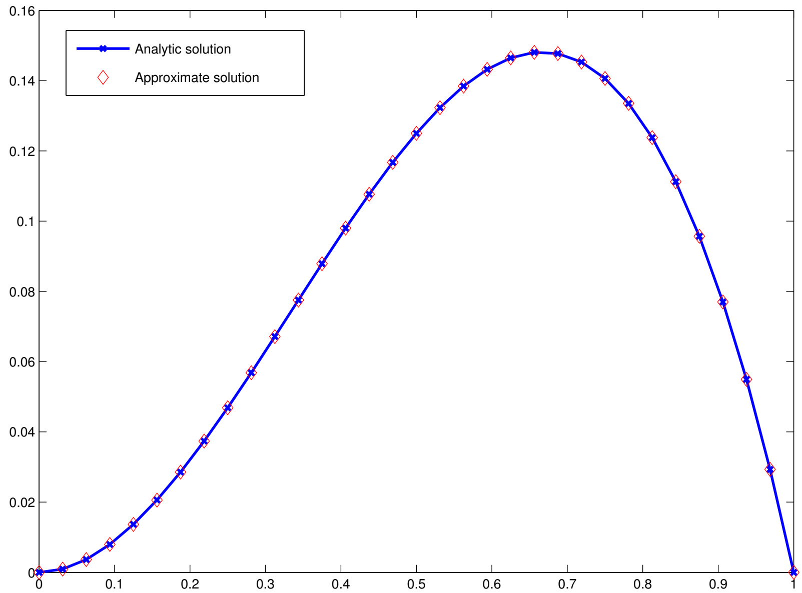

Figure 4: Analytic and approximate solutions for Example 3

Example 5. Now, we consider the non-linear model problem of oxygen diffusion with $\sigma = 0.76129, \mu = 0.03119, p = \beta = 5$ and $q = 1$.

Since the analytic solution for this problem is not available so the maximum absolute errors are obtained by using double mesh principle [16] and the error is defined by

$$E^n = \max_{i=0,1,\ldots,n} |y_i^n - y_{2i}^{2n}|,$$

where y_i^n and y_i^{2n} are the solutions obtained my taking n and $2n$ mesh points respectively. The maximum absolute errors in numerical solution and its derivatives are

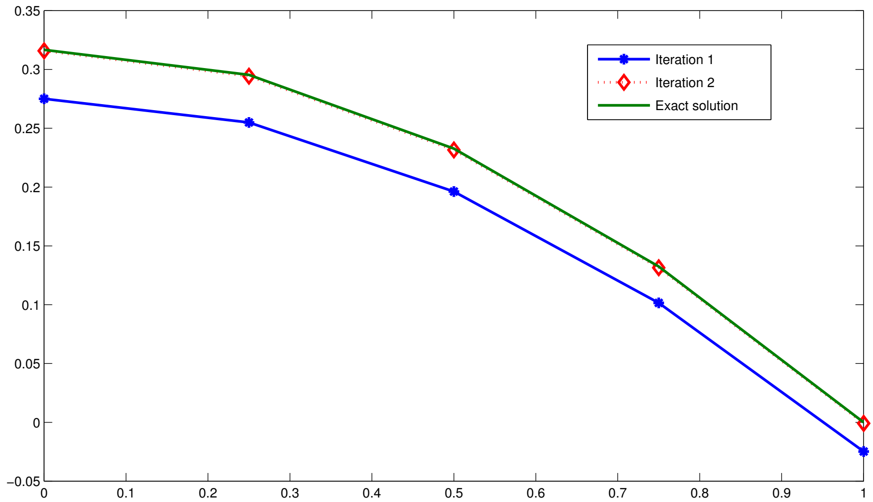

Figure 5: Analytic and approximate solutions (first two iterations) for Example 4

presented in Table 9. The numerical solution obtained by taking $n = 20$ is drawn in Figure 6.

Example 6. Finally, we consider the following non-linear SBVP:

$$y''(x) + \left(1 + \frac{k}{x}\right) y'(x) = \frac{5x^3(5x^5 \exp(y) - x - k - 4)}{x^5 + 4}, \quad 0 < x < 1,$$

with the boundary conditions

$$y'(0) = 0, \quad y(1) + 5y'(1) = \ln\left(\frac{1}{5}\right) - 5.$$

The analytic solution of this non-linear SBVP is given by $y(x) = \ln\left(\frac{1}{x^5+4}\right)$.

The maximum absolute errors of Example 6 are tabulated in Tables 10 for different values of n and k. Our results are compared with those obtained by non-polynomial cubic spline functions given in [33] and obtained by finite difference method given in [35]. As is evident from the table 10, our method is of high precision, and the results obtained by using our method are better than that of [33] and [35].

The graphs for first two iterations and the analytic solution are drawn in Figure 7. The graphs show that how good the proposed method approximates the solution just in two iterations.

Table 8: Comparison of numerical results for Example 4

n	Method in [8]	Proposed method	n	Method in [30]	Proposed method
20	3.16E-05	9.12E-06	8	2.0E-04	1.3E-04
40	7.87E-06	1.20E-06	16	5.0E-05	1.7E-05
60	3.50E-06	3.68E-07	32	1.2E-05	2.3E-06
90	1.55E-06	1.12E-07	64	3.1E-06	3.0E-07
111	1.04E-06	6.07E-08	128	7.8E-07	4.0E-08
161	4.91E-07	2.04E-08	256	*	5.2E-09

Table 9: Maximum absolute errors for Example 5

n	E^n	Errors in first derivative	Errors in second derivative
2	5.74E-04	1.69E-03	6.77E-03
4	6.05E-05	4.35E-04	3.48E-03
8	7.17E-06	1.09E-04	1.74E-03
16	8.77E-07	2.72E-05	8.69E-04
32	1.08E-07	6.79E-06	4.35E-04
64	1.35E-08	1.70E-06	2.17E-04
128	1.68E-09	4.24E-07	1.09E-04
256	2.07E-10	1.06E-07	5.43E-05
512	1.83E-11	2.65E-08	2.71E-05

6. CONCLUSION

Quintic B-spline functions are used to solve a class of SBVPs arising in physiology and other areas. To handle the problem at the singularity L'Hôpital's rule has been used. After removing the singularity, a quintic B-spline collocation method is used to approximate the solution over the interval $[0, 1]$. The advantage of the proposed method is that the solution can be estimated within the boundary interval (recall that the finite difference approach approximates the solution at the nodal points only). Another advantage of the proposed method over finite difference method is that it provides not only the approximate solution but also the derivatives of the solution within the given interval. Convergence analysis of the proposed method is discussed and the method is shown to be fourth-order convergent. The method applied to several linear as well as non-linear test problems and the absolute errors in the solution obtained by our method are compared with the several existing methods.

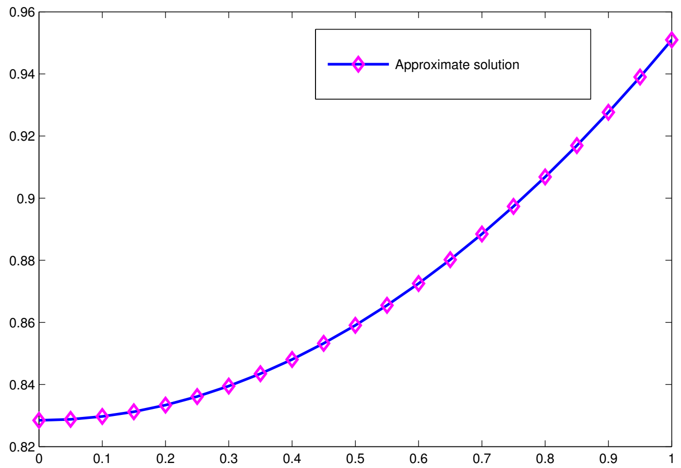

Figure 6: Approximate solution for Example 5

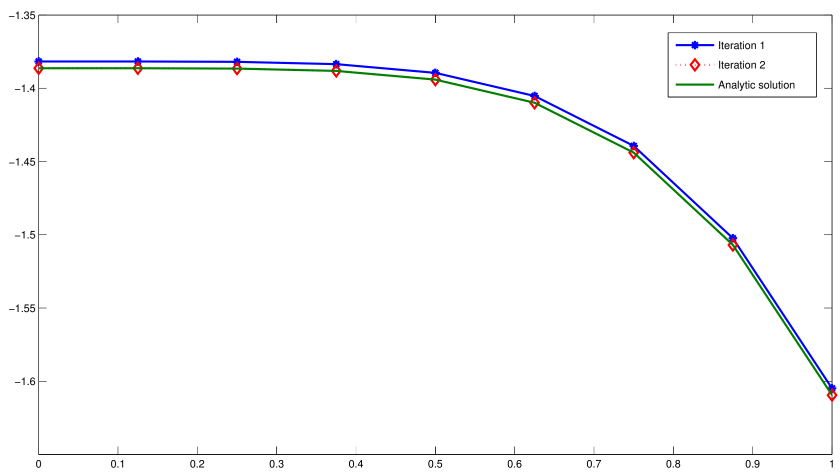

Figure 7: Analytic and approximate solutions (first two iterations) for Example 6

Table 10: Maximum absolute errors for Example 5

$k \downarrow$	$n \rightarrow$	16	32	64	128	256
	Method in [33]	1.17E-03	3.04E-04	7.67E-05	1.92E-05	4.81E-06
0.25	Method in [35]	2.47E-06	2.04E-06	3.09E-07	3.06E-07	3.03E-08
	Proposed Method	3.61E-06	2.27E-07	1.42E-08	8.93E-10	6.39E-11
	Method in [33]	1.37E-03	3.46E-04	8.69E-05	2.17E-05	5.44E-06
0.75	Method in [35]	1.27E-06	1.15E-06	1.56E-07	1.16E-07	1.65E-08
	Proposed Method	2.21E-06	1.41E-07	8.87E-09	5.57E-10	4.54E-11
	Method in [33]	1.46E-03	3.68E-04	9.20E-05	2.30E-05	5.75E-06
1.00	Method in [35]	4.66E-05	4.95E-06	4.92E-07	4.87E-07	4.84E-08
	Proposed Method	1.62E-06	1.05E-07	6.64E-09	4.17E-10	3.60E-11
	Method in [33]	1.82E-03	4.52E-04	1.13E-04	2.80E-05	7.00E-06
2.00	Method in [35]	7.46E-05	7.06E-06	6.85E-06	6.75E-07	6.71E-08
	Proposed Method	8.13E-07	4.25E-08	2.52E-09	1.54E-10	5.40E-12

Comparisons show that the errors obtained by our method are considerable accurate as compared to the existing numerical methods.

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