Neural, Parallel, and Scientific Computations, **26**, No. 1 (2018), 53-64 ISSN: 1056-2176

SYMMETRIC FUNCTIONS FOR FAMILIES OF GENERATING FUNCTIONS

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ABSTRACT: In this paper we show how the action of the symmetrizing operators $L^k_{e_1e_2}$ to the series $\sum_{j=0}^{\infty} S_j\,(-A)\, e_1^j z^j$ allows the obtention of an alternative approach for the determination of Fibonacci numbers and Chebychev polynomials of the first and second kind.

AMS Subject Classification: 05E05, 11B39

Key Words: symmetrizing operator, Chebychev polynomials, Fibonacci numbers

Received: November 12, 2017; **Accepted:** January 12, 2018;
Published: January 16, 2018. **doi:** 10.12732/npsc.v26i1.3
Dynamic Publishers, Inc., Acad. Publishers, Ltd. https://acadsol.eu/npsc

1. INTRODUCTION

Let F_n, U_n and T_n be the nth Fibonacci number, Chebyshev polynomials of the first kind and the second kind, respectively. Foata [12] used combinatorial techniques to give generating functions of squared Fibonacci numbers $\sum_{j=0}^{\infty} F_j^2 z^j$, and products of Chebyshev polynomials of first and second kinds. Foata also obtained the following series

$$\sum_{j=0}^{\infty} U_j F_j z^j, \ \sum_{j=0}^{\infty} T_j F_j z^j, \ \sum_{j=0}^{\infty} T_j U_j z^j.$$

Lascaux [14] later found an identity of Ramanujan by means of the divided differences.

Boussayoud, Kerada and Abderrezzak [11] have recovered the generating functions

$$\sum_{j=0}^{\infty} F_j z^j, \ \sum_{j=0}^{\infty} T_j z^j, \ \sum_{j=0}^{\infty} U_j z^j,$$

by using the definition of a symmetric operator $L^k_{e_1 e_2}$. Since then, more generating functions had been rediscovered [8, 10],

$$\sum_{j=0}^{\infty} F_j z^j, \ \sum_{j=0}^{\infty} T_j z^j, \ \sum_{j=0}^{\infty} U_j z^j, \sum_{j=0}^{\infty} F_j^2 z^j \ \sum_{j=0}^{\infty} U_j F_j z^j, \ \sum_{j=0}^{\infty} T_j F_j z^j, \ \sum_{j=0}^{\infty} T_j U_j z^j.$$

In this paper, we shall combine all these results in a unified way. All the results can be treated as special cases of the following theorem.

Theorem 1. *Given an alphabet set* $A = \{a_1, a_2, ...\}$, *we have*

$$\sum_{j=0}^{\infty} S_j(A) L^k_{e_1 e_2}(e_1^j) z^j =$$

$$\frac{\sum_{j=0}^{k-1} S_j(-A) e_1^j e_2^j L^k_{e_1 e_2}\left(\frac{1}{e_1^j}\right) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} S_{j+k+1}(-A) L^k_{e_1 e_2}(e_1) z^j}{\left(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j\right)}. \quad (1)$$

Theorem 1 generalizes several results published in [4, 5, 6, 7, 8, 9, 10, 11, 12]. In addition, we also find the generating function of the Stirling numbers of the second kind based on [2].

2. PRELIMINARIES

In the paper we need a lot of known results, quoted here for convenience to the reader.

Dfinition 2 ([1]). *Given two sets of indeterminate* A *and* B *(called alphabets), we define* $S_j(A - B)$ *as follows:*

$$\frac{\prod_{b \epsilon B}(1 - zb)}{\prod_{a \epsilon A}(1 - za)} = \sum_{j=0}^{\infty} S_j(A - B) z^j, \quad (2)$$

with $S_j(A - B) = 0$ *for* $j < 0$.

All the alphabets considered in this article are finite.

Remarque 3. *By taking* $A = 0$ *in* (2.1), *we obtain*

$$\prod_{b \epsilon B}(1 - zb) = \sum_{j=0}^{\infty} S_j(-B)z^j.$$

Proposition 4 ([2])**.** *Considering successively the case of* $A = \Phi$ *or* $B = \Phi$, *we can derive the following factorization*

$$\sum_{j=0}^{\infty} S_j(A - B)z^j = \sum_{j=0}^{\infty} S_j(A)z^j \sum_{j=0}^{\infty} S_j(-B)z^j.$$

Thus,

$$S_n(A - B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B).$$

The summation is in fact limited to a finite number of nonzero terms. In particular, we have

$$\prod_{b \epsilon B}(x - b) = S_n(x - B) = S_0(-B)x^n + S_1(-B)x^{n-1} + S_2(-B)x^{n-2} + \cdots + S_n(-B),$$

where $S_j(-B)$ are the coefficients of polynomials $S_n(x - B)$ for $0 < j < n$. We note that $S_j(-B) = 0$ for $j > n$.

Thus, the special case of $B = \{1, 1, 1, ..., 1\}$ gives the two binomial coefficients

$$S_j(-n) = (-1)^j \binom{n}{j} \text{ and } S_j(n) = \binom{n + j - 1}{j}.$$

Dfinition 5 ([16])**.** *Given a function* g *on* $\mathbb{R}^n$, *the divided difference operator is defined as follows:*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) - g(x_1, \cdots x_{i-1}, x_{i+1}, x_i, x_{i+2} \cdots x_n)}{x_i - x_{i+1}}.$$

Dfinition 6 ([8])**.** *Given a function* $g(e_1, e_2)$, *the symmetrizing operator* $L^k_{e_1 e_2}$ *is defined by*

$$L^k_{e_1 e_2} g(e_1, e_2) = \frac{e_1^k\, g(e_1, e_2) - e_2^k\, g(e_2, e_1)}{e_1 - e_2}.$$

Proposition 7 ([11, Proposition 14.1])**.** *Given an alphabet* $E = \{e_1, e_2\}$, *the operator* $L^k_{e_1 e_2}$ *satisfies the following formula*

$$L^k_{e_1 e_2} f(e_1) = S_{k-1}(e_1 + e_2)f(e_1) + e_2^k \partial_{e_1 e_2} f(e_1), \quad \textit{for all } k \in \mathbb{N}.$$

3. THE PROOF OF THEOREM 1

In this section, we present a proof of Theorem 1.

Proof of Theorem 1. Let

$$f(e_1) = \frac{1}{\prod_{a\in A}(1 - ae_1 z)}.$$

On one hand, since

$$f(e_1) = \sum_{j=0}^{\infty} S_j(A)e_1^j z^j,$$

we have that

$$\begin{aligned} L^k_{e_1 e_2} f(e_1) =& L^k_{e_1 e_2}\left(\sum_{j=0}^{\infty} S_j(A)e_1^j z^j\right) \\ =& \frac{e_1^k \sum_{j=0}^{\infty} S_j(A)e_1^j z^j - e_2^k \sum_{j=0}^{\infty} S_j(A_2)e_2^j z^j}{e_1 - e_2} \\ =& \sum_{j=0}^{\infty} S_j(A)\left(\frac{e_1^{j+k} - e_2^{j+k}}{e_1 - e_2}\right) z^j \\ =& \sum_{j=0}^{\infty} S_j(A) L^k_{e_1 e_2}(e_1^j) z^j, \end{aligned}$$

which is the left hand side of (1).

On the other hand, since

$$f(e_1) = \frac{1}{\sum_{j=0}^{\infty} S_j(-A)e_1^j z^j},$$

we have that

$$\begin{aligned} \partial_{e_1 e_2} f(e_1) &= \frac{1}{e_1 - e_2}\left(\frac{1}{\sum_{j=0}^{\infty} S_j(-A)e_1^j z^j} - \frac{1}{\sum_{j=0}^{\infty} S_j(-A)e_2^j z^j}\right) \\ &= \frac{\sum_{j=0}^{\infty} S_j(-A)\frac{e_2^j - e_1^j}{e_1 - e_2} z^j}{\left(\sum_{j=0}^{\infty} S_j(-A)e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A)e_2^j z^j\right)} \\ &= -\frac{\sum_{j=0}^{\infty} S_j(-A)S_{j-1}(e_1 + e_2)z^j}{\left(\sum_{j=0}^{\infty} S_j(-A)e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A)e_2^j z^j\right)}. \end{aligned}$$

By Proposition 7, it follows that

$$\begin{aligned} L^k_{e_1 e_2} f(e_1) =& S_{k-1}(e_1 + e_2) f(e_1) + e_2^k\, \partial_{e_1 e_2} f(e_1) \\ =& \frac{S_{k-1}(e_1 + e_2)}{\sum_{j=0}^{\infty} S_j(-A)e_1^j z^j} - \frac{\sum_{j=0}^{\infty} S_j(-A)S_{j-1}(e_1 + e_2)z^j}{\left(\sum_{j=0}^{\infty} S_j(-A)e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A)e_2^j z^j\right)} \end{aligned}$$

$$=\frac{\sum_{j=0}^{\infty} S_j(-A)\left(e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2)\right) z^j}{\left(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j\right)}.$$

Hence, we have that

$$\begin{aligned}
L_{e_1e_2}^k f(e_1) =& \frac{\sum_{j=0}^{k-1} S_j(-A)\left(e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2)\right) z^j}{\left(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j\right)} \\
&+ \frac{\sum_{j=k+1}^{\infty} S_j(-A)\left(e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2)\right) z^j}{\left(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j\right)} \\
=& \frac{\sum_{j=0}^{k-1} S_j(-A) e_1^j e_2^j L_{e_1e_2}^k\left(\frac{1}{e_1^j}\right) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} S_{j+k+1}(-A) L_{e_1e_2}^k(e_1) z^j}{\left(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j\right)},
\end{aligned}$$

which is the right hand side of (1). This completes the proof. □

4. APPLICATIONS TO THE GENERATING FUNCTIONS

In this section, we attempt to give results for some well-known generating functions. In fact, we will use Theorem 1 to derive Fibonacci numbers and Chebychev polynomials of second kind. Moreover, the generating functions for some special cases of Fibonacci numbers and Chebychev polynomials are given.

4.1. THE CASE $A = \{a_1\}$

If $k = 1$ and $A = \{a_1\}$, the next result gives a generating function [7, 11].

Corollaire 8. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1\}$, we have*

$$\sum_{j=0}^{\infty} a_1^j S_j(e_1+e_2) z^j = \frac{1}{(1-a_1e_1z)(1-a_1e_2z)}. \tag{3}$$

If $k = 2$ and $A = \{a_1\}$, the next result gives a generating function for Lucas numbers.

Corollaire 9. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1\}$, we have*

$$\sum_{j=0}^{\infty} a_1^j S_{j+1}(e_1+e_2) z^j = \frac{e_1+e_2-e_1e_2a_1z}{(1-a_1e_1z)(1-a_1e_2z)}. \tag{4}$$

If $a_1 = 1$, replacing e_2 by $(-e_2)$ in (3) and (4), we obtain

$$\sum_{j=0}^{\infty} S_j(e_1 + [-e_2])z^j = \frac{1}{(1 - ze_1)(1 + ze_2)}, \tag{5}$$

$$\sum_{j=0}^{\infty} S_{j+1}(e_1 + [-e_2])z^j = \frac{e_1 - e_2 + e_1e_2z}{(1 - ze_1)(1 + ze_2)}. \tag{6}$$

Choosing e_1 and e_2 such that

$$\begin{cases} e_1\, e_2 = 1, \\ e_1 - e_2 = 1, \end{cases}$$

and substituting in (5) and (6), we end up with

$$\sum_{j=0}^{\infty} S_j(e_1 + [-e_2])z^j = \frac{1}{1 - z - z^2}, \tag{7}$$

$$\sum_{j=0}^{\infty} S_{j+1}(e_1 + [-e_2])z^j = \frac{1 + z}{1 - z - z^2}, \tag{8}$$

which were given by Boussayoud et al. [7, 8, 11].

Remarque 10. *For all $j \in \mathbb{N}$,*

$$S_{j+1}(e_1 + [-e_2]) = S_j(e_1 + [-e_2]) + S_{j-1}(e_1 + [-e_2])).$$

Multiplying the equation (7) by 3 and subtracting it from (8), we obtain

$$\sum_{j=0}^{\infty} \left(3S_j(e_1 + [-e_2]) - S_{j+1}(e_1 + [-e_2])\right) z^j = \frac{2 - z}{1 - z - z^2},$$

which represents a generating function for Lucas numbers such that

$$L_j = 3S_j(e_1 + [-e_2]) - S_{j+1}(e_1 + [-e_2]).$$

On the other hand, when replacing e_1 and e_2 by $2e_1$ and $(-2e_2)$ respectively in (5) and (6), and under the condition $4e_1e_2 = -1$, we obtain, for $y = e_1 - e_2$, that

$$\sum_{j=0}^{\infty} S_j(2e_1 + [-2e_2])z^j = \frac{1}{1 - 2yz + z^2}, \tag{9}$$

$$\sum_{j=0}^{\infty} S_{j+1}(2e_1 + [-2e_2])z^j = \frac{2y - z}{1 - 2yz + z^2}, \tag{10}$$

where (9) represents a generating function for Chebychev polynomials of the second kind [7, 8, 11], and (10) represents a new generating function.

Remarque 11. *For all $j \in \mathbb{N}$:*

$$S_{j+1}(2e_1 + [-2e_2]) = 2yS_j(2e_1 + [-2e_2]) - S_{j-1}(2e_1 + [2e_2]).$$

Moreover, we deduce from (9) that

$$\sum_{j=0}^{\infty} \left(S_j(2e_1 + [-2e_2]) - yS_{j-1}(2e_1 + [-2e_2])\right) z^j = \frac{1 - yz}{1 - 2yz + z^2},$$

which represents a generating function for Chebychev polynomials of the first kind [7, 8, 11], such that

$$T_j(y) = S_j(2e_1 + [-2e_2]) - yS_{j-1}(2e_1 + [-2e_2]).$$

4.2. THE CASE $E = \{e_1, e_2\}$, $A = \{a_1, a_2\}$

If $k = 1$ and $A = \{a_1, a_2\}$, the next result gives a generating function [8, 10, 11, 12].

Corollaire 12 ([10, Theorem 4])**.** *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then*

$$\sum_{j=0}^{\infty} S_j(A)S_j(e_1 + e_2)z^j = \frac{1 - a_1a_2e_1e_2z^2}{\left(\sum_{j=0}^{\infty} S_j(-A)e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A)e_2^j z^j\right)}. \tag{11}$$

If $k = 2$ and $A = \{a_1, a_2\}$, the next result gives a new generating function for Stirling numbers of the second kind.

Corollaire 13. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, we have*

$$\sum_{j=0}^{\infty} L^k_{a_1a_2}(a_1)S_{j+1}(e_1 + e_2)z^j = \frac{e_1 + e_2 - e_1e_2(a_1 + a_2)z}{\left(\sum_{j=0}^{\infty} S_j(-A)e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A)e_2^j z^j\right)}. \tag{12}$$

Case 1: Substituting $e_1 = a_1 = 1$, $e_2 = x$ and $a_2 = y$ in (11), we obtain the following identity of Ramanujan [7, 9, 14]:

$$\sum_{j=0}^{\infty} S_j(1 + x)S_j(1 + y)z^j = \frac{1 - xyz^2}{(1 - z)(1 - zx)(1 - zy)(1 - zxy)}.$$

Case 2: Replacing e_2 by $(-e_2)$ and a_2 by $(-a_2)$ in (12) yields

$$\sum_{j=0}^{\infty} S_j(a_1 + [-a_2])S_j(e_1 + [-e_2])z^j = \frac{1 - e_1e_2a_1a_2z^2}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)}. \tag{13}$$

This case consists of three related parts.

Firstly, the substitutions of

$$\begin{cases} a_1 - a_2 = 1, \\ a_1\, a_2 = 1, \end{cases} \text{and} \begin{cases} e_1 - e_2 = 1, \\ e_1\, e_2 = 1, \end{cases}$$

in (13) give

$$\sum_{j=0}^{\infty} S_j(a_1 + [-a_2])S_j(e_1 + [-e_2])z^j = \frac{1 - z^2}{1 - z - 4z^2 - z^3 + z^4} = \sum_{j=0}^{\infty} F_j^2 z^j,$$

which represents a generating function for squared Fibonacci numbers [7, 10, 12], such that

$$F_j^2 = S_j(a_1 + [-a_2])S_j(e_1 + [-e_2]).$$

Secondly, making the substitution of

$$\begin{cases} e_1 - e_2 = 1, \\ e_1\, e_2 = 1, \\ 4\, a_1\, a_2 = -1, \end{cases}$$

in (13) and setting for ease on notations $x = a_1 - a_2$, we reach

$$\sum_{j=0}^{\infty} F_j\, U_j(x)z^j = \frac{1 + z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4},$$

which corresponds to a generating function for the product of Fibonacci numbers and Chebychev polynomials of the second kind [7, 10].

Thirdly, recall that for $y = e_1 - e_2$, the substitution of

$$\begin{cases} 4\, e_1\, e_2 = -1, \\ 4\, a_1\, a_2 = -1, \end{cases}$$

in (13) results in

$$\sum_{j=0}^{\infty} U_j(y)U_j(x)z^j = \frac{1 - z^2}{1 - 4yxz + (4x^2 + 4y^2 - 2)z^2 - 4yxz^3 + z^4},$$

which represents a generating function for Chebychev polynomials of the second kind [7, 9, 10].

According to formulas (9) and (11), and based on the fact that

$$S_{j-1}(2a_1 + [-2a_2]) = \frac{(2a_1)^j - (-2a_2)^j}{2a_1 + 2a_2},$$

we have

$$\sum_{j=0}^{\infty} U_j(y)T_j(x)z^j = \frac{1-2yxz+(2x^2-1)z^2}{1-4yxz+(4x^2+4y^2-2)z^2-4yxz^3+z^4},$$

which represents a generating function for the combined Chebychev polynomials of the second and first kinds.

Finally, we have

$$\sum_{j=0}^{\infty} T_j\,(y)\,T_j(x)z^j = \frac{1-3yxz+(2x^2+2y^2-1)z^2-yxz^3}{1-4yxz+(4x^2+4y^2-2)z^2-4yxz^3+z^4},$$

that corresponds to a generating function for Chebychev polynomials of the first kind [7, 9, 10].

Case 3: The Stirling numbers of the second kind $S(j,k)$ are defined by generating function

$$\sum_{j=0}^{\infty} S(j,k)z^{j-k} = \frac{1}{(1-z)(1-2z)\cdots(1-kz)}.$$

These numbers can be interpreted as the numbers of k partitions of a set of j elements. The Stirling numbers of the second kind $S(j,k)$ can be expressed as

$$S(j,k) = \frac{1}{k!}\sum_{s=0}^{k}(-1)^s\binom{k}{s}(k-s)^j.$$

Abderrezzak [1] showed that

$$S(j+k,k) = S_j(N_k),$$

with $N_k = \{1,2,...,k\}$. Thus,

$$\begin{aligned} S(j+1,1) =& S_j(1) = 1,\\ S(j+2,2) =& S_j(1+2) = 2^{j\ +1} - 1. \end{aligned}$$

If $a_1 = e_1 = 1$ and $a_2 = e_2 = 2$ in formulas (11) and (12), then new generating functions are derived,

$$S(j+1,1) + \sum_{j=1}^{\infty} S(j+2,2)^2 z^j = \frac{1+2z}{(1-z)(1-2z)(1-4z)},$$

$$S(j+1,1) + \sum_{j=1}^{\infty} S(j+2,2)\left(2^j-1\right)z^j = \frac{3}{(1-z)(1-2z)(1-4z)},$$

$$S(j+1,1) + \sum_{j=1}^{\infty} S(j+2,2)^3 z^j = \frac{1-84z+432z^2-672z^3-512z^4}{(1-z)(1-2z)^3(1-4z)^3(1-8z)}.$$

4.3. THE CASE $A = \{a_1, a_2, a_3\}$

Corollaire 14. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2, a_3\}$, we have*

$$\sum_{j=0}^{\infty} S_j(A) L^k_{e_1 e_2}(e_1) z^j$$
$$= \frac{1 - e_1 e_2 (a_1 a_2 + a_1 a_3 + a_2 a_3) z^2 + e_1 e_2 a_1 a_2 a_3 (e_1 + e_2) z^3}{\left(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j\right)\left(\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j\right)}.$$

Case 1: For $e_1 = a_1 = 1$, $a_2 = y$ and $e_2 = x$, $a_3 = \alpha$ in Corollary 14, we have

$$\sum_{j=0}^{\infty} S_j \ (1 + x) S_j (1 + y + \alpha) z^j$$
$$= \frac{1 - x(y + \alpha + \alpha y) z^2 + xy\alpha(1 + x) z^3}{(1 - z)(1 - zx)(1 - zy)(1 - zxy)(1 - \alpha z)(1 - x\alpha z)}.$$

Remarque 15. *Notice that for $\alpha = 0$, we find the identity* (11) *of Lascoux in [14].*

Case 2: By replacing e_2 by $(-e_2)$ and making the following specialization

$$\begin{cases} e_1\, e_2 = 1, \\ e_1 - e_2 = 1, \end{cases}$$

in Corollary 14, we obtain the following identity involving Fibonacci numbers and symmetric functions in several variables

$$\sum_{j=0}^{\infty} S_j(A) F_j z^j = \frac{1 + (a_1 a_2 + a_1 a_3 + a_2 a_3) z^2 + a_1 a_2 a_3 z^3}{(1 - a_1 z - a_1^2 z^2)(1 - a_2 z - a_2^2 z^2)(1 - a_3 z - a_3^2 z^2)}.$$

Case 3: By replacing e_1 by $2e_1$ and e_2 by $(-2e_2)$ making the following specialization $4e_1 e_2 = -1$ in Corollary 14, gives us an identity involving Chebyshev polynomials of second kind and the symmetric functions in several variables, as follows for $y = e_1 - e_2$,

$$\sum_{j=0}^{\infty} S_j(A) U_j(y) z^j = \frac{1 - (a_1 a_2 + a_1 a_3 + a_2 a_3) z^2 + 2 a_1 a_2 a_3 y z^3}{(1 - 2a_1 yz - a_1^2 z^2)(1 - 2a_2 yz - a_2^2 z^2)(1 - 2a_3 yz - a_3^2 z^2)}.$$

5. CONCLUSION

In this paper, we proposed a new theorem (Theorem 1) to determine certain generating functions, which is based on the concepts of symmetric functions. The results are consistent with results obtained in some previous work [7, 8, 9, 10, 11, 12]. The results obtained in this work are promising, but there are other perspectives to follow in the field. Future work should be based on the extension of the alphabet E and the study of k parameter values.

REFERENCES

[1] A. Abderrezzak, Généralisation de la transformation d'Euler d'une série formelle, *Adv. Math.*, **103** (1994), 180-195.

[2] A. Abderrezzak, Généralisation d'identités de Carlitz, Howard et Lehmer, *Aequationes Math.*, **49** (1995), 36-46.

[3] A. Boussayoud, M. Kerada, N. Harrouche, On the k-Lucas numbers and Lucas Polynomials, *Turkish Journal of Analysis and Number.*, **5**, No. 3 (2017), 121-125.

[4] A. Boussayoud, A. Abderrezzak, On Some Identities and Generating Functions for Hadamard Product, *Electron. J. Math. Analysis Appl.***5**, No. 2 (2017), 89-97.

[5] A. Boussayoud, On some identities and generating functions for Pell-Lucas numbers, *Online.J. Anal. Comb.*, **12** (2017), 1-10.

[6] A. Boussayoud, M. Boulyer, M. Kerada, A simple and accurate method for determination of some generalized sequence of numbers, *Int. J. Pure Appl. Math.*, **108** (2016), 503-511.

[7] A. Boussayoud, A. Abderrezzak and M. Kerada, Some applications of symmetric functions, *Integers.*, **15**, A#48 (2015), 1-7.

[8] A. Boussayoud and R. Sahali, The application of the operator $L_{b_1 b_2}^{-k}$ in the series $\sum_{j=0}^{+\infty} a_j b_1^j z^j$, *J. Adv. Res. Appl. Math.*, **7** (2015), 68-75.

[9] A. Boussayoud and M. Kerada, Symmetric and Generating Functions, *Int. Electron. J. Pure Appl. Math.*, **7** (2014), 195-203.

[10] A. Boussayoud, M. Kerada, R. Sahali and W. Rouibah, Some Applications on Generating Functions, *J. Concr. Appl. Math.* **12** (2014), 321-330.

[11] A. Boussayoud, M. Kerada and A. Abderrezzak, A Generalization of Some Orthogonal Polynomials, *Springer Proc. Math. Stat.* **41** (2013), 229-235.

[12] D. Foata and G.-N. Han, Nombres de Fibonacci et Polynômes Orthogonaux. *in M. Morelli and M. Tangheroni, eds., Leonardo Fibonacci: Il Tempo, Le Opere, L'Eredit a Scienti ca, Pacini, Rome* (1994), 179-200.

[13] D. Foata, Combinatoire et représentation du Groupe symétrique, Lect. Notes Math. **579** (1977), 1-339.

[14] A. Lascoux, Addition of ± 1: Application to Arithmetic. *Sémin Lothar Comb.* **52** (2004), 1-9.

[15] A. Lascoux and A. M. Fu, Partition Analysis and Symmetrizing Operators, *J. Comb. Theory, Ser. A.*, **109** (2005), 339-343.

[16] I.G. Macdonald, *Symmetric Functions and Hall Polynomias*, Oxford University Press, 1979.