SYMmetric functions for
families of Generating functions

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Abstract: In this paper we show how the action of the symmetrizing operators
\( L^k_{\epsilon_1, \epsilon_2} \) to the series \( \sum_{j=0}^{\infty} S_j (-A) e^{\epsilon_1 j} z^j \) allows the obtention of an alternative approach
for the determination of Fibonacci numbers and Chebychev polynomials of the first and
second kind.

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1. Introduction

Let \( F_n, U_n \) and \( T_n \) be the \( n \)th Fibonacci number, Chebyshev polynomials of the first
kind and the second kind, respectively. Foata [12] used combinatorial techniques to
give generating functions of squared Fibonacci numbers \( \sum_{j=0}^{\infty} F^2_j z^j \), and products of
Chebyshev polynomials of first and second kinds. Foata also obtained the following
series
\[ \sum_{j=0}^{\infty} F_j z^j, \sum_{j=0}^{\infty} T_j z^j, \sum_{j=0}^{\infty} U_j z^j. \]

Lascaux [14] later found an identity of Ramanujan by means of the divided differences.
Boussayoud, Kerada and Abderrezzak [11] have recovered the generating functions
\[ \sum_{j=0}^{\infty} F_j z^j, \sum_{j=0}^{\infty} T_j z^j, \sum_{j=0}^{\infty} U_j z^j, \]
by using the definition of a symmetric operator \( L_{e_1 e_2}^k \). Since then, more generating
functions had been rediscovered [8, 10],
\[ \sum_{j=0}^{\infty} F_j z^j, \sum_{j=0}^{\infty} T_j z^j, \sum_{j=0}^{\infty} U_j z^j, \sum_{j=0}^{\infty} F_j^2 z^j, \sum_{j=0}^{\infty} T_j z^j, \sum_{j=0}^{\infty} T_j U_j z^j. \]

In this paper, we shall combine all these results in a unified way. All the results
can be treated as special cases of the following theorem.

**Theorem 1.** Given an alphabet set \( A = \{a_1, a_2, \ldots\} \), we have
\[ \sum_{j=0}^{\infty} S_j(A) L_{e_1 e_2}^k (e_1^j) z^j = \]
\[ \frac{\sum_{j=0}^{k-1} S_j(-A) e_1^j e_2^j L_{e_1 e_2}^k \left( \frac{1}{e_1^j} \right) z^j - e_1^k e_2^{k+1} \sum_{j=0}^{\infty} S_{j+k+1}(-A) L_{e_1 e_2}^k (e_1^j) z^j}{\left( \sum_{j=0}^{\infty} S_j(-A) e_1^j z^j \right) \left( \sum_{j=0}^{\infty} S_j(-A) e_2^j z^j \right)}. \] (1)

Theorem 1 generalizes several results published in [4, 5, 6, 7, 8, 9, 10, 11, 12]. In
addition, we also find the generating function of the Stirling numbers of the second
kind based on [2].

2. PRELIMINARIES

In the paper we need a lot of known results, quoted here for convenience to the reader.

**Definition 2 ([1]).** Given two sets of indeterminate \( A \) and \( B \) (called alphabets), we
define \( S_j(A - B) \) as follows:
\[ \frac{\prod_{b \in B}(1 - zb)}{\prod_{a \in A}(1 - za)} = \sum_{j=0}^{\infty} S_j(A - B) z^j, \] (2)
with \( S_j(A - B) = 0 \) for \( j < 0 \).
All the alphabets considered in this article are finite.

**Remarque 3.** By taking $A = 0$ in (2.1), we obtain
\[
\prod_{b \in B} (1 - zb) = \sum_{j=0}^{\infty} S_j(-B) z^j.
\]

**Proposition 4 ([2]).** Considering successively the case of $A = \Phi$ or $B = \Phi$, we can derive the following factorization
\[
\sum_{j=0}^{\infty} S_j(A - B) z^j = \sum_{j=0}^{\infty} S_j(A) z^j \sum_{j=0}^{\infty} S_j(-B) z^j.
\]

Thus,
\[
S_n(A - B) = \sum_{k=0}^{n} S_{n-k}(A) S_k(-B).
\]

The summation is in fact limited to a finite number of nonzero terms. In particular, we have
\[
\prod_{b \in B} (x - b) = S_n(x - B) = S_0(-B)x^n + S_1(-B)x^{n-1} + S_2(-B)x^{n-2} + \cdots + S_n(-B),
\]

where $S_j(-B)$ are the coefficients of polynomials $S_n(x - B)$ for $0 < j < n$. We note that $S_j(-B) = 0$ for $j > n$.

Thus, the special case of $B = \{1, 1, 1, \ldots, 1\}$ gives the two binomial coefficients
\[
S_j(-n) = (-1)^j \binom{n}{j} \quad \text{and} \quad S_j(n) = \binom{n+j-1}{j}.
\]

**Definition 5 ([16]).** Given a function $g$ on $\mathbb{R}^n$, the divided difference operator is defined as follows:
\[
\partial_{x_i, x_{i+1}}(g) = \frac{g(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - g(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2} \ldots, x_n)}{x_i - x_{i+1}}.
\]

**Definition 6 ([8]).** Given a function $g(e_1, e_2)$, the symmetrizing operator $L^k_{e_1 e_2}$ is defined by
\[
L^k_{e_1 e_2} g(e_1, e_2) = \frac{e_1^k g(e_1, e_2) - e_2^k g(e_2, e_1)}{e_1 - e_2}.
\]

**Proposition 7 ([11, Proposition 14.1]).** Given an alphabet $E = \{e_1, e_2\}$, the operator $L^k_{e_1 e_2}$ satisfies the following formula
\[
L^k_{e_1 e_2} f(e_1) = S_{k-1}(e_1 + e_2)f(e_1) + e_2^k \partial_{e_1 e_2} f(e_1), \quad \text{for all } k \in \mathbb{N}.
\]
3. THE PROOF OF THEOREM 1

In this section, we present a proof of Theorem 1.

*Proof of Theorem 1.* Let

$$f(e_1) = \frac{1}{\prod_{a \in A} (1 - ae_1 z)}.$$  

On one hand, since

$$f(e_1) = \sum_{j=0}^{\infty} S_j(A) e_1^j z^j,$$

we have that

$$L^{k}_{\varepsilon_1, \varepsilon_2} f(e_1) = L_{\varepsilon_1, \varepsilon_2}^k \left( \sum_{j=0}^{\infty} S_j(A) e_1^j z^j \right)$$

$$= \sum_{j=0}^{\infty} S_j(A) \left( \frac{e_1^{j+k} - e_2^{j+k}}{e_1 - e_2} \right) z^j$$

$$= \sum_{j=0}^{\infty} S_j(A) L_{\varepsilon_1, \varepsilon_2}^k (e_1^j) z^j,$$

which is the left hand side of (1).

On the other hand, since

$$f(e_1) = \sum_{j=0}^{\infty} S_j(-A) e_1^j z^j,$$

we have that

$$\partial_{\varepsilon_1, \varepsilon_2} f(e_1) = \frac{1}{e_1 - e_2} \left( \sum_{j=0}^{\infty} S_j(-A) e_1^j z^j \right) - \frac{1}{\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j}$$

$$= \frac{\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j}{(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j)} - \frac{\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j}{(\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j)}$$

$$= -\frac{\sum_{j=0}^{\infty} S_j(-A) S_{j-1}(e_1 + e_2) z^j}{(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j)(\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j)}.$$  

By Proposition 7, it follows that

$$L^{k}_{\varepsilon_1, \varepsilon_2} f(e_1) = S_{k-1}(e_1 + e_2) f(e_1) + e_2^k \partial_{\varepsilon_1, \varepsilon_2} f(e_1)$$

$$= \frac{S_{k-1}(e_1 + e_2)}{\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j} - \frac{\sum_{j=0}^{\infty} S_j(-A) S_{j-1}(e_1 + e_2) z^j}{(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j)(\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j)}.$$
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\[ \sum_{j=0}^{\infty} S_j(-A) \left( e_2^j S_{k-1}(e_1 + e_2) - e_2^j S_{j-1}(e_1 + e_2) \right) z^j = \frac{\sum_{j=0}^{\infty} S_j(-A) (e_1^j z^j)}{\left( \sum_{j=0}^{\infty} S_j(-A) e_1^j z^j \right) \left( \sum_{j=0}^{\infty} S_j(-A) e_2^j z^j \right)} \]

Hence, we have that

\[ L_{e_1e_2}^k f(e_1) = \sum_{j=0}^{\infty} S_j(-A) \left( e_2^j S_{k-1}(e_1 + e_2) - e_2^j S_{j-1}(e_1 + e_2) \right) z^j \]

which is the right hand side of (1). This completes the proof. \qed

4. APPLICATIONS TO THE GENERATING FUNCTIONS

In this section, we attempt to give results for some well-known generating functions. In fact, we will use Theorem 1 to derive Fibonacci numbers and Chebychev polynomials of second kind. Moreover, the generating functions for some special cases of Fibonacci numbers and Chebychev polynomials are given.

4.1. THE CASE \( A = \{a_1\} \)

If \( k = 1 \) and \( A = \{a_1\} \), the next result gives a generating function [7, 11].

Corollaire 8. Given two alphabets \( E = \{e_1, e_2\} \) and \( A = \{a_1\} \), we have

\[ \sum_{j=0}^{\infty} a_1^j S_j(e_1 + e_2) z^j = \frac{1}{(1-a_1e_1 z)(1-a_1e_2 z)}. \]  \hspace{1cm} (3)

If \( k = 2 \) and \( A = \{a_1\} \), the next result gives a generating function for Lucas numbers.

Corollaire 9. Given two alphabets \( E = \{e_1, e_2\} \) and \( A = \{a_1\} \), we have

\[ \sum_{j=0}^{\infty} a_1^j S_{j+1}(e_1 + e_2) z^j = \frac{e_1 + e_2 - e_1 e_2 a_1 z}{(1-a_1 e_1 z)(1-a_1 e_2 z)}. \]  \hspace{1cm} (4)
If \( a_1 = 1 \), replacing \( e_2 \) by \((-e_2)\) in (3) and (4), we obtain

\[
\sum_{j=0}^{\infty} S_j (e_1 + [-e_2]) z^j = \frac{1}{(1 - ze_1)(1 + ze_2)},
\]

(5)

\[
\sum_{j=0}^{\infty} S_{j+1}(e_1 + [-e_2]) z^j = \frac{e_1 - e_2 + e_1e_2z}{(1 - ze_1)(1 + ze_2)}.
\]

(6)

Choosing \( e_1 \) and \( e_2 \) such that

\[
\begin{cases}
  e_1 e_2 = 1, \\
  e_1 - e_2 = 1,
\end{cases}
\]

and substituting in (5) and (6), we end up with

\[
\sum_{j=0}^{\infty} S_j (e_1 + [-e_2]) z^j = \frac{1}{1 - z - z^2},
\]

(7)

\[
\sum_{j=0}^{\infty} S_{j+1}(e_1 + [-e_2]) z^j = \frac{1 + z}{1 - z - z^2},
\]

(8)

which were given by Boussayoud et al. \[7, 8, 11\].

\textbf{Remarque 10.} For all \( j \in \mathbb{N} \),

\[
S_{j+1}(e_1 + [-e_2]) = S_j(e_1 + [-e_2]) + S_{j-1}(e_1 + [-e_2]).
\]

Multiplying the equation (7) by 3 and subtracting it from (8), we obtain

\[
\sum_{j=0}^{\infty} (3S_j(e_1 + [-e_2]) - S_{j+1}(e_1 + [-e_2])) z^j = \frac{2 - z}{1 - z - z^2},
\]

which represents a generating function for Lucas numbers such that

\[
L_j = 3S_j(e_1 + [-e_2]) - S_{j+1}(e_1 + [-e_2]).
\]

On the other hand, when replacing \( e_1 \) and \( e_2 \) by \( 2e_1 \) and \((-2e_2)\) respectively in (5) and (6), and under the condition \( 4e_1e_2 = -1 \), we obtain, for \( y = e_1 - e_2 \), that

\[
\sum_{j=0}^{\infty} S_j (2e_1 + [-2e_2]) z^j = \frac{1}{1 - 2yz + z^2},
\]

(9)

\[
\sum_{j=0}^{\infty} S_{j+1}(2e_1 + [-2e_2]) z^j = \frac{2y - z}{1 - 2yz + z^2},
\]

(10)

where (9) represents a generating function for Chebychev polynomials of the second kind \[7, 8, 11\], and (10) represents a new generating function.
**Remarque 11.** For all \( j \in \mathbb{N} \):

\[
S_{j+1}(2e_1 + [-2e_2]) = 2yS_j(2e_1 + [-2e_2]) - S_{j-1}(2e_1 + [2e_2]).
\]

Moreover, we deduce from (9) that

\[
\sum_{j=0}^{\infty} (S_j(2e_1 + [-2e_2]) - yS_{j-1}(2e_1 + [-2e_2])) z^j = \frac{1 - yz}{1 - 2yz + z^2},
\]

which represents a generating function for Chebyshev polynomials of the first kind [7, 8, 11], such that

\[
T_j(y) = S_j(2e_1 + [-2e_2]) - yS_{j-1}(2e_1 + [-2e_2]).
\]

### 4.2. THE CASE \( E = \{e_1, e_2\}, A = \{a_1, a_2\} \)

If \( k = 1 \) and \( A = \{a_1, a_2\} \), the next result gives a generating function [8, 10, 11, 12].

**Corollaire 12** ([10, Theorem 4]). Given two alphabets \( E = \{e_1, e_2\} \) and \( A = \{a_1, a_2\} \), then

\[
\sum_{j=0}^{\infty} S_j(A)S_j(e_1 + e_2)z^j = \frac{1 - a_1a_2e_1e_2z^2}{\left(\sum_{j=0}^{\infty} S_j(-A)e_1^jz^j\right)\left(\sum_{j=0}^{\infty} S_j(-A)e_2^jz^j\right)}.
\]  

If \( k = 2 \) and \( A = \{a_1, a_2\} \), the next result gives a new generating function for Stirling numbers of the second kind.

**Corollaire 13.** Given two alphabets \( E = \{e_1, e_2\} \) and \( A = \{a_1, a_2\} \), we have

\[
\sum_{j=0}^{\infty} L_{a_1a_2}^k(a_1)S_{j+1}(e_1 + e_2)z^j = \frac{e_1 + e_2 - e_1e_2(a_1 + a_2)z}{\left(\sum_{j=0}^{\infty} S_j(-A)e_1^jz^j\right)\left(\sum_{j=0}^{\infty} S_j(-A)e_2^jz^j\right)}.
\]  

**Case 1:** Substituting \( e_1 = a_1 = 1, e_2 = x \) and \( a_2 = y \) in (11), we obtain the following identity of Ramanujan [7, 9, 14]:

\[
\sum_{j=0}^{\infty} S_j(1 + x)S_j(1 + y)z^j = \frac{1 - xzy^2}{(1 - z)(1 - zx)(1 - zy)(1 - zxy)}.
\]

**Case 2:** Replacing \( e_2 \) by \((-e_2)\) and \( a_2 \) by \((-a_2)\) in (12) yields

\[
\sum_{j=0}^{\infty} S_j(a_1 + [-a_2])S_j(e_1 + [-e_2])z^j
\]

\[
= \frac{1 - e_1e_2a_1a_2z^2}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)}.
\]
This case consists of three related parts.

**Firstly**, the substitutions of

\[
\begin{cases}
  a_1 - a_2 = 1, \\
  a_1 a_2 = 1,
\end{cases}
\quad \text{and} \quad
\begin{cases}
  e_1 - e_2 = 1, \\
  e_1 e_2 = 1,
\end{cases}
\]

in (13) give

\[
\sum_{j=0}^{\infty} S_j (a_1 + [-a_2]) S_j (e_1 + [-e_2]) z^j = \frac{1 - z^2}{1 - z - 4z^2 - z^3 + z^4}
\]

\[
= \sum_{j=0}^{\infty} F_j^2 z^j,
\]

which represents a generating function for squared Fibonacci numbers [7, 10, 12], such that

\[
F_j^2 = S_j(a_1 + [-a_2]) S_j(e_1 + [-e_2]).
\]

**Secondly**, making the substitution of

\[
\begin{cases}
  e_1 - e_2 = 1, \\
  e_1 e_2 = 1, \\
  4a_1 a_2 = -1,
\end{cases}
\]

in (13) and setting for ease on notations \( x = a_1 - a_2 \), we reach

\[
\sum_{j=0}^{\infty} F_j U_j (x) z^j = \frac{1 + z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4},
\]

which corresponds to a generating function for the product of Fibonacci numbers and Chebychev polynomials of the second kind [7, 10].

**Thirdly**, recall that for \( y = e_1 - e_2 \), the substitution of

\[
\begin{cases}
  4e_1 e_2 = -1, \\
  4a_1 a_2 = -1,
\end{cases}
\]

in (13) results in

\[
\sum_{j=0}^{\infty} U_j (y) U_j (x) z^j = \frac{1 - z^2}{1 - 4yxz + (4x^2 + 4y^2 - 2)z^2 - 4yxz^3 + z^4},
\]

which represents a generating function for Chebychev polynomials of the second kind [7, 9, 10].

According to formulas (9) and (11), and based on the fact that

\[
S_{j-1}(2a_1 + [-2a_2]) = \frac{(2a_1)^j - (-2a_2)^j}{2a_1 + 2a_2},
\]
we have

\[
\sum_{j=0}^{\infty} U_j(y) T_j(x) z^j = \frac{1 - 2y x z + (2x^2 - 1)z^2}{1 - 4y x z + (4x^2 + 4y^2 - 2)z^2 - 4y x z^3 + z^4},
\]

which represents a generating function for the combined Chebychev polynomials of the second and first kinds.

Finally, we have

\[
\sum_{j=0}^{\infty} T_j(y) T_j(x) z^j = \frac{1 - 3y x z + (2x^2 + 2y^2 - 1)z^2 - y x z^3}{1 - 4y x z + (4x^2 + 4y^2 - 2)z^2 - 4y x z^3 + z^4},
\]

that corresponds to a generating function for Chebychev polynomials of the first kind [7, 9, 10].

**Case 3:** The Stirling numbers of the second kind \(S(j, k)\) are defined by generating function

\[
\sum_{j=0}^{\infty} S(j, k) z^{j-k} = \frac{1}{(1-z)(1-2z) \cdots (1-kz)}.
\]

These numbers can be interpreted as the numbers of \(k\) partitions of a set of \(j\) elements. The Stirling numbers of the second kind \(S(j, k)\) can be expressed as

\[
S(j, k) = \frac{1}{k!} \sum_{s=0}^{k} (-1)^s \binom{k}{s} (k-s)^j.
\]

Abderrezzak [1] showed that

\[
S(j + k, k) = S_j(N_k),
\]

with \(N_k = \{1, 2, ..., k\}\). Thus,

\[
S(j + 1, 1) = S_j(1) = 1,
\]

\[
S(j + 2, 2) = S_j(1 + 2) = 2^{j+1} - 1.
\]

If \(a_1 = e_1 = 1\) and \(a_2 = e_2 = 2\) in formulas (11) and (12), then new generating functions are derived,

\[
S(j + 1, 1) + \sum_{j=1}^{\infty} S(j + 2, 2) z^j = \frac{1 + 2z}{(1-z)(1-2z)(1-4z)},
\]

\[
S(j + 1, 1) + \sum_{j=1}^{\infty} S(j + 2, 2) (2^j - 1) z^j = \frac{3}{(1-z)(1-2z)(1-4z)},
\]

\[
S(j + 1, 1) + \sum_{j=1}^{\infty} S(j + 2, 2) z^j = \frac{1 - 84z + 432z^2 - 672z^3 - 512z^4}{(1-z)(1-2z)^3(1-4z)^3(1-8z)}.
\]
4.3. THE CASE $A = \{a_1, a_2, a_3\}$

**Corollaire 14.** Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2, a_3\}$, we have

$$\sum_{j=0}^{\infty} S_j(A) L_{e_1 e_2}(e_1) z^j = \frac{1 - e_1 e_2 (a_1 a_2 + a_1 a_3 + a_2 a_3) z^2 + e_1 e_2 a_1 a_2 a_3 (e_1 + e_2) z^3}{(\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j) (\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j)}.$$

**Case 1:** For $e_1 = a_1 = 1$, $a_2 = y$ and $e_2 = x$, $a_3 = \alpha$ in Corollary 14, we have

$$\sum_{j=0}^{\infty} S_j (1 + x) S_j(1 + y + \alpha) z^j = \frac{1 - x(y + \alpha + \alpha y) z^2 + x y \alpha (1 + x) z^3}{(1 - z) (1 - z x) (1 - z y) (1 - z y z) (1 - z x z)(1 - x z)}.$$

**Remarque 15.** Notice that for $\alpha = 0$, we find the identity (11) of Lascoux in [14].

**Case 2:** By replacing $e_2$ by $(-e_2)$ and making the following specialization

$$\begin{cases} e_1 e_2 = 1, \\ e_1 - e_2 = 1, \end{cases}$$

in Corollary 14, we obtain the following identity involving Fibonacci numbers and symmetric functions in several variables

$$\sum_{j=0}^{\infty} S_j(A) F_j z^j = \frac{1 + (a_1 a_2 + a_1 a_3 + a_2 a_3) z^2 + a_1 a_2 a_3 z^3}{(1 - a_1 z - a_1^2 z^2) (1 - a_2 z - a_2^2 z^2) (1 - a_3 z - a_3^2 z^2)}.$$

**Case 3:** By replacing $e_1$ by $2e_1$ and $e_2$ by $(-2e_2)$ making the following specialization $4e_1 e_2 = -1$ in Corollary 14, gives us an identity involving Chebyshev polynomials of second kind and the symmetric functions in several variables, as follows for $y = e_1 - e_2$,

$$\sum_{j=0}^{\infty} S_j(A) U_j(y) z^j = \frac{1 - (a_1 a_2 + a_1 a_3 + a_2 a_3) z^2 + 2a_1 a_2 a_3 y z^3}{(1 - 2a_1 y z - a_1^2 z^2) (1 - 2a_2 y z - a_2^2 z^2) (1 - 2a_3 y z - a_3^2 z^2)}.$$

5. CONCLUSION

In this paper, we proposed a new theorem (Theorem 1) to determine certain generating functions, which is based on the concepts of symmetric functions. The results
are consistent with results obtained in some previous work [7, 8, 9, 10, 11, 12]. The results obtained in this work are promising, but there are other perspectives to follow in the field. Future work should be based on the extension of the alphabet $E$ and the study of $k$ parameter values.

REFERENCES


