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AN ALTERNATIVE ELEMENTARY METHOD FOR APPROXIMATION OF INVARIANT MEASURES FOR RANDOM MAPS

MD SHAFIQUL ISLAM

School of Mathematical and Computational Sciences
University of Prince Edward Island
550 University Ave, Charlottetown, PE, C1A 4P3, CANADA

ABSTRACT: In this paper we describe an alternative elementary method of approximating invariant measures for random maps. Instead of computing Ulam's matrices associated with the Frobenious-Perron operator for random map we compute matrices which approximate Ulam's matrices.

Let $T = \{\tau_1(x), \tau_2(x), \dots, \tau_K(x); p_1, p_2, \dots, p_K\}$ be a random map which posses a unique absolutely continuous invariant measure $\hat{\mu}$ with probability density function $\hat{f}$. With our elementary method it is possible to develop and implement algorithms for the approximation of the invariant measure $\hat{\mu}$ with a given bound on the error of the approximation. One of the main advantages of our method is that we do not need to deal with the inverse of the component maps of the random maps. Our result is a generalization of the result of Galatolo and Nisoli (see the paper [12] **Galatolo, S. and Nisoli, I**, *An elementary approch to rigorous approximation of Invariant measures,* SIAM J. Appl. Dynamical Systems, Vol. 13, N0. 2, pp 958–985, 2014) of single piecewise expanding maps to results of random maps. We present a numerical example.

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1. INTRODUCTION

The existence and properties of absolutely continuous invariant measures for random maps reflect their long time behavior and play an important role in understanding their chaotic nature [3, 10, 13]. Absolutely continuous measures which are invariant under the random map $T = \{\tau_1(x), \tau_2(x), \dots, \tau_K(x); p_1, p_2, \dots, p_K\}$ are fixed points of an operator $\mathcal{D}$ on the space of measures (see Eq. (2.3)). Equivalently, a fixed point of the Frobenius-Perron operator P_T (see Eq. (2.2)) of a random map is the invariant density f of an absolutely continuous invariant measure μ. Thus, the problem of the computation of absolutely continuous invariant measures for the random map T reduces to the problem of computing fixed point of the transfer operator $\mathcal{D}$ or computing fixed point of the Frobenius – Perron operator P_T (see [23, 10, 13]). The transfer operator equation or the the Frobenius–Perron equation for a random map is a more complicated equation than the transfer operator equation or the Frobenius-Perron equation (respectively) for a single map and it is difficult to solve these functional equations except in some simple cases. The numerical approximation of (absolutely continuous) invariant measures of dynamical systems (single maps or random maps) is of practical importance in the application of ergodic theory and dynamical systems to various applied areas. A number of methods have, therefore, been developed to approximate (absolutely continuous) invariant measures for dynamical systems. Ulam's method which was suggested by Ulam [28] is one of the simplest, most used and best understood method. For a single piecewise C^2, piecewise expanding maps of interval satisfying $|\tau'| > \alpha > 2$ (see [21]), Li [22] first proved the convergence of Ulam's approximation. Since then, Ulam's method have been applied to one and higher dimensional single transformations (see for example, [5, 6]). The computation of invariant measures for random maps is not as simple as the computation of invariant densities for single maps. In [10], Froyland extended Ulam's method for a single transformation to a method for random map with constant probabilities (see [24]). Góra and Boyarsky (see [13]) proved the convergent of Ulam's method for position dependent random maps. For Markov switching position dependent random maps we developed Ulam's method in [16]. Recently, Froyland et al. have studied stability and approximation of random invariant densities for Lasota–Yorke map cocycles (see [11]). Almost all of the results in the literature on the approximation of invariant measures provided proofs for the convergence of the corresponding methods. Moreover, asymptotic estimates on the rate of convergence are also provided on some of the results mentioned above. However, results with explicit (rigorous) bound on the error for position dependent random maps are very few.

In an Ulam's method, first finite dimensional approximations of linear operators are found for the transfer operator or the Frobenius–Perron operator, then eigenvectors of corresponding matrix representation of fine dimensional approximation operators are found. The calculation of the (i, j)-th entry of these matrices involve the calculation of portion of the pre-image (inverse image) of the interval partition set I_j under the corresponding on the partition set I_i. For the error in an Ulam's method, the distance between the fixed point of the discretization approximation operator and the finxed point of the real operator are found using the stability result in [19]. The method requires some estimation which cannot be trivially done in a rigorous way in a reasonable time. In this paper, we describe an approach which requires simpler assumptions and estimations. Our method provides algorithms to approximate invariant measures with a specific bound on the error, that is, we can keep our approximation as sharp as possible. One of the other advantages of our method is that we do not need to use the inverse of corresponding transformations. Our approach is a generalization of the approach of Galatolo and Nisoli in [12] of one dimensional single dynamical systems to an approach for random maps.

The paper is organized in the following way. In Section 2, a present a review for random maps, invariant measures, transfer operator, the Frobenius–Perron operator and the existence of absolutely continuous invariant measures is presented. In Section 3, abstract results on the fixed points of operators are presented. An elementary method for the approximation of invariant measures with error bounds for Ulam method is presented in Section 4. The implementation of the elementary method is presented in Section 5. Numerical examples are presented in Section 6.

2. RANDOM MAPS, THE FROBENIUS-PERRON OPERATOR AND INVARIANT MEASURES

2.1. RANDOM MAPS WITH CONSTANT PROBABILITIES

Random maps with constant probabilities are an important special case of skew products. Let $(X, \mathcal{B}, \lambda)$ be a measure space and $\Omega = \{1, 2, 3, \ldots, K\}^{\{0,1,2,\ldots\}} = \{\omega = \{\omega_i\}_{i=0}^{\infty} : \omega_i \in \{1, 2, 3, \ldots, K\}\}$ be the set of set of all one sided infinite sequences . Let $\tau_k : X \to X, k = 1, 2, \ldots, K$ be nonsingular piecewise one-to-one transformations and $p_1, p_2, \ldots, p_K$ be constant probabilities such that $\sum_{i=1}^{K} p_i = 1$. The topology on Ω is the product of the discrete topology on $\{1, 2, 3, \ldots, n\}$ and the Borel probability measure μ_p on Ω is defined as $\mu_p(\{\omega : \omega_0 = i_0, \omega_1 = i_1, \ldots, \omega_n = i_n\}) = p_{i_0} p_{i_1} \ldots p_{i_n}$. Let $\sigma : \Omega \to \Omega$ be the left shift. Now consider the skew product $S : \Omega \times X \to \Omega \times X$ defined by

$$S(\omega, x) = (\sigma(\omega), \tau_{\omega_0}(x)), \omega \in \Omega, x \in X.$$

Now,

$$S^2(\omega, x) = \left(\sigma^2(\omega), \tau_{\omega_1} \circ \tau_{\omega_0}(x)\right)$$

and for any integer $N \geq 1$,

$$S^N(\omega, x) = \big(\sigma^N(\omega), \tau_{\omega_{N-1}} \circ \tau_{\omega_{N-2}} \circ \ldots \circ \tau_{\omega_1} \circ \tau_{\omega_0}(x)\big)$$

A random map

$$T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\},$$

with constant probabilities $p_1, p_2, \ldots, p_K$ is defined as follows: for any $x \in X, T(x) = \tau_k(x)$ with probability p_k and for any non-negative integer N, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \ldots \circ \tau_{k_1}(x)$ with probability $\Pi_{j=1}^N p_{k_j}$. $T^N(x)$ can be viewed as the second component of the S^N of the skew product S. It can be easily shown that a measure μ is $T-$invariant if and only if the measure $\mu_p \times \mu$ is $S-$invariant. Pelikan [32] defined a $T-$invariant measure μ as follows:

Definition 2.1. Let T be a random map on X and μ be a measure on X. The measure μ is invariant under the random map T if

$$\mu(E) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(E)), \tag{2.1}$$

for any measurable set $E \in \mathcal{B}$.

Lemma 2.2. *Let μ be a measure on X. Let μ_p be the Borel probability measure on $\Omega = \{1, 2, 3, \ldots, K\}^{\{0,1,2,\ldots\}}$. Then μ is $T-$invariant if and only if the measure $\mu_p \times \mu$ on $\mathcal{B}(\Omega) \times \mathcal{B}$ is S invariant.*

Proof. By definition of S and μ_p,

$$\begin{aligned}(\mu_p \times \mu)(S^{-1}(A \times B)) &= \sum_{k=1}^{K} p_k \mu_p(A) \mu(\tau_k^{-1}(B)) \\ &= \mu_p(A) \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(B))\end{aligned}$$

If μ is T invariant, then

$$(\mu_p \times \mu)(S^{-1}(A \times B)) = \mu_p(A)\mu(B).$$

Thus, $\mu_p \times \mu$ is S invariant. The proof of converse is easy. □

Let f be the density of μ. Then $d\mu = f \cdot d\lambda$. Let $A \times B$ be a measurable subset of $\Omega \times X$. Then

$$(\mu_p \times \mu)(S^{-1}(A \times B)) = \sum_{k=1}^{K} p_k \mu_p(A) \mu(\tau_k^{-1}(B))$$

$$\begin{aligned} &= \sum_{k=1}^{K} p_k \mu_p(A) \int_B P_{\tau_k} f d\lambda \\ &= \mu_p(A) \sum_k p_k \int_B P_{\tau_k} f d\lambda. \end{aligned}$$

Thus, the density on the second component is $\sum_i p_i P_{\tau_i} f$. Hence the Perron-Frobenius operator P_T for the random map T is given by

$$P_T f = \sum_{k=1}^{K} p_k P_{\tau_k} f, \tag{2.2}$$

where P_{τ_k} is the Frobenius–Perron operator of the transformation τ_k. The operator $\mathcal{D}$ on measures on $(I, \mathcal{B})$ defined by

$$\mathcal{D}\mu(A) = \sum_{k=1}^{K} p_k \mu\left(\tau_k^{-1}(A)\right), A \in \mathcal{B} \tag{2.3}$$

is known as the transfer operator of the random map T. It can be easily shown that (i) $P_T : L^1([0,1]) \to L^1([0,1])$ is a linear operator; (ii) P_T is non-negative, i.e., $f \in L^1([0,1])$ and $f \geq 0 \implies P_T f \geq 0$; (iii) P_T is a contraction, i.e., $\| P_T f \|_1 \leq \| f \|_1$, for any $f \in L^1([0,1])$; (iv) P_T satisfies the composition property, i.e., if T and R are two position dependent random maps on $[0,1]$, then $P_{T\circ R} = P_T \circ P_R$. In particular, for any $n \geq 1, P_T^n = P_{T^n}$;

Lemma 2.3. *$P_T f^* = f^*$ if and only if $\mu = f^* \lambda$ is T invariant.*

Proof. Assume that $\mu(A) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(A))$, for any $A \in \mathcal{B}$. Then

$$\begin{aligned} \int_A f^* d\lambda &= \sum_{k=1}^{K} p_k \int_{\tau_k^{-1}(A)} f^* d\lambda \\ &= \sum_{k=1}^{K} p_k \int_A P_{\tau_k} f^* d\lambda \\ &= \int_A \sum_{k=1}^{K} p_k P_{\tau_k} f^* d\lambda \\ &= \int_A P_T f^* d\lambda. \end{aligned}$$

Therefore, $P_T f^* = f^*$.

Conversely, assume that $P_T f^* = f^*$ almost everywhere. Then

$$\mu(A) = \int_A f^* d\lambda = \int_A P_T f^* d\lambda$$

$$
\begin{aligned}
&= \int_A \sum_{k=1}^{K} p_k P_{\tau_k} f^* d\lambda \\
&= \sum_{k=1}^{K} p_k \int_A P_{\tau_k} f^* d\lambda \\
&= \sum_{k=1}^{K} p_k \int_{\tau_k^{-1}(A)} f^* d\lambda \\
&= \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(A))
\end{aligned}
$$

□

2.2. EXISTENCE OF INVARIANT MEASURES FOR RANDOM MAPS

Let $\mathcal{T}_0(I)$ denote the class of transformations $\tau : I = [0,1] \to I$ that satisfy the following conditions:

(i) τ is piecewise monotonic, i.e., there exists a partition $\mathcal{J} = \{J_i = [x_{i-1}, x_i], i = 1, 2, \ldots, q\}$ of I such that $\tau_i = \tau|J_i$ is C^1, and

$$|\tau_i'(x)| \geq \alpha > 0, \tag{2.4}$$

for any i and for all $x \in (x_{i-1}, x_i)$;

(ii) $g(x) = \frac{1}{|\tau_i'(x)|}$ is a function of bounded variation, where $\tau_i'(x)$ is the appropriate one-sided derivative at the end points of $\mathcal{J}$.

We say that $\tau \in \mathcal{T}_1(I)$ if $\tau \in \mathcal{T}_0(I)$ and $\alpha > 1$ in condition (2.4), i.e., τ is piecewise expanding.

Lemma 2.4. *[32] Let $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ be a random map, where $\tau_k \in \mathcal{T}_0(I)$, with the common partition $\mathcal{J} = \{J_1, J_2, \ldots, J_q\}$, $k = 1, 2, \ldots, K$. If, for all $x \in [0,1]$, the following Pelikan's condition*

$$\sum_{k=1}^{K} \frac{p_k}{|\tau_k'(x)|} \leq \gamma < 1, \tag{2.5}$$

is satisfied, then, for any $f \in BV(I)$,

$$V_I P_T f \leq A V_I f + B \parallel f \parallel_1, \text{where } 0 < \text{A} < 1, \text{ and B} > 0 \tag{2.6}$$

Theorem 2.5. *Let $T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\}$ be a random map, where $\tau_k \in \mathcal{T}_0(I)$, with the common partition $\mathcal{J} = \{J_1, J_2, \dots, J_q\}$, $k = 1, 2, \dots, K$. If, for all $x \in [0,1]$, the following Pelikan's condition*

$$\sum_{k=1}^{K} \frac{p_k}{|\tau_k'(x)|} \le \gamma < 1, \tag{2.7}$$

is satisfied, then for all $f \in L^1 = L^1([0,1], \lambda)$:

(i) The limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} P_T^i(f) = f^* \quad \text{exists in} \quad \mathrm{L}^1;$$

(ii) $P_T(f^*) = f^*$;

(iii) $V_{[0,1]}(f^*) \le C \cdot \|f\|_1$, for some constant $C > 0$, which is independent of $f \in L^1$.

3. ABSTRACT RESULTS ON THE FIXED POINTS OF OPERATORS

Let $\mathcal{M}(I)$ be the space of all measures on $(I, \mathcal{B})$. The transfer operator $\mathcal{D}$ in (2.3) or in (**??**) which is defined in Section 2 is a an operator $\mathcal{D} : \mathcal{M}(I) \to \mathcal{M}(I)$. Let $\mathcal{H}$ be an invariant normed subspace of $\mathcal{M}$. Consider a restiction of $\mathcal{D}$ from $\mathcal{H}$ into $\mathcal{H}$. For simplicity, we denote the restricted operator again by $\mathcal{D}$. Let $\|\cdot\|_{\mathcal{H}}$ denotes the norm on $\mathcal{H}$. For $\delta \in \mathbb{R}$, let $\mathcal{D}_\delta$ be a finite dimensional approximation of $\mathcal{D}$ and we assume that we can compute the fixed points of $\mathcal{D}_\delta$. The parameter δ measures the accuracy of the approximation (for example, the size of a grid). Let $\nu, \nu_\delta \in \mathcal{H}$ be the fixed point of $\mathcal{D}$ and $\mathcal{D}_\delta$ respectively. In our approach of approximation ν, first, we want to get as much information as possible for the operator $\mathcal{D}_\delta$ and use these information to approximate ν. Recall the following abstract result which was proved in [12]:

Theorem 3.1. *Suppose that*

1. $\| D_\delta \nu - D\nu \|_{\mathcal{H}} < \infty$,
2. *there exists a positive integer $\bar{N}$ such that* $\| D_\delta^{\bar{N}}(\nu_\delta - \nu) \|_{\mathcal{H}} < \frac{1}{2} \| \nu_\delta - \nu \|_{\mathcal{H}}$,
3. *for each i, there exists C_i such that for all $g \in \mathcal{H}$,* $\| D_\delta^i g \|_{\mathcal{H}} < C_i \| g \|_{\mathcal{H}}$.

Then

$$\| \nu_\delta - \nu \|_{\mathcal{H}} \le 2 \| D_\delta \nu - D\nu \|_{\mathcal{H}} \sum_{i \in [0, \bar{N}-1]} C_i. \tag{3.1}$$

4. AN ELEMENTARY METHOD FOR ERROR BOUND ESTIMATION OF ULAM'S METHOD FOR RANDOM MAPS

4.1. ULAM'S METHOD FOR RANDOM MAPS

In this subsection, we consider position dependent random map

$$T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\}$$

satisfying the following assumptions:

(B) T has a unique acim μ with density f^*.

Now, we describe Ulam's method for T. Let n be a positive integer. Let $\mathcal{P}^{(n)} = \{I_1, I_2, \dots, I_n\}$ be a partition of the interval $[0,1]$ into n equal subintervals. We assume that $\max_{I_i \in \mathcal{P}^{(n)}} \lambda(I_i)$ goes to 0 as $n \to \infty$. Let F_n be the σ-algebra generated by the partition $\mathcal{P}^{(n)}$. For each $1 \le k \le K$, create the matrix

$$\mathbb{M}_k^{(n)} = \left(\frac{\lambda\left(\tau_k^{-1}(I_j) \cap I_i\right)}{\lambda(I_i)} \right)_{1 \le i,j \le n}.$$

Let $L^{(n)} \subset L^1([0,1], \lambda)$ be a subspace of L^1 consisting of functions which are constant on elements of the partition $\mathcal{P}^{(n)}$. We will represent functions in $L^{(n)}$ as vectors: vector $f = [f_1, f_2, \dots, f_n]$ corresponds to the function $f = \sum_{i=1}^n f_i \chi_{I_i}$. Let $Q^{(n)}$ be the isometric projection of L^1 onto $L^{(n)}$:

$$Q^{(n)}(f) = \sum_{i=1}^n \left(\frac{1}{\lambda(I_i)} \int_{I_i} f d\lambda \right) \chi_{I_i} = \left[\frac{1}{\lambda(I_1)} \int_{I_1} f d\lambda, \dots, \frac{1}{\lambda(I_n)} \int_{I_n} f d\lambda \right].$$

It can be easily shown that $Q^n f = \mathbb{E}(f|F_n), f \in L^1$. We define the operator $P_T^{(n)} : L^{(n)} \to L^{(n)}$ by

$$P_T^{(n)} = \sum_{k=1}^K p_k \left(\mathbb{M}_k^{(n)} \right)^{\mathrm{C}}, \tag{4.1}$$

where C denotes the transpose of a matrix. Note that $P_T^{(n)}$ is a finite dimensional approximation to the operator P_T. It can be shown that

$$P_T^{(n)} = Q^n \circ P_T \circ Q^n.$$

Equivalently, for $f \in L^1$,

$$P_T^{(n)} f = \mathbb{E}\left(P_T \left(\mathbb{E}\left(f | F_n \right) \right) | F_n \right),$$

where F_n is the $\sigma-$algebra associated to the partition $\mathcal{P}^{(n)}$. Ulam's matrix for position dependent random map with respect to the partition $\mathcal{P}^{(n)}$ is

$$\mathbb{M}^{*(n)}_{\mathcal{P}^{(n)}} = \sum_{k=1}^{K} \left(\mathbb{M}^{(n)}_{k}\right)^{C} \left(\left[p^{(n)}_{k,1}, p^{(n)}_{k,2}, \ldots, p^{(n)}_{k,n}\right]\right)^{C}.$$

Note that if the probabilities does not depend on position x, then the Ulam's matrix $\mathbb{M}^{*(n)}_{\mathcal{P}^{(n)}}$ reduces to

$$\mathbb{M}^{*(n)}_{\mathcal{P}^{(n)}} = \sum_{k=1}^{K} p_k \left(\mathbb{M}^{(n)}_{k}\right)^{C}.$$

In this way the random map is approximated by the finite state Markov chain with transition probabilities $m_{ij}, 1 \leq i, j \leq n$, where m_{ij} is the (i, j)th element of the Ulam's matrix $\mathbb{M}^{*(n)}_{\mathcal{P}^{(n)}}$.

4.2. ESTIMATION

We are interested in connecting Theorem 3.1 for explicit estimation for approximation error in Ulam's descretization method with L^1 norm. We assume that the norm $\| \nu \|_{BV}$ can be estimated and there is an estimation for the norm $\| P^{(n)}_T - P_T \|_{BV\to L^1}$. Thus, the left hand side of condition (1) in Theorem 3.1 reduces to

$$\| P^{(n)}_T \nu - P_T \nu \|_{L^1} \leq \| P^{(n)}_T - P_T \|_{BV\to L^1} \| \nu \|_{BV}.$$

Note that $\| \nu \|_{BV}$ is possible if P_T satisfies condtions similar to (2.6). With these assumptions we have from Theorem 3.1 that

$$\| \nu_\delta - \nu \|_{BV} \leq 2 \sum_{i\in[0,N-1]} C_i \| P^{(n)}_T - P_T \|_{BV\to L^1} \| \nu \|_{L^1}.$$

Our main goal in this section is to determine $\bar{N}$ in the Theorem 3.1. Recall that $P^{(n)}_T$ is the Ulam's approximation of the Frobenius–Perron operator P_T. Set

$$V_0 = \{f \in L_1([0, 1]) : \int f d\lambda = 0\}.$$

Note that $\nu - \nu_\delta \in V_0$. Therefore, if we prove $\| \left(P^{(n)}_T\right)^{\bar{N}} |V_0 \|_{L^1\to L^1} < \frac{1}{2}$ we imply condition (2) in Theorem 3.1. We consider the set of functions in $L^{(n)}$ with integral zero and for convenience we denote this set by V_0. In order to determine $\bar{N}$ in the Theorem 3.1, we consider the Ulam matrix $\mathbb{M}^{*(n)}_{\mathcal{P}^{(n)}}|V_0$ restricted to the set V_0. Consider the matrix norm of the Ulam's matrix, $\| \mathbb{M}^{*(n)}_{\mathcal{P}^{(n)}}|V_0 \|_1 = \sup_{|x|_1=1}|\mathbb{M}^{*(n)}_{\mathcal{P}^{(n)}}(x)|$. Let $\mathcal{I} : \mathbb{R}^n \to L^1$ is the trivial identification of a vector in $\mathbb{R}^n$ with a piecewise constant function given by the choice of the basis. Then, we have the following (see Section 4.1 in [12]) :

$$\| P_T^{(n)} \|_{L_1 \to L_1} \leq \| \mathbb{M}^{*(n)}_{\mathcal{P}(n)} \|_1;$$
$$\| P_T^{(n)}|V_0 \|_{L_1} \leq \| \mathbb{M}^{*(n)}_{\mathcal{P}(n)}|I^{-1}(V_0) \|_1;$$
$$\| \left(P_T^{(n)}\right)^{\bar{N}} |V_0 \|_{L_1} = \| \left(\mathbb{M}^{*(n)}_{\mathcal{P}(n)}\right)^{\bar{N}} |I^{-1}(V_0) \|_1 .$$

In this way, we can have an estimation of $\| \left(P_T^{(n)}\right)^{\bar{N}} |_{V_0} \|_{L_1 \to L_1}$ by computing a matrix $\hat{\mathbb{M}}^{*(n)}_{\mathcal{P}(n)}$ approximating $\mathbb{M}^{*(n)}_{\mathcal{P}(n)}|I^{-1}(V_0)$ and $\| \left(\hat{\mathbb{M}}^{*(n)}_{\mathcal{P}(n)}\right)^{\bar{N}} \|_1$. Now, compute $\| \left(\hat{\mathbb{M}}^{*(n)}_{\mathcal{P}(n)}\right)^{j} \|_1$ for each $j > 0$ iteratively form $\left(\hat{\mathbb{M}}^{*(n)}_{\mathcal{P}(n)}\right)^{j-1}$ until it finds some j for which it can deduce that $\| \left(\mathbb{M}^{*(n)}_{\mathcal{P}(n)}\right)^{j} |V_0 \|_1 \leq \frac{1}{2}$. This j will the the output as $\bar{N}$ required in Theorem 3.1.

4.2.1. ALGORITHM

1. Input the random map and the partition.
2. Compute the matrix $\hat{\mathbb{M}}^{*(n)}_{\mathcal{P}(n)}$ approximating $P_T^{(n)}|V_0$ and the the corresponding approximated fixed point $\hat{f}_n$ upto some required approximation ϵ_1.
3. Compute ΔL, an estimate for $\| P_T^{(n)} f - P_T f \|$ upto some error ϵ_2
4. Compute $\bar{N}$ in Theorem 3.1 which is described above.
5. If all computations ends successfully, output $\hat{f}_n$.

We have the following lemma:

Lemma 4.1. *$I^{-1}(\hat{f}_n)$ is an approximation the invariant measure upto an error ϵ given by*

$$\epsilon \leq \epsilon_1 + 2\bar{N}(\Delta L + \epsilon_2)$$

4.3. EXPLICIT ESTIMATION OF THE COEFFICIENTS OF THE LASOTA – YORKE INEQUALITY FOR RANDOM MAPS

In this section we present explicit estimation of the co-efficient of the Lasota-Yorke inequality for random maps with respect to the transfer operator $\mathcal{D}$ (see Eq. (2.3) and Eq. (**??**)).

Consider the following semi-norm for measures on $(I, \mathcal{B})$.

$$\| \mu \| = \sup_{\phi \in C^1, |\phi|_\infty = 1} |\mu(\phi')|. \tag{4.2}$$

Moreover, consider the space of measures, $\mathcal{M}' = \{\mu :\| \mu \|< \infty\}$.

Theorem 4.2. *If $\| \mu \|< \infty$, then μ is absolutely continuous with respect to the Lebesgue measure.*

Proof. See Lemma 1.1 in [23]. □

4.3.1. EXPLICIT ESTIMATION OF THE COEFFICIENTS FOR RANDOM MAPS WITH CONSTANT PROBABILITIES

We consider random maps with constant probabilities.

Theorem 4.3. *Let $T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\}$ be an i.i.d. random map on $I = [0,1]$, where $\tau_k, k = 1, 2, \dots, K$ are piecewise C^2 Lasota-Yorke maps on a common partion $0 = d_1, d_2, \dots, d_n = 1$. If the random map T satisfies the Pelikan's average expanding condition (**??**) and μ is a measure on $[0,1]$, then*

$$\| \mathcal{D}\mu \| \leq \lambda \| \mu \| + B'|\mu|_1, \tag{4.3}$$

where

$$\lambda = \left(\sum_{k=1}^{K} \frac{2p_k}{\inf \tau_k'} \right), \; B' = \left(\sum_{k=1}^{K} p_k \left(\frac{2}{\min(d_i - d_{i+1})} + 2 | \frac{\tau_k''}{(\tau')^2} |_\infty \right) \right)$$

and $|\mu|_1$ is defined in (4.4).

Proof.

$$\mathcal{D}\mu(\phi') = \sum_{Z \in \{(d_i, d_{i+1}) | i \in \{1,2,\dots,n-1\}\}} \mathcal{D}\mu(\phi' \chi_Z).$$

For each Z define ϕ_Z to be linear such that $\phi_Z = \phi$ on the boundary of Z. Moreover, define $\psi_Z = \phi - \phi_Z$ on Z and extend it to $[0,1]$ by setting it to zero outside Z. In this way, we obtain a continuous function. Moreover, for each $x \in Z$,

$$|\phi_Z'|_\infty \leq \frac{2|\phi|_\infty}{\min(d_i - d_{i+1})}.$$

Now,

$$\begin{aligned} \mathcal{D}\mu(\phi') &= \sum_{Z \in \{(d_i, d_{i+1}) | i \in \{1,2,\dots,n-1\}\}} \mathcal{D}\mu(\phi' \chi_Z) \\ &= \sum_Z \mathcal{D}\mu(\psi_Z' \chi_Z) + \sum_Z \mathcal{D}\mu(\phi_Z' \chi_Z) \end{aligned}$$

Thus,

$$|\mathcal{D}\mu(\phi')| = | \sum_{k=1}^{K} p_k \sum_Z \left(\mu \left(\psi_Z' \circ \tau_k \chi_{\tau_k^{-1}(Z)} \right) + \mu \left(\phi_Z' \circ \tau_k \chi_{\tau_k^{-1}(Z)} \right) \right) |.$$

It can be easily shown that, on Z,

$$\psi_Z' \circ \tau_k = \left(\frac{\psi_Z \circ \tau_k}{\tau_k'}\right) + \frac{(\psi_Z \circ \tau_k)\,\tau_k''}{(\tau_k')^2}, k = 1, 2, \dots, K.$$

Thus,

$$\begin{aligned}|\mathcal{D}\mu(\phi')| \leq &\sum_{k=1}^{K} p_k \left(|\sum_Z \mu\left(\left(\frac{\psi_Z \circ \tau_k}{\tau_k'}\right)' \chi_{\tau_k^{-1}(Z)}\right)| \right.\\ &\left. + |\sum_Z \mu\left(\frac{(\psi_Z \circ \tau_k)\,\tau_k''}{(\tau_k')^2}\chi_{\tau_k^{-1}(Z)}\right)| + \frac{2|\phi|_\infty}{\min(d_i - d_{i+1})}\mu(1)\right) \\ q \leq &\sum_{k=1}^{K} p_k \left(|\mu\left(\left(\frac{\psi_Z \circ \tau_k}{\tau_k'}\right)'\right)| + 2|\phi|_\infty \mu\left(|\frac{\tau_k''}{(\tau')^2}|\right) + \frac{2|\phi|_\infty}{\min(d_i - d_{i+1})}\mu(1)\right).\end{aligned}$$

The function $\sum_Z \frac{\psi_Z \circ \tau_k}{\tau_k'}$ is not C^1, because its derivative has a finite number of points of discontinuity. This function can be approximated by a C^1 function ψ_ϵ such that $|\psi_\epsilon - \sum_Z \frac{\psi_Z \circ \tau_k}{\tau_k'}|$ and $\mu\left(|\psi_\epsilon - \sum_Z \frac{\psi_Z \circ \tau_k}{\tau_k'}|\right)$ are as small as wanted. It is shown in [12] (see also [23]) that

$$|\mu\left(\left(\frac{\psi_Z \circ \tau_k}{\tau_k'}\right)'\right)| \leq \| \mu \| \frac{2}{\inf \tau_k'}|\phi|_\infty, k = 1, 2, \dots, K$$

Thus,

$$|\mathcal{D}\mu(\phi')| \quad \leq \quad \sum_{k=1}^{K} p_k \left(\| \mu \| \frac{2}{\inf \tau_k'}|\phi|_\infty + 2|\phi|_\infty \mu\left(|\frac{\tau_k''}{(\tau_k')^2}|\right) + \frac{2|\phi|_\infty}{\min(d_i - d_{i+1})}\mu(1)\right).$$

Now,

$$\| \mathcal{D}\mu \| \quad \leq \quad \sum_{k=1}^{K} p_k \left(\frac{2}{\inf \tau_k'} \| \mu \| + 2\mu\left(|\frac{\tau_k''}{(\tau_k')^2}|\right) + \frac{2}{\min(d_i - d_{i+1})}\mu(1)\right)$$

Define

$$|\mu|_1 = \sup_{|\phi|_\infty = 1} |\mu(\phi)|. \tag{4.4}$$

Then,

$$\begin{aligned}\| \mathcal{D}\mu \| \quad &\leq \quad \sum_{k=1}^{K} p_k \left(\frac{2}{\inf \tau_k'} \| \mu \| + \left(\frac{2}{\min(d_i - d_{i+1})} + 2|\frac{\tau_k''}{(\tau_k')^2}|_\infty \right) |\mu|_1 \right) \\ &= \quad \left(\sum_{k=1}^{K} \frac{2p_k}{\inf \tau_k'} \right) \| \mu \| + \left(\sum_{k=1}^{K} p_k \left(\frac{2}{\min(d_i - d_{i+1})} + 2|\frac{\tau_k''}{(\tau_k')^2}|_\infty \right) \right) |\mu|_1\end{aligned}$$

Let

$$\lambda = \left(\sum_{k=1}^{K} \frac{2p_k}{\inf \tau_k'} \right), \; B' = \left(\sum_{k=1}^{K} p_k \left(\frac{2}{\min(d_i - d_{i+1})} + 2 | \frac{\tau_k''}{(\tau_k')^2} |_\infty \right) \right).$$

Then,

$$\| \mathcal{D}\mu \| \quad \leq \quad \lambda \| \mu \| + B' |\mu|_1.$$

□

It is not difficult to show that for any integer $l \geq 1$,

$$\| \mathcal{D}^l \mu \| \quad \leq \quad \lambda^l \| \mu \| + \frac{1}{1-\lambda} B' |\mu|_1.$$

Let $B = \frac{B'}{1-\lambda}$. Then

$$\| \mathcal{D}^l \mu \| \quad \leq \quad \lambda^l \| \mu \| + B |\mu|_1. \tag{4.5}$$

Lemma 4.4. *If f is a fixed point of the Frobenius-Perron operator P_T, then*

$$\| P_T f - P_T^{(n)} \|_{L^1} \leq \frac{2}{n} \| f \|_{L^1}$$

Proof.

$$\begin{aligned} \| P_T f - P_T^{(n)} f \|_{L^1} = & \| P_T^{(n)} f - P_T f \|_{L^1} = \| P_T^{(n)} f - E(P_T f | \mathcal{F}_n) \\ & + E(P_T f|\mathcal{F}_n) - P_T f \|_{L^1} \\ & \leq \| P_T^{(n)} f - E(P_T f|\mathcal{F}_n) \|_{L^1} + \| E(P_T f|\mathcal{F}_n) - P_T f \|_{L^1} \\ & = \| \mathbb{E} \left(P_T \left(\mathbb{E} \left(f|F_n \right) \right) |F_n \right) - E(P_T f|\mathcal{F}_n) \|_{L^1} \\ & \; + \| E(P_T f|\mathcal{F}_n) - P_T f \|_{L^1} \\ & = \| \mathbb{E} \left[P_T \left(\mathbb{E} \left(f|\mathcal{F}_n \right) \right) |\mathcal{F}_n \right] \|_{L^1} + \| E(P_T f|\mathcal{F}_n) - P_T f \|_{L^1} \\ & \leq \| E(P_T f|\mathcal{F}_n) - P_T f \|_{L^1} + \| E(P_T f|\mathcal{F}_n) - P_T f \|_{L^1} \\ & = 2 \| E(P_T f|\mathcal{F}_n) - P_T f \|_{L^1} . \end{aligned}$$

Note that

$$\sum_i |\sup_{I_i}(f) - \inf_{I_i}(f)| \leq \| f \|_{L^1} .$$

Moreover,

$$\inf_{I_i}(f) \leq \mathbb{E} \left(f|I_i \right) \leq \sup_{I_i}(f),$$

where I_i are varius intervals composing the sigma algebra $\mathcal{F}$. Thus,

$$\int_{I_i} |\mathbb{E} \left(f|\mathcal{F}_n \right) - f| \leq \frac{1}{n} |\sup_{I_i}(f) - \inf_{I_i}(f)|$$

and hence

$$\| \mathbb{E}(f|\mathcal{F}_n) - f \|_{L^1} \leq \frac{1}{n} \| f \|_{L^1} .$$

Therefore,

$$\| P_T f - P_T^{(n)} \|_{L^1} \leq \frac{2}{n} \| f \|_{L^1}$$

□

Note that if f is a fixed point of P_T, then $\| P_T f \|_{L^1} \leq \| f \|_{L^1}$ and $\| \mathbb{E}(f|\mathcal{F}_n) \|_{L^1} \leq \| f \|_{L^1}$. Note also that $P_T^{(n)}$ is a composition of P_T and $\mathbb{E}$. Thus, it is not difficult to show that each of the constants C_i in Theorem 3.1 is 1.

Theorem 4.5. *If the random map $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ satisfies the Pelikan's average expanding condition (**??**) and T has a unique invariant measure μ, then it is possible to approximate the invariant measure at any precision with the above algorithm.*

Proof. The proof follows from the proof of Theorem 5.7 in [12]. For the convenient of readers we repeat the proof. Both P_T and $P_T^{(n)}$ satisfies the same L-Y inequality and $\| P_T - P_T^{(n)} \|_{BV \to L^1} \to 0$ as $\delta \to 0$. By the result of Liverani (Proposition 3.1 and Lemma 6.1) the spectral gap of P_T combined with the stability of the spectral picture implies that there are $A^*, \beta \in \mathbb{R}, \beta < 1$, independent of n such that for n large enough, $P_T^{(n)}$ satisfies $\| \left(P_T^{(n)}\right)^l |V \|_{BV \to BV} \leq A^* \beta^l$. Since $\| \mathbb{E}(g|\mathcal{F}_n) \geq 2n \| P_T^{(n)} \|_{BV \to BV}$, this implies that

$$\| \left(P_T^{(n)}\right)^l |V \|_{L^1 \to L^1} \leq 2n \| \left(P_T^{(n)}\right)^l \|_{BV \to BV} \leq 2nA^* \beta^l.$$

Hence if $l \geq \frac{\log(\frac{1}{4nA^*})}{\log \beta}$, then $\| \left(P_T^{(n)}\right)^l |V \|_{L^1 \to L^1} \leq \frac{1}{2}$ and the algorithm stop. Moreover, upto multiplying constants, the error will be $O(\frac{\log n}{n})$ (see the proof of Theorem 5.7 in [12]) and can be made as small as possible. □

5. IMPLEMENTING THE ALGORITHM

5.1. COMPUTING THE ULAM APPROXIMATION

Let $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ be a random map on a common partition $\{J_1, J_2, \ldots, J_n\}$ of $[0, 1]$ statisfying Pelikan's condition (2.7) in Theorem 2.5. Thereofore, there exists an acim μ^* with a density f^*. Moreover, we assume that μ^* is unique. In the following we present the implimentation of our algorithm.

Let N be a multiple of n and $\mathcal{P}^{(N)} = \{I_1, I_2, \ldots, I_N\}$ be a partion of $[0,1]$. Then each of the component map $\tau_k, k = 1, 2, \ldots, K$ is monotonic on $I_i, i = 1, 2, \ldots, N$. Recall that the Ulam's matrix (see Section 4.1) with respect to the partition $\mathcal{P}^{(N)}$ is

$$\mathbb{M}^{*(N)}_{\mathcal{P}^{(N)}} = \sum_{k=1}^{K} p_k \left(\mathbb{M}^{(N)}_k\right)^C,$$

where $\left(\mathbb{M}^{(N)}_k\right)^C$ is the matrix representation of the Frobenius-Perron operator of $\tau_k, k = 1, 2, \ldots, K$. The random map $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ is approximated by the finite state Markov chain with transition probabilities $m_{ij}, 1 \le i, j \le N$, where m_{ij} is an element of the Ulam's matrix $\mathbb{M}^{*(N)}_{\mathcal{P}^{(N)}}$. Our algorithm does not calculate the Ulam's matrix directly. The main target of our algorithm is to compute a rigorous approximation of the related Markov chain and a faster method to rigorously approximate its steady state. With our rigorous algorithm we compute a matrix $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$ which approximate the Ulam matrix $\mathbb{M}^{*(N)}_{\mathcal{P}^{(N)}}$. We use the following algorithm to compute the matrix $\hat{\mathbb{M}}'^{*(N)}_{\mathcal{P}^{(N)}}$ which is preliminary to compute $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$.

Algorithm for computing $\hat{\mathbb{M}}'^{*(N)}_{\mathcal{P}^{(N)}}$: Let $\hat{\mathbb{M}}'^{*(N)}_{\mathcal{P}^{(N)}} = \left(\hat{m_{ij}}'^{*(N)}_{1\le i,j\le N}\right)$. In the following we describe an algorithm for random maps $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ with constant probabilities $p_1, p_2, \ldots, p_K$.

step 1: Set $\hat{m_{ij}}'^{*(N)} = 0$ for $i = 1, 2, \ldots, N, j = 1, 2, \ldots, N$.

step 2: for $j = 1, 2, \ldots, N$ do

for $i = 1, 2, \ldots, N$, partition $I_i, i = 1, 2, \ldots, N$ into m intervals $I_{i,l}, l = 1, 2, \ldots, m$.

for i from 1 to N do for k from 1 to K do

set **sum** $= 0$

for l from 1 to m do

compute $\tau_k(I_{i,l})$.

if $\tau_k(I_{i,l}) \subset I_j$ then **sum** $=$ **sum** $+ \lambda(\mathrm{I_{i,l}})$

if $\tau_k(I_{i,l}) \subset (I_j)^C$ then go to the next step.

if $\tau_k(I_{i,l}) \cap I_j \neq \emptyset$ and $\tau_k(I_{i,l}) \cap (I_j)^C \neq \emptyset$ and $\lambda(I_{i,l}) > \nu$ then divide $I_{i,l}$ into m intervals

and follow the above steps

if $\tau_k(I_{i,l}) \cap I_j \neq \emptyset$ and $\tau_k(I_{i,l}) \cap (I_j)^C \neq \emptyset$ and $\lambda(I_{i,l}) < \nu$ then add $\lambda(I_{i,l})$ to $\epsilon_{k,ij}$, the

error corrosponding τ_k and $I_{i,l}$ and go to the next step

end do.

end do.

sub[k]=sum.

end do.

$\hat{m}'^{*(N)}_{ij,\mathcal{P}^{(N)}} = p_1 sub[1] + p_2 sub[2] + \cdots + p_K sub[K]$ and $error_{ij} = p_1\epsilon_{1,ij} + p_2\epsilon_{2,ij} \cdots + p_K\epsilon_{K,ij}$

end do.

By applying the above algorithm, we obtain the matrix $\hat{\mathbb{M}}'^{*(N)}_{\mathcal{P}^{(N)}}$. Let $\epsilon = \max_{i,j} error_{ij}$. Note that the matrix $\hat{\mathbb{M}}'^{*(N)}_{\mathcal{P}^{(N)}}$ is not a stochastic matrix. For the rest of the algorithm we closely follow [12]. For each row, we split the difference of the absolute value of the sum of the nonzero entiries and 1 equaly. In this way we obtain a stochastic matrix $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$. Let ϵ be the maximum of errors $|\hat{\mathbb{M}}'^{*(N)}_{ij,\mathcal{P}^{(N)}} - \mathbb{M}^{*(N)}_{ij,\mathcal{P}^{(N)}}|$ and let nnz_i be the number of nonzero elements of the row. It is easy to see that for each row i the sum of its entries differs from 1 by at most $nnz_i \cdot \epsilon$. Thus, the stochastic matrix $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$ satisfy

$$|\hat{\mathbb{M}}^{*(N)}_{ij,\mathcal{P}^{(N)}} - \mathbb{M}^{*(N)}_{ij,\mathcal{P}^{(N)}}| < 2\epsilon.$$

Let $NNZ = \max_i nnz_i$, then the matrix $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$ is such that

$$\| \mathbb{M}^{*(N)}_{\mathcal{P}^{(N)}} - \hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}} \|_1 < 2 \cdot NNZ \cdot \epsilon.$$

The largest eigenvalue of $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$ is 1 because $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$ is a stochastic matrix. Theorem 3.1 allows us to have a rigorous estimate of the L^1 distance between the eigenvectors of $\mathbb{M}^{*(N)}_{\mathcal{P}^{(N)}}$ and $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$. Note that, Remark 8.1 of [12] implies that

$$NNZ \leq \sum_{k=1}^{K} \sup|\tau_k'| + 4K.$$

In the following two sections (Sec. 5.2 and Sec. 5.3 below) we follow [12] very closley. The derivations in these two sections are very similar to the drivations of Section 8.3 and Section 8.4 in [12]. For the convenience of the reader we present the derivations.

5.2. COMPUTING RIGOROUSLY THE STEADY STATE VECTOR AND THE ERROR

In the following we use the power iteration method for the steady state of $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$. Let b_0 be any initial condition. The power iteration method states that if $b_{l+1} = b_l \cdot \hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$, then b_l converges to the steady state of $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}$.

For $x \in \mathbb{R}^N$, define the norm $\| x \|$ by $\| x \| = \sum_{i=1}^{N} |x_i|$. Moreover, let $\triangle = \{x \in \mathbb{R}^N | x_i \geq 0, i = 1, 2, \ldots, N, \| x \|_1 = 1\}$ be the nonnegative $(N-1)$-dimensional simplex. From the proof of the Perron-Fobenius theorem [**?**] it is well known that a Markov matrix (aperiodic, irreducible) contracts the simplex $\triangle$ of vectors v having norm $\| v \|$ is equal to 1. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis of the simplex. Then the simples is given by the convex combination of the vectors of the base. Let Diam$-$ be the diameter in the distance induced by the norm $\| \cdot \|_1$. Then,

$$\begin{aligned}\mathrm{Diam}(\mathrm{A}^1\triangle) &\leq \max_{i,j} \| A^l(e_i - e_j) \|_1 \leq \max_{i,j} \| A^l(e_1 - e_j) \|_1 + \max_{i,j} \| A^l(e_1 - e_i) \|_1 \\ &\leq 2 \max_{i} \| A^l(e_1 - e_i) \|_1 .\end{aligned}$$

Now, we fix an input threshold ϵ_{num}. Then, we iterate the vectors $\{v_1 - v_j\}_{j=2}^{k}$ and look at their norm until we find an l such that $\mathrm{Diam}(\mathrm{A}^1\triangle) < \epsilon_{\text{num}}$. For any initial condition b_0 iterating it l times, we get a vector contained in $A^l(\triangle)$ whose numerical error is enclosed by ϵ_{num}.

5.3. ESTIMATION OF THE RIGOROUS ERROR FOR THE INVARIANT MEASURE

Now, we compute the number of iteration $\bar{N}$ needed for the Ulam approximation $P_T^{(n)}$ to contract to $\frac{1}{2}$ the space of average 0 vectors. Note that the vectors $\{e_1 - e_j\}_{j=1}^{k}$ are a base for the space of average 0 vectors. Now,

$$\| \left(P_T^{(n)}\right)^j |_V \|_1 \quad \leq \quad \| \left(\left(\mathbb{M}^{*(N)}_{\mathcal{P}^{(N)}}\right)^j - \left(\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}\right)^j + \left(\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}^{(N)}}\right)\right)^j |_V \|_1$$

$$\leq \quad \| \left(\left(\mathbb{M}^{*(N)}_{\mathcal{P}(N)} \right)^j - \left(\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}(N)} \right)^j \right) |_V \|_1 + \| \left(\left(\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}(N)} \right) \right)^j |_V \|_1 .$$

Moreover,

$$\begin{aligned} \| \left(\mathbb{M}^{*(N)}_{\mathcal{P}(N)} \right)^j - \left(\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}(N)} \right)^j |_V \|_1 \leq & \sum_{i=1}^{j} \| \left(\mathbb{M}^{*(N)}_{\mathcal{P}(N)} \right)^{j-i} |_V \|_1 \\ & \cdot \| \mathbb{M}^{*(N)}_{\mathcal{P}(N)} - \mathbb{M}^{*(N)}_{\mathcal{P}(N)} |_V \|_1 \cdot \| \left(\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}(N)} \right)^{i-1} |_V \|_1 \\ \leq & 2 \cdot j \cdot \mathrm{NNZ} \cdot \epsilon, \end{aligned}$$

because $\| \left(\mathbb{M}^{*(N)}_{\mathcal{P}(N)} \right)^j |_V \|_1 \leq 1$ and $\| \left(\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}(N)} \right)^{jh} |_V \|_1 \leq 1$ for every j and h. Thereofore,

$$\| \left(\mathbb{M}^{*(N)}_{\mathcal{P}(N)} \right)^j |_V \|_1 \leq 2 \cdot j \cdot \mathrm{NNZ} \cdot \epsilon + \| \left(\hat{\mathbb{M}}^{*(\mathrm{N})}_{\mathcal{P}(\mathrm{N})} \right)^{\mathrm{j}} |_{\mathrm{V}} \|_1 .$$

Thus, if ϵ and j are small enough then we can estimate the number $\bar{N}$ of iterates needed for $\mathbb{M}^{*(N)}_{\mathcal{P}(N)}$ to contract the space V_0 by the number of iterates needed by the matrix $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}(N)}$.

Theorem 5.1. *Let $f, v_N, \hat{v}_N$ be the fixed point of $P_T, \mathbb{M}^{*(N)}_{\mathcal{P}(N)}, \hat{\mathbb{M}}^{*(N)}_{\mathcal{P}(N)}$ respectively and $\underline{v}$ be numerical approximation of $\hat{v}_N$, then*

$$\| f - \underline{v} \|_1 \leq 2\bar{N}\frac{2B}{N} + 4N_\epsilon \cdot \mathrm{NNZ} \cdot \epsilon + \epsilon_{\mathrm{num}}.$$

Proof.

$$\| f - \underline{v} \|_1 \leq \| f - v_N \|_1 + \| v_N - \hat{v}_N \|_1 + \| \hat{v}_N - \underline{v} \|_1 .$$

Let N_ϵ be number of iterates needed for $\hat{\mathbb{M}}^{*(N)}_{\mathcal{P}(N)}$ to contract to $\frac{1}{2}$ the space of average zero vectors. Then by Theorem 3.1,

$$\| v_N - \hat{v}_N \|_1 \leq 2N_\epsilon \| \mathbb{M}^{*(N)}_{\mathcal{P}(N)} - \hat{\mathbb{M}}^{*(N)}_{\mathcal{P}(N)} \|_1 \| v_N \|_1 \leq 4N_\epsilon \cdot \mathrm{NNZ} \cdot \epsilon.$$

Thus,

$$\| f - \underline{v} \|_1 \leq 2\bar{N}\frac{2B}{N} + 4N_\epsilon \cdot \mathrm{NNZ} \cdot \epsilon + \epsilon_{\mathrm{num}}.$$

□

6. NUMERICAL EXPERIMENT

Example 6.1. Consider the random map $T = \{\tau_1(x), \tau_2(x); p_1, p_2\}$, where $\tau_1, \tau_2 : [0,1] \to [0,1]$ are defined by

$$\tau_1(x) = \begin{cases} \frac{17}{5}x, & 0 \le x < \frac{5}{17}, \\ \frac{17}{5}x - 1, & \frac{5}{17} \le x < \frac{10}{17}, \\ \frac{17}{5}x - 2, & \frac{10}{17} \le x < \frac{15}{17}, \\ \frac{17}{5}x - 3, & \frac{15}{17} \le x \le 1, \end{cases}$$

$$\tau_2(x) = \begin{cases} 2x, & 0 \le x < \frac{5}{17}, \\ 2x - \frac{5}{17}, & \frac{5}{17} \le x < \frac{10}{17}, \\ 2x - \frac{20}{17}, & \frac{10}{17} \le x < \frac{15}{17}, \\ \frac{15}{2}x - \frac{225}{34}, & \frac{15}{17} \le x \le 1, \end{cases}$$

$$p_1 = \frac{2}{5}, p_2 = \frac{3}{5}$$

It is easy to show that the random map T satisfies Pelikan's average expanding condition (2.7). Thus, T has an acim $\hat{\mu}$ with density $\hat{f}$. It is easy to show that both τ_1 and τ_2 has unique acim. Thus, the random map $T = \{\tau_1(x), \tau_2(x); p_1, p_2\}$ also has a unique acim (see Proposition 1 in [13]) and thus $\hat{\mu}$ is unique with density $\hat{f}$. Here, $\lambda = 2 \cdot \sum_{k=1}^{2} \frac{p_k}{\inf \tau_k'} = \frac{10}{17}$, $B' = \left(\sum_{k=1}^{K} p_k \left(\frac{2}{\min(d_i - d_{i+1})} + 2|\frac{\tau_k''}{(\tau')^2}|_\infty\right)\right) = 17$ and $B = \frac{B'}{1-\lambda} = 7$. By choosing appropreiate $N, \epsilon, \epsilon_{\text{num}}$ one can find N_ϵ, l and $\bar{N}$ as outputs. Using these inputs and outputs, one can estimate $\| \hat{f} - \underline{v} \|_1$.

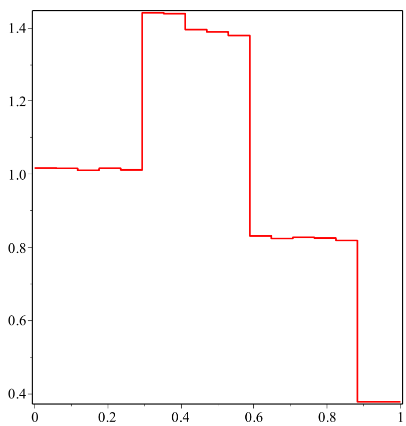

Figure 1: The graph of an approximate density $\underline{v}$ of the actual density $\hat{f}$ of the random map T in Example 6.1.

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