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SOME ANALYTICAL SOLUTIONS FOR THE EULER AND EULER-POISSON EQUATIONS

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ABSTRACT. In this article, we obtain a family of exact solutions for the 3-dimensional non-isentropic compressible pressureless Euler equations the Euler-Poisson equations with pressure in both the attractive and repulsive cases. Here, the exact solutions for the pressureless Euler equations are with arbitrary function parameters.

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Key Words: Euler Equations, Euler-Poisson Equations, Exact Solution, Symmetry Reduction, Function Parameter

1. Introduction and Main Results

In fluid dynamics, the 3-dimensional non-isentropic compressible Euler or Euler-Poisson equations are expressed as follows:

$$(1)\qquad \begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) &= 0 \\ \rho[\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}] + \nabla P &= \delta\rho\nabla\Phi \\ S_t + \mathbf{u}\cdot\nabla S &= 0 \\ \Delta\Phi &= \rho, \end{cases}$$

where $\rho = \rho(t, \mathbf{x})$ denotes the density of the fluid, $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) = (u_1, u_2, u_3) \in \mathbf{R}^3$ is the velocity, $\mathbf{x} = (x, y, z) \in \mathbf{R}^3$. The first equation of (1) is called the mass equation which is derived from the law of conservation of mass; the second equations are called the momentum equations which are a consequence of the law of conservation of momentum of the fluid; the third equation is the entropy equation which is a result of the law of conservation of energy; the fourth equation is called the Poisson equation which is related to the balance of forces of the substance involved.

The function P denotes the pressure function, which is given by the γ-law, that is

$$(2)\qquad P = Ke^S\rho^\gamma,$$

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where $K \geq 0$ and $\gamma \geq 1$ are constants related to the state of the gas or fluid involved.

The constant δ can be either -1, 0 or 1.

When $\delta = -1$ or 1, the system (1) is called the non-isentropic compressible Euler-Poisson equations. More precisely, when $\delta = -1$ the system is attractive and can be used to model fluids such as gaseous starts. Moreover, the system is a non-relativistic descriptions of a galaxy in astrophysics. Reader may refer to [2] and [4] for more details. When $\delta = 1$, the system (1) is repulsive and and can be viewed as a semiconductor model. See [5] and [10] for more information.

When $\delta = 0$, the system $(1)_{1-3}$ is called non-isentropic compressible Euler equations which is a fundamental model in fluid mechanics [10, 6]. The Euler equations are also the special case of the noted Navier-Stokes equations, whose problem of whether there is a formation of singularity is still open and long-standing. In addition, if $K = 0$, then the pressure vanishes and the system is called pressureless Euler equations which be regarded as simple models of cosmology [18] or plasma physics [3]. For more information of studies of the pressureless Euler equations, readers can refer to [1, 7, 8, 13].

The importance of constructing analytical or exact solutions in mathematical physics and applied mathematics is that they can be used to classify and understand the nonlinear phenomena. In this area, Makino first obtained the solutions for the Euler equations in $\mathbf{R}^N$ in radial symmetry in 1993 [12]. A number of special solutions for these equations [15, 16] were subsequently obtained. In particular, in 2015, Yuen obtained a class of rotational solutions for the compressible Euler equations in [17]. While for the solutions for the Euler-Poisson equations, there exists some corresponding results [9, 11, 14].

In this article, we present two families of exact solutions for the pressureless Euler equations and the Euler-Poisson equations with pressure in $\mathbf{R}^3$. To be specific, we have the following two theorems.

Theorem 1. *For $\delta = 0$ and $K = 0$, the 3-dimensional pressureless Euler equations, $(1)_1$, $(1)_2$ and $(1)_3$, have the following family of exact solutions,*

$$(3) \qquad \begin{cases} \rho = \dfrac{f(s,z)}{a^{m+n}} \\ \mathbf{u} = \left(\dfrac{\dot{a}}{a}(mx+ny+g(z)), \dfrac{\dot{a}}{a}(mx+ny+g(z)), 0\right) \\ S = h\,(s,z) \\ a = \begin{cases} e^{\alpha_1+\alpha_2 t}, & \text{if } 2-m-n=0 \\ (\beta_1+\beta_2 t)^{\frac{1}{2-m-n}}, & \text{if } 2-m-n \neq 0, \end{cases} \end{cases}$$

where f, g and h are C^1 functions in addition that $f \geq 0$ with $s := \frac{mx+ny+g(z)}{a^{m+n}}$; and m, n, α_1, α_2, $\beta_1 > 0$ and β_2 are arbitrary constants. In particular,

(1) *if* $2-m-n\neq 0$ *and* $\beta_2<0$, *the solutions* (3) *blow up on* $-\frac{\beta_1}{\beta_2}$;
(2) *Otherwise, the solutions* (3) *globally exist.*

Theorem 2. *For* $\delta\neq 0$, *the* 3*-dimensional Euler-Poisson equations with pressure* (1) *have the following family of analytical solutions,*

$$\begin{cases} \rho=\max(f(\eta),0)\\ u_1=u_2=u_3=g(\xi)\\ S=\ln f^{\alpha}(\eta)\\ \Phi=\frac{K(\alpha+\gamma)}{\delta(\alpha+\gamma-1)}f^{\alpha+\gamma-1}(\eta), \end{cases} \tag{4}$$

where $\eta:=b_1x+b_2y+b_3z$ *and* $\xi:=a_1x+a_2y+a_3z$ *with* $\sum_{k=1}^{3}b_k=\sum_{k=1}^{3}a_k=0$, a_k *are not all zero,* b_k *are not all zero;* α *is a constant such that* $\alpha+\gamma\neq 1,\ 0$; *and* $f=f(\eta)$ *satisfies the following ordinary differential equation*

$$\begin{cases} ff''+(\alpha+\gamma-2)(f')^2=\frac{\delta}{K(\alpha+\gamma)(b_1^2+b_2^2+b_3^2)}\frac{1}{f^{\alpha+\gamma-4}}\\ f(0)>0,\quad \dot{f}(0)=f_1, \end{cases} \tag{5}$$

here f_1 *is an any constant.*

The remaining sections are organized as follows. In section 2, we present the proof of Theorem 1 and the proof of Theorem 2 is given in section 3. Both of the proofs are explained by plugging the solutions forms directly and checking that the mass equation $(1)_1$, the momentum equations $(1)_2$, the entropy equation $(1)_3$ and the Poisson equation $(1)_4$ are satisfied.

2. The Pressureless Euler Equations

In this section, we give the proof of Theorem 1.

Proof of Theorem 1. For the mass equation, namely, the equation $(1)_1$, we have

$$\rho_t+\rho\nabla\cdot\mathbf{u}+\mathbf{u}\cdot\nabla\rho \tag{6}$$

$$=\frac{\partial}{\partial t}\rho+\left[\frac{\partial}{\partial x}\frac{\dot{a}}{a}(mx+ny+g(z))\right]\rho+\left[\frac{\partial}{\partial y}\frac{\dot{a}}{a}(mx+ny+g(z))\right]\rho \tag{7}$$

$$+\left[\frac{\partial}{\partial z}0\right]\rho+\frac{\dot{a}}{a}(mx+ny+g(z))\frac{\partial}{\partial x}\rho+\frac{\dot{a}}{a}(mx+ny+g(z))\frac{\partial}{\partial y}\rho+0\cdot\frac{\partial}{\partial z}\rho \tag{8}$$

$$=\frac{\partial}{\partial t}\rho+(m+n)\frac{\dot{a}}{a}\rho+\frac{\dot{a}}{a}(mx+ny+g(z))\left(\frac{\partial}{\partial x}\rho+\frac{\partial}{\partial y}\rho\right) \tag{9}$$

$$=f(s,z)\left(\frac{-(m+n)\dot{a}}{a^{m+n+1}}\right)+\frac{1}{a^{m+n}}\frac{\partial f(s,z)}{\partial s}\frac{\partial s}{\partial t}+(m+n)\frac{\dot{a}}{a}\frac{f(s,z)}{a^{m+n}} \tag{10}$$

$$+\frac{\dot{a}}{a}(mx+ny+g(z))\left(\frac{\frac{\partial f(s,z)}{\partial s}}{a^{m+n}}\frac{\partial s}{x}+\frac{\frac{\partial f(s,z)}{\partial s}}{a^{m+n}}\frac{\partial s}{y}\right) \tag{11}$$

$$= \frac{1}{a^{m+n}} \frac{\partial f(s,z)}{\partial s} \frac{\partial s}{\partial t} + \frac{\dot{a}}{a}(mx+ny+g(z)) \left(\frac{\frac{\partial f(s,z)}{\partial s}}{a^{m+n}} \frac{\partial s}{\partial x} + \frac{\frac{\partial f(s,z)}{\partial s}}{a^{m+n}} \frac{\partial s}{\partial y} \right) \tag{12}$$

$$= \frac{\frac{\partial f(s,z)}{\partial s}}{a^{m+n}} \left[\frac{\partial s}{\partial t} + \frac{\dot{a}}{a}(mx+ny+g(z)) \left(\frac{\partial s}{\partial x} + \frac{\partial s}{\partial y} \right) \right] \tag{13}$$

$$= \frac{\frac{\partial f(s,z)}{\partial s}}{a^{m+n}} \Bigg[-(m+n)\dot{a} \frac{(mx+ny+g(z))}{a^{m+n+1}} \tag{14}$$

$$+ \frac{\dot{a}}{a}(mx+ny+g(z)) \left(\frac{m}{a^{m+n}} + \frac{n}{a^{m+n}} \right) \Bigg]$$

$$= 0, \tag{15}$$

where $s := \frac{mx+ny+g(z)}{a^{m+n}}$.

For the first momentum equation, we have

$$u_{1t} + u_1 u_{1x} + u_2 u_{1y} + u_3 u_{1z} \tag{16}$$

$$= \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right)(mx+ny+g(z)) + \frac{\dot{a}}{a}(mx+ny+g(z)) \left(\frac{\dot{a}}{a} m \right) \tag{17}$$

$$+ \frac{\dot{a}}{a}(mx+ny+g(z)) \left(\frac{\dot{a}}{a} n \right)$$

$$= (mx+ny+g(z)) \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2} m + \frac{\dot{a}^2}{a^2} n \right) \tag{18}$$

$$= (mx+ny+g(z)) \left[\frac{\ddot{a}}{a} - (m+n-1) \frac{\dot{a}^2}{a^2} \right] \tag{19}$$

$$= (mx+ny+g(z)) \frac{1}{a^{2-m-n}} \left[a^{1-m-n} \ddot{a} + (1-m-n) a^{-m-n} \dot{a}^2 \right] \tag{20}$$

$$= (mx+ny+g(z)) \frac{1}{a^{2-m-n}} \frac{\partial}{\partial t} \left[a^{1-m-n} \dot{a} \right]. \tag{21}$$

If $2-m-n=0$, then the equation (21) becomes

$$(mx+ny+g(z)) \frac{1}{a^0} \frac{\partial}{\partial t} \left(a^{-1} \dot{a} \right) \tag{22}$$

$$= (mx+ny+g(z)) \frac{\partial^2}{\partial t^2} \ln a \tag{23}$$

$$= (mx+ny+g(z)) \frac{\partial^2}{\partial t^2} (\alpha_1 + \alpha_2 t) \tag{24}$$

$$= 0. \tag{25}$$

If $2-m-n \neq 0$, then the equation (21) becomes

$$(mx+ny+g(z)) \frac{1}{a^{2-m-n}} \frac{\partial}{\partial t} \left(a^{1-m-n} \dot{a} \right) \tag{26}$$

$$= (mx+ny+g(z)) \frac{1}{a^{2-m-n}} \frac{\partial^2}{\partial t^2} \left(\frac{1}{2-m-n} a^{2-m-n} \right) \tag{27}$$

$$= (mx+ny+g(z))\frac{1}{a^{2-m-n}}\frac{\partial^2}{\partial t^2}(\beta_1+\beta_2 t) \tag{28}$$

$$= 0. \tag{29}$$

For the second momentum equation, we have

$$u_{2t}+u_1u_{2x}+u_2u_{2y}+u_3u_{2z} \tag{30}$$

$$= u_{1t}+u_1u_{1x}+u_2u_{1y}+u_3u_{1z} \tag{31}$$

$$= 0. \tag{32}$$

For the third momentum equation, we have

$$u_{3t}+u_1u_{3x}+u_2u_{3y}+u_3u_{3z} \tag{33}$$

$$= 0+0+0+0 \tag{34}$$

$$= 0. \tag{35}$$

For the entropy equation, we have

$$S_t+\mathbf{u}\cdot\nabla S \tag{36}$$

$$= S_t+u_1S_x+u_2S_y+u_3S_z \tag{37}$$

$$= \frac{\partial S}{\partial t}+\frac{\dot{a}}{a}(mx+ny+g(z))\left(\frac{\partial S}{\partial x}+\frac{\partial S}{\partial y}\right) \tag{38}$$

$$= \frac{\partial h(s,z)}{\partial s}\frac{\partial s}{\partial t}+\frac{\dot{a}}{a}(mx+ny+g(z))\left(\frac{\partial h(s,z)}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial h(s,z)}{\partial s}\frac{\partial s}{\partial y}\right) \tag{39}$$

$$= \frac{\partial h(s,z)}{\partial s}\left[\frac{\partial s}{\partial t}+\frac{\dot{a}}{a}(mx+ny+g(z))\left(\frac{\partial s}{\partial x}+\frac{\partial s}{\partial y}\right)\right] \tag{40}$$

$$= \frac{\partial h(s,z)}{\partial s}\left[\frac{-(m+n)(mx+ny+g(z))\dot{a}}{a^{m+n+1}}\right. \tag{41}$$
$$\left.+\frac{\dot{a}}{a}(mx+ny+g(z))\left(\frac{m}{a^{m+n}}+\frac{n}{a^{m+n}}\right)\right]$$

$$= 0. \tag{42}$$

Based on the solution $(3)_4$, it is clear to have the following results. In particular,

(1) if $2-m-n\neq 0$ and $\beta_2<0$, the solutions (3) blow up on $-\frac{\beta_1}{\beta_2}$;
(2) Otherwise, the solutions (3) globally exist.

The proof is completed. □

3. **The Euler-Poisson Equations**

In this section, we present the proof of Theorem 2.

Proof of Theorem 2. For the mass equation $(1)_1$, we have

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) \tag{43}$$
$$= 0 + \nabla \cdot (\rho \mathbf{u}) \tag{44}$$
$$= \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho \tag{45}$$
$$= f(\eta)\left[g_x(\xi) + g_y(\xi) + g_z(\xi)\right] + g f_x(\eta) + g f_y(\eta) + g f_z(\eta) \tag{46}$$
$$= (a_1 + a_2 + a_3) f g' + (b_1 + b_2 + b_3) g f' \tag{47}$$
$$= 0 + 0 \tag{48}$$
$$= 0, \tag{49}$$

where $\eta := b_1 x + b_2 y + b_3 z$ and $\xi := a_1 x + a_2 y + a_3 z$.

For the entropy equation $(1)_3$, we have

$$S_t + \mathbf{u} \cdot \nabla S \tag{50}$$
$$= 0 + \mathbf{u} \cdot \nabla S \tag{51}$$
$$= u_1 S_x + u_2 S_y + u_3 S_z \tag{52}$$
$$= g(a_1 x + a_2 y + a_3 z)\left(\frac{\partial}{\partial x}\alpha \ln f(\eta) + \frac{\partial}{\partial y}\alpha \ln f(\eta) + \frac{\partial}{\partial z}\alpha \ln f(\eta)\right) \tag{53}$$
$$= g(a_1 x + a_2 y + a_3 z)\alpha \frac{1}{f(\eta)} f'(\eta)\left(\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial z}\right) \tag{54}$$
$$= g(a_1 x + a_2 y + a_3 z)\alpha \frac{1}{f(\eta)} f'(\eta)\,(b_1 + b_2 + b_3) \tag{55}$$
$$= 0. \tag{56}$$

We can rewrite the momentum equations $(1)_2$ as

$$[\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}] + \frac{1}{\rho}\nabla P = \delta \nabla \Phi. \tag{57}$$

For the first term in the first momentum equation in the equations (57), we have

$$u_{1t} + u_1 u_{1x} + u_2 u_{1y} + u_3 u_{1z} \tag{58}$$
$$= 0 + u_1(u_{1x} + u_{1y} + u_{1z}) \tag{59}$$
$$= g(\xi)[g_x(\xi) + g_y(\xi) + g_z(\xi)] \tag{60}$$
$$= (a_1 + a_2 + a_3) g g' \tag{61}$$
$$= 0. \tag{62}$$

Similarly, the first terms in the second and third momentum equations in the equations (57) vanish. That is,

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0. \tag{63}$$

On the other hand, the second term in the first momentum equation becomes

$$\frac{1}{\rho}P_x \tag{64}$$

$$= \frac{1}{f(\eta)}\left[Ke^S\rho^\gamma\right]_x \tag{65}$$

$$= \frac{1}{f(\eta)}\left[Kf^{\alpha+\gamma}(\eta)\right]_x \tag{66}$$

$$= \left[\frac{K(\alpha+\gamma)}{\alpha+\gamma-1}f^{\alpha+\gamma-1}\right]_x \tag{67}$$

$$= \delta\Phi_x. \tag{68}$$

by the equations $(4)_{3-4}$. Similarly, we have

$$\frac{1}{\rho}P_y = \delta\Phi_y, \tag{69}$$

and

$$\frac{1}{\rho}P_z = \delta\Phi_z. \tag{70}$$

That is,

$$\frac{1}{\rho}\nabla P = \delta\nabla\Phi. \tag{71}$$

From the equations (63) and the equations (71), we have

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla P = \delta\nabla\Phi. \tag{72}$$

For the Poisson equation $(1)_4$, we have, by the ordinary differential equation (5),

$$\Delta\Phi \tag{73}$$

$$= \Phi_{xx}(\eta) + \Phi_{yy}(\eta) + \Phi_{zz}(\eta) \tag{74}$$

$$= (b_1^2+b_2^2+b_3^2)\Phi''(\eta) \tag{75}$$

$$= (b_1^2+b_2^2+b_3^2)\left[\frac{K\gamma}{\delta(\alpha+\gamma-1)}(\alpha+\gamma-1)f^{\alpha+\gamma-2}f'\right]' \tag{76}$$

$$= \frac{(b_1^2+b_2^2+b_3^2)K(\alpha+\gamma)}{\delta}\left[f^{\alpha+\gamma-2}f'\right]' \tag{77}$$

$$= \frac{(b_1^2+b_2^2+b_3^2)K(\alpha+\gamma)}{\delta}\left[f^{\alpha+\gamma-2}(f'')^{\alpha+\gamma-3}(f')^2\right] \tag{78}$$

$$= \frac{(b_1^2+b_2^2+b_3^2)K(\alpha+\gamma)}{\delta}f^{\alpha+\gamma-3}\left[ff''+(\gamma-2)(f')^2\right] \tag{79}$$

$$= \frac{(b_1^2+b_2^2+b_3^2)K(\alpha+\gamma)}{\delta}f^{\alpha+\gamma-3}\frac{\delta}{(b_1^2+b_2^2+b_3^2)K(\alpha+\gamma)}\frac{1}{f^{\alpha+\gamma-4}} \tag{80}$$

$$= f \tag{81}$$

$$= \rho. \tag{82}$$

Thus, the Poisson equation $(1)_4$ is satisfied.

The proof is completed. □

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