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FINDING THE CRITICAL DOMAIN OF MULTI-DIMENSIONAL QUENCHING PROBLEMS WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. Let Ω be a disc in R^2 with the center $(0,0)$ and radius a, $\partial\Omega$ and $\bar{\Omega}$ be its boundary and closure, respectively. Suppose that u is a function of τ, χ, and ζ. Further, assume that β is a positive number. In this paper, we investigate the multi-dimensional parabolic quenching problems with the second initial-boundary condition:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \chi^2} + \frac{\partial^2 u}{\partial \zeta^2} + \frac{1}{1-u} \text{ for } (\chi, \zeta, \tau) \in \Omega \times (0, \infty),$$

$$u(\chi, \zeta, 0) = u_0(\chi, \zeta) \text{ for } (\chi, \zeta) \in \bar{\Omega}, \quad \frac{\partial u(\chi, \zeta, \tau)}{\partial n} = -\frac{\beta}{a} \text{ for } \tau > 0 \text{ and } (\chi, \zeta) \in \partial\Omega,$$

where $u_0 \in C^2(\bar{\Omega})$ and $u_0(\chi, \zeta) < 1$ for $(\chi, \zeta) \in \bar{\Omega}$, and $\partial u / \partial n$ is the outward normal derivative of u. We shall determine an approximated critical domain of some $u_0(\chi, \zeta)$ of the above problem by using a numerical method.

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1. INTRODUCTION

Let β and a be positive numbers, and Ω be a disc in R^2 with the center $(0,0)$ and radius a. We also let $\partial\Omega$ and $\bar{\Omega}$ be its boundary and closure, respectively. Suppose that u is a function of τ, χ, and ζ, where τ, χ, and ζ are independent variables. In this paper, we determine an approximated critical domain of the following multi-dimensional parabolic quenching problems with Neumann boundary condition:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \chi^2} + \frac{\partial^2 u}{\partial \zeta^2} + \frac{1}{1-u} \text{ for } (\chi, \zeta, \tau) \in \Omega \times (0, \infty), \tag{1.1}$$

$$u(\chi, \zeta, 0) = u_0(\chi, \zeta) \text{ for } (\chi, \zeta) \in \bar{\Omega}, \quad \frac{\partial u(\chi, \zeta, \tau)}{\partial n} = -\frac{\beta}{a} \text{ for } \tau > 0 \text{ and } (\chi, \zeta) \in \partial\Omega, \tag{1.2}$$

where $u_0(\chi, \zeta) \in C^2(\bar{\Omega})$ and $\partial u / \partial n$ is the outward normal derivative of u. Further, we assume that $u_0(\chi, \zeta) < 1$ for $(\chi, \zeta) \in \bar{\Omega}$ and satisfies $\partial u_0(\chi, \zeta) / \partial n = -\beta / a$ for $(\chi, \zeta) \in \partial\Omega$.

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If the Problem (1.1)–(1.2) is an one-dimensional problem with $\Omega = (0, l)$ and is subject to the first boundary condition with $u_0 \equiv 0$ on $\bar{\Omega}$, then Kawarda [6] used this problem to describe a polarization phenomenon in ionic conductors. He proved that the solution u quenches if $l > 2\sqrt{2}$. When the forcing term is a constant λ instead of $1/(1-u)$, Barles and Lio [1] called this problem as the boundary ergodic control problem. The forcing term λ is the ergodic cost and the solution u is the value function of the control problem. From the paper of Lio [7], the problem can be used as a homogenization of elliptic and parabolic partial differential equations. He studied the large time behavior of the Problem (1.1)–(1.2) when $t \to \infty$. If the forcing term is $\sigma e^{\nu u}$ instead of $1/(1-u)$, this mathematical problem describes a polarization model on the boundary surface of a diffusion medium [9, pp. 286–287]. The investigation of Problem (1.1)–(1.2) helps us understand the existence of the solution and blow-up property of u.

The critical domain Ω^* of the Problem (1.1)–(1.2) is a domain such that u exists for all time when $\Omega \subsetneq \Omega^*$ and also there is a finite time Γ such that

$$\max\left\{u(\chi, \zeta, \tau) : (\chi, \zeta) \in \bar{\Omega}\right\} \to 1^- \text{ as } \tau \to \Gamma^-$$

when $\Omega^* \subsetneq \Omega$. The aim of this paper is to determine an approximated critical domain of the quenching Problem (1.1)–(1.2) through studying the steady state problem. This steady state solution will be represented in a form of an integral equation. This solution will be a limiting solution of a sequence of linear integral equation.

Chan [3] studied the critical domain of the Equation (1.1) with a two-dimensional elliptic plate subject to the zero first initial-boundary condition. This two-dimensional domain is given by

$$\left\{(x, y) : \frac{x^2}{b_1^2} + \frac{y^2}{b_2^2} < 1\right\},$$

where b_1 and b_2 are positive constants with $b_1 \neq b_2$. He developed a computational method to determine an approximated critical domain through solving the integral equation of the steady state equation. Recently, Chan [4] calculated approximated critical domains of a coupled parabolic quenching problem with square-shaped domains. His approach is to use a power series to approximate the Green's function in solving the integral equation.

This paper is organized as follows. In Section 2, we shall prove that if u exists globally, then $u(\chi, \zeta, \tau)$ will approach the steady state solution $U(\chi, \zeta)$ when $\tau \to \infty$. In Section 3, we shall prove that the solution to the Problem (1.1)–(1.2) will quench in a finite time and the quenching set will be a compact subset of $\bar{\Omega}$. In Section 4, we shall report an approximated critical domain of the Problem (1.1)–(1.2) for some β and u_0.

2. CONVERGENCE OF THE TIME-DEPENDENT SOLUTION

In this section, we shall prove that the time-dependent solution converges to the steady state solution. We transform the Problem (1.1)–(1.2) from the domain Ω to a unit disc. Let D be a unit disc with the center $(0,0)$, ∂D and $\bar{D}$ be its boundary and closure, respectively. Our substitution for the independent variable is $\tau = a^2 t$, $\chi = ax$, and $\zeta = ay$. Then, the Problem (1.1)–(1.2) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{a^2}{1-u} \text{ for } (x,y,t) \in D \times (0,\infty), \tag{2.1}$$

(2.2)

$$u(x,y,0) = u_0(x,y) \text{ for } (x,y) \in \bar{D} \text{ and } \frac{\partial u(x,y,t)}{\partial n} = -\beta \text{ for } t > 0 \text{ and } (x,y) \in \partial D.$$

In this section, we also assume that $u_0(x,y)$ satisfies

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{a^2}{1-u_0} \geq 0 \text{ on } \bar{D}.$$

Finding the critical domain of the Problem (2.1)–(2.2) is equivalent to determine the critical value a^* such that $U(x,y)$ exists when $a < a^*$ where $U(x,y)$ is the steady state solution to the following boundary-valued Laplace problem:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -\frac{a^2}{1-U} \text{ for } (x,y) \in D, \tag{2.3}$$

$$\frac{\partial U(x,y)}{\partial n} = -\beta \text{ for } (x,y) \in \partial D. \tag{2.4}$$

In order to show $u(x,y,t)$ converging $U(x,y)$ when $t \to \infty$, we prove the following two lemmas.

Lemma 1. *$u_t(x,y,t) \geq 0$ on $\bar{D} \times [0,\infty)$.*

Proof. Since $u(x,y,0) = u_0(x,y)$ for $(x,y) \in \bar{D}$ and $u_0(x,y)$ satisfies

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{a^2}{1-u_0} \geq 0 \text{ on } \bar{D},$$

and $\partial u_0(x,y)/\partial n = -\beta$ for $(x,y) \in \partial D$, by the comparison theorem $u_0(x,y) \leq u(x,y,t)$ for $t \geq 0$ on $\bar{D}$. Let h be a positive real number and $v(x,y,t) = u(x,y,t+h)$. $v(x,y,t)$ is the solution to the following problem:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{a^2}{1-v} \text{ for } (x,y,t) \in D \times (0,\infty),$$

$$v(x,y,0) = u(x,y,h) \text{ for } (x,y) \in \bar{D} \text{ and } \frac{\partial v(x,y,t)}{\partial n} = -\beta \text{ for } t > 0 \text{ and } (x,y) \in \partial D.$$

Then, $v(x,y,t) - u(x,y,t)$ satisfies the equation below:

$$\frac{\partial (v-u)}{\partial t} = \frac{\partial^2 (v-u)}{\partial x^2} + \frac{\partial^2 (v-u)}{\partial y^2} + \frac{a^2}{1-v} - \frac{a^2}{1-u} \text{ for } (x,y,t) \in D \times (0,\infty).$$

By the mean value theorem, we obtain

$$\frac{\partial (v-u)}{\partial t} = \frac{\partial^2 (v-u)}{\partial x^2} + \frac{\partial^2 (v-u)}{\partial y^2} + \frac{a^2}{(1-p)^2}(v-u) \text{ for } (x,y,t) \in D \times (0,\infty),$$

where p is between u and v. Further, $v(x,y,0) - u(x,y,0) = u(x,y,h) - u_0(x,y) \geq 0$ for $(x,y) \in \bar{D}$, and $\partial (v(x,y,t) - u(x,y,t))/\partial n = -\beta - (-\beta) = 0$ for $t > 0$ and $(x,y) \in \partial D$. Then, by the comparison theorem $v(x,y,t) \geq u(x,y,t)$ on $\bar{D} \times [0,\infty)$. Therefore, $u_t(x,y,t) \geq 0$ on $\bar{D} \times [0,\infty)$. □

Let ρ be a positive real number less than 1. In below, we prove that $u(x,y,t)$ exists globally by constructing an upper solution if a is sufficiently small.

Lemma 2. *If* $a^2 < 2\beta(1-\rho)$, *there is a global solution to the Problem* (2.1)–(2.2).

Proof. Let $g(x,y) = \rho - bx^2 - by^2$ and b be a positive real number such that $u_0(x,y) \leq g(x,y) < 1$ for $(x,y) \in \bar{D}$. For $a^2 < 2\beta(1-\rho)$, we have $a^2/[4(1-\rho)] < \beta/2$. Choose b such that $a^2/[4(1-\rho)] < b < \beta/2$, then $-2b > -\beta$ and $a^2 < 4b(1-\rho)$. This gives $\partial g/\partial n = -2b > -\beta$ on ∂D. By substituting this $g(x,y)$ into (2.1), it yields

$$g_t - g_{xx} - g_{yy} - \frac{a^2}{1-g} = 4b - \frac{a^2}{1-g}.$$

Since $a^2 < 4b(1-\rho) < 4b(1-g)$, we get

$$g_t - g_{xx} - g_{yy} - \frac{a^2}{1-g} \geq 0.$$

By the comparison theorem, $g(x,y) \geq u(x,y,t)$ on $\bar{D} \times [0,\infty]$. Therefore, $u(x,y,t)$ exists globally. □

Since $u(x,y,t)$ is bounded by $u_0(x,y)$ and $g(x,y)$ and $u_t(x,y,t) \geq 0$ on $\bar{D} \times [0,\infty)$, $\lim_{t\to\infty} u(x,y,t) = U(x,y)$ exists on $\bar{D}$. Further, $g(x,y) \geq U(x,y) \geq u_0(x,y)$ on $\bar{D}$. We follow Theorem 5.4.2 of Pao [9, p. 200] to obtain the following result.

Theorem 3. *If* $u(x,y,t) < 1$ *for* $\bar{D} \times [0,\infty]$, *then* u *converges to the classical solution* U *to the Problem* (2.3)–(2.4).

3. QUENCHING OF THE SOLUTION

In this section, we prove that u quenches in a finite time. Let ϕ_1 be the first eigenfunction and λ_1 be its eigenvalue of the following boundary-valued problem:

$$\phi_{xx} + \phi_{yy} + \lambda\phi = 0 \text{ in } D, \quad \partial\phi/\partial n = 0 \text{ on } \partial D, \tag{3.1}$$

$\int\int_D \phi_1 dxdy = 1$. By Theorem 3.1.2 of Pao [9, p. 97], λ_1 is real and nonnegative, and ϕ_1 is positive in D.

Lemma 4. *If* $a^2 > \lambda_1$ *and* $\left(a^2 - \beta \int_{\partial D} \phi_1 ds\right) / \left(a^2 - \lambda_1\right) + \int\int_D u_0 \phi_1 dxdy > 0$, *then* u *quenches in a finite time.*

Proof. Multiply ϕ_1 on both sides of Equation (2.1) and integrate the expression over the domain D, we obtain

$$\int\int_D u_t \phi_1 dxdy - \int\int_D \left(u_{xx} + u_{yy}\right) \phi_1 dxdy = a^2 \int\int_D \frac{\phi_1}{1-u} dxdy.$$

Using the Green's second identity,

$$\left(\int\int_D u\phi_1 dxdy\right)_t - \left[\int\int_D \left(\left(\phi_1\right)_{xx} + \left(\phi_1\right)_{yy}\right) udxdy + \int_{\partial D} \left(\phi_1 \frac{\partial u}{\partial n} - u\frac{\partial \phi_1}{\partial n}\right) ds\right]$$
$$= a^2 \int\int_D \frac{\phi_1}{1-u} dxdy.$$

By (3.1) and the boundary condition of Equation (2.2),

$$\left(\int\int_D u\phi_1 dxdy\right)_t - \left(\int\int_D -\lambda_1 \phi_1 udxdy + \int_{\partial D} -\beta\phi_1 ds\right) = a^2 \int\int_D \frac{\phi_1}{1-u} dxdy.$$

Equivalently,

$$\left(\int\int_D u\phi_1 dxdy\right)_t = -\int\int_D \lambda_1 \phi_1 udxdy - \int_{\partial D} \beta\phi_1 ds + a^2 \int\int_D \frac{\phi_1}{1-u} dxdy. \tag{3.2}$$

According to the Maclaurin series, for $|q| < 1$,

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots \geq 1 + q.$$

Then, (3.2) leads to the following inequality

$$\left(\int\int_D u\phi_1 dxdy\right)_t \geq -\lambda_1 \int\int_D u\phi_1 dxdy + a^2 \int\int_D u\phi_1 dxdy$$
$$+ a^2 \int\int_D \phi_1 dxdy - \beta \int_{\partial D} \phi_1 ds.$$

Let $R(t) = \int\int_D u(x,y,t)\phi_1 dxdy$ and $A = a^2 - \beta\int_{\partial D} \phi_1 ds$. Then, the above inequality becomes

$$\frac{d}{dt} R \geq \left(a^2 - \lambda_1\right) R + A.$$

Integrate the above inequality over $[0,t]$, we get

$$R(t) - \frac{R(0)}{e^{-(a^2-\lambda_1)t}} \geq \frac{A}{e^{-(a^2-\lambda_1)t}\left(a^2 - \lambda_1\right)} \left[1 - e^{-\left(a^2-\lambda_1\right)t}\right].$$

Equivalently,

$$R(t) \geq \left[\frac{A}{\left(a^2 - \lambda_1\right)} + R(0)\right] e^{\left(a^2-\lambda_1\right)t} - \frac{A}{\left(a^2 - \lambda_1\right)}.$$

By assumption $a^2 > \lambda_1$ and $A/\left(a^2 - \lambda_1\right) + R(0) > 0$, if $R(t)$ exists for all $t > 0$, then $R(t) \to \infty$ as $t \to \infty$. However, $R(t)$ is bounded above by 1 for $t > 0$. It shows that $R(t)$ reaches 1 in a finite time T. Thus, u quenches in a finite time. □

To prove the quenching set of the solution of the Problem (2.1)–(2.2) being a compact subset of $\bar{D}$, we rewrite the Problem (2.1)–(2.2) in the polar coordinates with $x = r\cos\theta$ and $y = r\sin\theta$. The Problem (2.1)–(2.2) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{a^2}{1-u} \text{ in } D \times (0,\infty), \tag{3.3}$$

$$u(r,\theta,0) = u_0(r\cos\theta, r\sin\theta) \text{ for } r \in [0,1] \text{ and } \theta \in [0,2\pi),$$

$$\left.\frac{\partial u(r,\theta,t)}{\partial r}\right|_{r=1} = -\beta \text{ for } \theta \in [0,2\pi) \text{ and } t > 0.$$

If we differentiate Equation (3.3) with respect to r, it yields

$$\frac{\partial u_r}{\partial t} = \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - r^{-2}u_r - 2r^{-3}\frac{\partial^2 u}{\partial \theta^2} + \frac{a^2}{(1-u)^2}u_r \text{ in } D \times (0,\infty).$$

Then, u_r satisfies the following initial-boundary value problem

$$\frac{\partial u_r}{\partial t} = \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^3}\frac{\partial^2 u}{\partial \theta^2} + \left[\frac{a^2}{(1-u)^2} - \frac{1}{r^2}\right]u_r \text{ in } D \times (0,\infty), \tag{3.4}$$

$$u_r(r,\theta,0) = \frac{\partial u_0}{\partial r} \text{ for } r \in [0,1] \text{ and } \theta \in [0,2\pi),$$

$$\left.\frac{\partial u(r,\theta,t)}{\partial r}\right|_{r=1} = -\beta \text{ for } \theta \in [0,2\pi) \text{ and } t > 0.$$

We remark that $\partial u_0(r,\theta)/\partial r = -\beta$ for $r = 1$ and $\theta \in [0,2\pi)$. We modify Theorem 2.2 of Deng and Levine [5] to obtain the following result.

Theorem 5. *The quenching set of u is a compact subset of $\bar{D}$.*

Proof. Let $w(r,\theta,t) = ru_r(r,\theta,t)$. As D is a radial symmetric domain, we set the angle θ to be a fixed value. Then, $\partial^2 u_r/\partial\theta^2 = 0$. From (3.4), we have

$$\frac{\partial w}{\partial t} + \frac{1}{r}w_r - w_{rr} = \frac{a^2}{(1-u)^2}w.$$

From the boundary condition $\partial u/\partial r|_{r=1} = -\beta$, $w < 0$ when $r = 1$ for $t \geq 0$. By the continuity of u_r in $D \times (0,\infty)$, there exists a neighborhood (S) of $r = 1$ such that $\partial u(r,\theta,t)/\partial r < 0$ for $r \in S$ and $t > 0$. This tells us that there exists a $t_1 > 0$ such that $w(r,\theta,t_1) < -\sigma$ for some positive number σ and $r \in S$. Let r_1 be a positive constant less than 1, $r_1 \in S$, and $Q(r,\theta,t) = w + c(1-r)$ for $r \in [r_1,1]$ where c is a positive constant. At $t = t_1$, choose c such that $Q < 0$ when $r \in [r_1,1)$. Clearly, $Q < 0$ when $r = 1$ for $t > 0$. Furthermore, $Q_r = w_r - c$, $Q_{rr} = w_{rr}$, and $Q_t = w_t$. Substitute them in the above differential equation

$$Q_t + \frac{1}{r}Q_r - Q_{rr} = \frac{a^2}{(1-u)^2}w - \frac{c}{r} < 0 \text{ in } S \times (t_1,\infty).$$

Then, by the maximum principle, $Q \le 0$ for $S \times [t_1, \infty)$. That is, $ru_r + c(1-r) \le 0$, which yields

$$u_r \le -\frac{c}{r}(1-r) = -c\left(\frac{1}{r} - 1\right).$$

Integrate both sides from r_2 to r_3 where $r_1 < r_2 < r_3 < 1$,

$$\int_{u(r_2,t)}^{u(r_3,t)} du \le -c \int_{r_2}^{r_3} \left(\frac{1}{r} - 1\right) dr$$

$$u(r_3,t) - u(r_2,t) \le -c\left[(\ln r_3 - r_3) - (\ln r_2 - r_2)\right].$$

As $\ln r - r$ is increasing for $r \in (0,1)$, we have $(\ln r_3 - r_3) - (\ln r_2 - r_2) > 0$ when $r_2 < r_3$. This tells us that

$$\begin{aligned} u(r_3,t) &\le u(r_2,t) - c\left[(\ln r_3 - r_3) - (\ln r_2 - r_2)\right] \\ &< 1 - c\left[(\ln r_3 - r_3) - (\ln r_2 - r_2)\right] \text{ for } t > 0. \end{aligned}$$

Thus, r_3 is not a quenching point. Hence, the quenching set is a compact subset of $\bar{D}$. □

4. NUMERICAL RESULTS

Refer to Section 2, finding the critical domain of the Problem (1.1)–(1.2) is equivalent to calculate the critical value a^* of the boundary-valued Laplace equation (2.3)–(2.4):

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -\frac{a^2}{1-U} \text{ for } (x,y) \in D,$$

$$\frac{\partial U(x,y)}{\partial n} = -\beta \text{ for } (x,y) \in \partial D.$$

From the results of Begehr [2] and Nehari [8], the Neumann function $N(z,\gamma)$ for the two-dimensional Laplace operator for the unit disc D is given by

$$N(z,\gamma) = \frac{-1}{2}\log\left|(\gamma - z)(1 - z\bar{\gamma})\right|^2,$$

where $z \ne \gamma$. $z = x + iy$ and $\gamma = \xi + i\eta$ are some points locating inside D and are represented in a complex number form. This Neumann function $N(x,y;\xi,\eta)$ satisfies

$$-\frac{\partial^2 N}{\partial x^2} - \frac{\partial^2 N}{\partial y^2} = \delta(x-\xi)\,\delta(y-\eta) \text{ for } (x,y) \text{ and } (\xi,\eta) \in D,$$

$$\frac{\partial N}{\partial n} = -1 \text{ for } (x,y) \in \partial D,$$

where $(x,y) \ne (\xi,\eta)$ and δ is the Dirac δ-function. By the Green's second identity, $U(x,y)$ is given by

$$U(x,y) = \int_{\partial D}\left(N\frac{\partial U}{\partial n} - U\frac{\partial N}{\partial n}\right) ds - \int\int_D N \Delta U \, d\xi d\eta.$$

From $\partial N/\partial n = -1$ and $\partial U/\partial n = -\beta$ on ∂D,

$$U(x,y) = \int_{\partial D} (-\beta N + U)\, ds + \int\int_D N \frac{a^2}{1-U} d\xi d\eta.$$

The approximated solution of $U(x,y)$ is obtained by solving the following iterative equation

$$U_{j+1}(x,y) = \int_{\partial D} (-\beta N + U_j)\, ds + \int\int_D N \frac{a^2}{1-U_j} d\xi d\eta,$$

with $j = 0, 1, 2, \ldots$ and $U_0(x,y) = u_0(x,y)$. The point γ in the polar form is represented as $\gamma = r\cos\theta + ir\sin\theta$. For $(x,y) \in D$, the above iterative scheme becomes

(4.1)
$$\begin{cases} U_{j+1}(x,y) \\ = \int_0^{2\pi} \left\{ \frac{\beta}{2} \log \left| \left[(\cos\theta + i\sin\theta) - (x+iy) \right] \left[1 - (x+iy)(\cos\theta - i\sin\theta) \right] \right|^2 \right. \\ \left. + U_j(\cos\theta, \sin\theta) \right\} d\theta \\ + \int_0^{2\pi} \int_0^1 \left\{ \frac{-1}{2} \log \left| \left[(r\cos\theta + ir\sin\theta) - (x+iy) \right] \left[1 - (x+iy)(r\cos\theta - ir\sin\theta) \right] \right|^2 \right. \\ \left. \times \frac{a^2}{1 - U_j(r\cos\theta, r\sin\theta)} \right\} r dr d\theta. \end{cases}$$

To obtain an estimate of $U_{j+1}(x,y)$ when $(x,y) = (\cos\mu, \sin\mu) \in \partial D$, we use $\partial U/\partial n = \partial U/\partial r = -\beta$ which is approximated by the finite difference

$$\frac{U_{j+1}(\cos\mu, \sin\mu) - U_{j+1}((1-\varepsilon)\cos\mu, (1-\varepsilon)\sin\mu)}{\varepsilon} = -\beta,$$

where ε is a small positive number less than 1. Therefore,

$$U_{j+1}(\cos\mu, \sin\mu) = U_{j+1}((1-\varepsilon)\cos\mu, (1-\varepsilon)\sin\mu) - \varepsilon\beta. \tag{4.2}$$

The value of $U_{j+1}((1-\varepsilon)\cos\mu, (1-\varepsilon)\sin\mu)$ is given by Equation (4.1).

The region D is divided uniformly with 289 grid points. The procedure for computing the critical value a^* is below.

Step 1: Use (4.1) to compute the solution $U_{j+1}(x,y)$ for $j = 0, 1, 2, \ldots$ with $U_0(r\cos\theta, r\sin\theta) \equiv u_0(r\cos\theta, r\sin\theta)$. At the boundary point $(x,y) = (\cos\mu, \sin\mu)$, we set $\varepsilon = 0.0001$. From (4.2), it gives

$$U_{j+1}(\cos\mu, \sin\mu) = -0.0001\beta + U_{j+1}(0.9999\cos\mu, 0.9999\sin\mu).$$

Step 2: Choose some a to be a_1 and some value for a_2. At $a = a_1$, $U_{j+1} < 1$ on $\bar{D}$ and satisfies the following condition:

$$\left| \max U_{j+1}(x,y) - \max U_j(x,y) \right| < 1 \times 10^{-4}.$$

At $a = a_2$, U_{j+1} does not exist (that is, $\max U_{j+1} \geq 1$). Determine $a_3 = (a_1 + a_2)/2$. Then, compute U_{j+1} at $a = a_3$.

Step 3: Set $a_1 = a_3$ if $U_{j+1} < 1$ on $\bar{D}$ and

$$|\max U_{j+1}(x,y) - \max U_j(x,y)| < 1 \times 10^{-4}.$$

Set $a_2 = a_3$ when U_{j+1} does not exist. The iterative procedure stops when $|a_1 - a_3| < 1 \times 10^{-6}$ (or $|a_2 - a_3| < 1 \times 10^{-6}$). Then, set $a^* = a_3$. Otherwise, repeat Step 3.

We use Mathematica 8.0 to compute the critical domain. The initial condition $u_0(x,y) = \beta/2\,(1 - x^2 - y^2)$. The numerical result for some values of β is below:

β	$(a^*)^2$	Critical Domain $= \pi\,(a^*)^2$
0.2	0.0592727	0.1862107
0.1	0.0307251	0.0965257

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