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EXPECTED NUMBER OF REAL ROOTS OF CERTAIN GAUSSIAN RANDOM TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. Let $D_n(\theta) = \sum_{k=0}^{n}(A_k \cos k\theta + B_k \sin k\theta)$ be a random trigonometric polynomial where the coefficients $A_0, A_1, \ldots, A_n$, and $B_0, B_1, \ldots, B_n$, form sequences of Gaussian random variables. Moreover, we assume that the increments $\Delta_k^1 = A_k - A_{k-1}$, $\Delta_k^2 = B_k - B_{k-1}$, $k = 0, 1, 2, \ldots, n$, are independent, with conventional notation of $A_{-1} = B_{-1} = 0$. The coefficients $A_0, A_1, \ldots, A_n$, and $B_0, B_1, \ldots, B_n$, can be considered as n consecutive observations of a Brownian motion. In this paper we provide the asymptotic behavior of the expected number of real roots of $D_n(\theta) = 0$ as order $\frac{2\sqrt{2}n}{\sqrt{3}}$. Also by the symmetric property assumption of coefficients, i.e., $A_k \equiv A_{n-k},\ B_k \equiv B_{n-k}$, we show that the expected number of real roots is of order $\frac{2n}{\sqrt{3}}$.

Key words: Random Trigonometric Polynomials, Brownian motion, Symmetric Property.

1. PRELIMINARIES

There are two different forms of random trigonometric polynomials previously studied.

$$T_n(\theta) = \sum_{k=0}^{n} A_k \cos(k\theta)$$

and

$$D_n(\theta) = \sum_{k=0}^{n}(A_k \cos k\theta + B_k \sin k\theta), \tag{1.1}$$

Dunnage [2] first studied the classical random trigonometric polynomials $T(\theta)$. He showed that in the case of identically and normally distributed coefficients $A_0, A_1, \ldots,$ A_n with mean zero and variances 1, the expected number of real roots in the interval $(0, 2\pi)$, outside of an exceptional set of measure zero, is $\frac{2n}{\sqrt{3}} + O\{n^{11/3}(\log n^{2/3})\}$, when n is large. In Farahmand [3, 4, 5], it is shown the asymptotic formula for the expected number of K-level crossings remain valid when the level K increases. The work of Sambandham and Renganathan [13] and Farahmand [6] among other obtained this result for different assumption on the distribution of the coefficients. For various

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aspects on random polynomials see Bharucha-Reid and Sambandham [1], which includes a comprehensive reference. Farahmand and Sambandham [8] study a case of coefficients with different mean and variances, which shows an interesting results for the expected number of level crossing in the interval $(0, 2\pi)$. Farahmand and T. Li [9] obtained asymptotic behavior for the expected number of real roots of two different forms of random trigonometric polynomials $T_n(\theta)$ and $D_n(\theta)$, where the coefficients of polynomials are normally distributed random variables with different means and variances. Also They studied a case of reciprocal random polynomials for $T_n(\theta)$ and $D_n(\theta)$. We consider the classical forms of random trigonometric polynomials $D_n(\theta)$ where the coefficients $A_0, A_1, \ldots, A_n$ and $B_0, B_1, \ldots, B_n$ be a mean zero Gaussian random sequence in which the increments $\Delta_k^{(1)} = A_k - A_{k-1}$ and $\Delta_k^{(2)} = B_k - B_{k-1}$, $k = 0, 1, 2, \ldots,$ are independent, $A_{-1} = 0,\ B_{-1} = 0.$ The sequence $A_0, A_1, \ldots$ and $B_0, B_1, \ldots$ may be considered as successive Brownian points, i.e., $A_k = W_1(t_k)$, $B_k = W_2(t_k)$,$k = 0, 1, \ldots, n$, where $t_0 < t_1 < \cdots$ and $\{W_i(t_k), t \geq 0\}$, $i = 1, 2$, are the standard Brownian motion. In this physical interpretation, $\mathrm{Var}(\Delta_k^{(i)})$ is the distance between successive times t_{k-1}, t_k. We note that

$$A_k = \Delta_0^{(1)} + \Delta_1^{(1)} + \cdots + \Delta_k^{(1)}, \quad B_k = \Delta_0^{(2)} + \Delta_1^{(2)} + \cdots + \Delta_k^{(2)} \quad k = 0, 1, \ldots, n,$$

where $\Delta_k^{(i)} \sim N(0, \sigma_i^2)$, $k = 0, 1, \ldots, n$, $i = 1, 2$, and $\Delta_k^{(i)}$ are independent. Thus

$$\begin{aligned} D_n(\theta) &= \sum_{k=0}^{n} \left[\left(\sum_{j=k}^{n} \cos j\theta \right) \Delta_k^{(1)} + \left(\sum_{j=k}^{n} \sin j\theta \right) \Delta_k^{(2)} \right] \\ &= \sum_{k=0}^{n} \left(a_{k1}(\theta)\Delta_k^{(1)} + b_{k1}(\theta)\Delta_k^{(2)} \right) \end{aligned}$$

and

$$\begin{aligned} D_n'(\theta) &= \sum_{k=0}^{n} \left[\left(-\sum_{j=k}^{n} j \sin j\theta \right) \Delta_k^{(1)} + \left(\sum_{j=k}^{n} j \cos j\theta \right) \Delta_k^{(2)} \right] \\ &= \sum_{k=0}^{n} \left(c_{k1}(\theta)\Delta_k^{(1)} + d_{k1}(\theta)\Delta_k^{(2)} \right) \end{aligned}$$

where

$$a_{k1}(\theta) = \sum_{j=k}^{n} \cos j\theta = \frac{\sin(2n+1)\frac{\theta}{2} - \sin(2k-1)\frac{\theta}{2}}{2\sin(\frac{\theta}{2})},$$

$$b_{k1}(\theta) = \sum_{j=k}^{n} \sin j\theta = \frac{\cos(2k-1)\frac{\theta}{2} - \cos(2n+1)\frac{\theta}{2}}{2\sin(\frac{\theta}{2})},$$

$$c_{k1}(\theta) = -\sum_{j=k}^{n} j \sin j\theta = \left(\frac{\sin(2n+1)\frac{\theta}{2} - \sin(2k-1)\frac{\theta}{2}}{2\sin(\frac{\theta}{2})} \right)',$$

$$d_{k1}(\theta)=\sum_{j=k}^{n} j\cos j\theta=\left(\frac{\cos(2k-1)\frac{\theta}{2}-\cos(2n+1)\frac{\theta}{2}}{2\sin(\frac{\theta}{2})}\right) \tag{1.2}$$

Now given $D_n(\theta)$ in (1.1) with a symmetric property of coefficients, i.e., $A_k \equiv A_{n-k}$ and $B_k \equiv B_{n-k}$ for $k=0,1,\ldots,n$, we can write $Q_n(\theta)$ for odd n's as follows:

$$Q_n(\theta)=\sum_{k=0}^{\frac{n-1}{2}}[A_k(\cos k\theta+\cos(n-k)\theta)+B_k(\sin k\theta+\sin(n-k)\theta)] \tag{1.3}$$

The polynomials will have one additional term for even n's and we will not discuss this case here.

$$\begin{aligned}
Q_n(\theta)&=\sum_{k=0}^{\frac{n-1}{2}}[A_k(\cos k\theta+\cos(n-k)\theta)+B_k(\sin k\theta+\sin(n-k)\theta)]\\
&=\sum_{k=0}^{\frac{n-1}{2}}\left[\left(\sum_{j=k}^{\frac{n-1}{2}}(\cos j\theta+\cos(n-j)\theta)\right)\Delta_k^{(1)}+\left(\sum_{j=k}^{\frac{n-1}{2}}(\sin j\theta+\sin(n-j)\theta)\right)\Delta_k^{(2)}\right]\\
&=\sum_{k=0}^{\frac{n-1}{2}}\left(a_{k2}(\theta)\Delta_k^{(1)}+b_{k2}(\theta)\Delta_k^{(2)}\right),\\
Q'_n(\theta)&=\sum_{k=0}^{\frac{n-1}{2}}\left[\left(-\sum_{j=k}^{\frac{n-1}{2}}(j\sin j\theta+(n-j)\sin(n-j)\theta)\right)\Delta_k^{(1)}\right.\\
&\quad\left.+\left(\sum_{j=k}^{\frac{n-1}{2}}(j\cos j\theta+(n-j)\cos(n-j)\theta)\right)\Delta_k^{(2)}\right]\\
&=\sum_{k=0}^{\frac{n-1}{2}}\left(c_{k2}(\theta)\Delta_k^{(1)}+d_{k2}(\theta)\Delta_k^{(2)}\right)
\end{aligned}$$

where by using this results

$$a_{k2}(\theta)=\sum_{j=k}^{\frac{n-1}{2}}(\cos j\theta+\cos(n-j)\theta)=\frac{\sin(2n-2k+1)\frac{\theta}{2}-\sin(2k-1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}},$$

$$b_{k2}(\theta)=\sum_{j=k}^{\frac{n-1}{2}}(\sin j\theta+\sin(n-j)\theta)=\frac{\cos(2k-1)\frac{\theta}{2}-\cos(2n-2k+1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}},$$

$$c_{k2}(\theta)=-\sum_{j=k}^{\frac{n-1}{2}}(j\sin j\theta+(n-j)\sin(n-j)\theta)=\left(\frac{\sin(2n-2k+1)\frac{\theta}{2}-\sin(2k-1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right)',$$

(1.4)

$$d_{k2}(\theta)=\sum_{j=k}^{\frac{n-1}{2}}(j\cos j\theta+(n-j)\cos(n-j)\theta)=\left(\frac{\cos(2k-1)\frac{\theta}{2}-\cos(2n-2k+1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\right)'$$

2. Kac-Rice Formula

Let $N(0, 2\pi)$ be denotes the number of real roots of the random trigonometric polynomials in the interval $(0, 2\pi)$ and $E(N(0, 2\pi))$ be its expected value. To deal with the asymptotic behavior of the expected number of real roots of $D_n(\theta) = 0$ and $Q_n(\theta) = 0$, we refer to Kac-Rice formula [10, 11], which is defined as

$$E(N(0, 2\pi)) = \int_0^{2\pi} \frac{\Delta}{\pi A^2} d\theta, \tag{2.1}$$

where $\Delta^2 = A^2 B^2 - C^2$. For $D_n(\theta)$ given in (1.1) we have

$$A_D^2 = \mathrm{Var}(D_n(\theta)) = \sum_{k=0}^{n} (a_{k1}^2(\theta)\sigma_1^2 + b_{k1}^2(\theta)\sigma_2^2),$$

$$B_D^2 = \mathrm{Var}(D_n'(\theta)) = \sum_{k=0}^{n} (c_{k1}^2(\theta)\sigma_1^2 + d_{k1}^2(\theta)\sigma_2^2),$$

$$C_D = \mathrm{Cov}(D_n(\theta), D_n'(\theta)) = \sum_{k=0}^{n} (a_{k1}(\theta)c_{k1}(\theta)\sigma_1^2 + b_{k1}(\theta)d_{k1}(\theta)\sigma_2^2), \tag{2.2}$$

where $a_{k1}(\theta)$, $b_{k1}(\theta)$, $c_{k1}(\theta)$ and $d_{k1}(\theta)$ are defined in (1.2). For $Q_n(\theta)$ given in (1.3) we have

$$A_Q^2 = \mathrm{Var}(Q_n(\theta)) = \sum_{k=0}^{\frac{n-1}{2}} (a_{k2}^2(\theta)\sigma_1^2 + b_{k2}^2(\theta)\sigma_2^2),$$

$$B_Q^2 = \mathrm{Var}(Q_n'(\theta)) = \sum_{k=0}^{\frac{n-1}{2}} (c_{k2}^2(\theta)\sigma_1^2 + d_{k2}^2(\theta)\sigma_2^2),$$

$$C_Q = \mathrm{Cov}(Q_n(\theta), Q_n'(\theta)) = \sum_{k=0}^{\frac{n-1}{2}} (a_{k2}(\theta)c_{k2}(\theta)\sigma_1^2 + b_{k2}(\theta)d_{k2}(\theta)\sigma_2^2), \tag{2.3}$$

where $a_{k2}(\theta)$, $b_{k2}(\theta)$, $c_{k2}(\theta)$ and $d_{k2}(\theta)$ are defined in (1.4).

As in algebraic case the above identities are not well behaved around 0, π and 2π. Therefore we first consider the intervals $(\varepsilon, \pi - \varepsilon)$, $(\pi + \varepsilon, 2\pi - \varepsilon)$, where ε is any positive constant, smaller than π and arbitrary at this point to be chosen later. It should be positive and small enough to facilitate handling the roots in the intervals $(\varepsilon, \pi - \varepsilon)$, $(\pi + \varepsilon, 2\pi - \varepsilon)$ and for roots inside this two intervals, we use (2.1). For the real roots lying in the intervals $(0, \varepsilon)$, $(\pi - \varepsilon, \pi + \varepsilon)$ and $(2\pi - \varepsilon, 2\pi)$, which it so happens, are negligible, we will use a different method based on the Jensen's theorem.

We now define some functions to make the estimations, define $S(\theta) = \sin(2n + 1)\theta/\sin\theta$, see from [5, page 74] which is continuous at $\theta = j\pi$ and will occur frequently in follows. Since for $\theta \in (\varepsilon, \pi - \varepsilon)$ and $\theta \in (\pi + \varepsilon, 2\pi - \varepsilon)$, we have $|S(\theta)| < 1/\sin\varepsilon$. Hence, we can obtain

$$S(\theta) = O\left(\frac{1}{\varepsilon}\right)$$

Further

$$S'(\theta) = O\left(\frac{n}{\varepsilon}\right), \quad S''(\theta) = O\left(\frac{n^2}{\varepsilon}\right)$$

We can show

$$\sum_{k=0}^{n} \cos k\theta = \frac{\sin(2n+1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}} + \frac{1}{2} = \frac{S(\frac{\theta}{2}) + 1}{2} = O\left(\frac{1}{\varepsilon}\right), \tag{2.4}$$

and

$$\sum_{k=0}^{n} k \sin k\theta = -\frac{S'(\frac{\theta}{2})}{4} = O\left(\frac{n}{\varepsilon}\right), \quad \sum_{k=0}^{n} k^2 \cos k\theta = -\frac{S''(\frac{\theta}{2})}{8} = O\left(\frac{n^2}{\varepsilon}\right), \tag{2.5}$$

In similar way, we define $P(\theta) = \cos\theta - \frac{\cos(2n+1)\theta}{2\sin\theta}$, we also have $|P(\theta)| < 1/\sin\varepsilon$. Hence, we can obtain

$$P(\theta) = O\left(\frac{1}{\varepsilon}\right)$$

Further

$$P'(\theta) = O\left(\frac{n}{\varepsilon}\right), \quad P''(\theta) = O\left(\frac{n^2}{\varepsilon}\right)$$

We can show

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos\frac{\theta}{2} - \cos(2n+1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = P\left(\frac{\theta}{2}\right) = O\left(\frac{1}{\varepsilon}\right), \tag{2.6}$$

and

$$\sum_{k=0}^{n} k \cos k\theta = \frac{P'(\frac{\theta}{2})}{4} = O\left(\frac{n}{\varepsilon}\right), \quad \sum_{k=0}^{n} k^2 \sin k\theta = -\frac{P''(\frac{\theta}{2})}{8} = O\left(\frac{n^2}{\varepsilon}\right), \tag{2.7}$$

Now, using the above identities, we are able to evaluate the characteristics required in using the Kac-Rice formula in (2.1).

3. Asymptotic Behavior of $E(N(0, 2\pi))$

This section includes two subsection. We evaluate the asymptotic behavior of the expected number of real roots of $D_n(\theta) = 0$ in the intervals $(\varepsilon, \pi - \varepsilon)$, $(\pi + \varepsilon, 2\pi - \varepsilon)$ in subsection 3.1 and in the intervals $(0, \varepsilon)$, $(\pi - \varepsilon, \pi + \varepsilon)$, $(2\pi - \varepsilon, 2\pi)$ in subsection 3.2.

3.1. **Results On the Intervals $(\varepsilon, \pi-\varepsilon)$, $(\pi+\varepsilon, 2\pi-\varepsilon)$.** In this part, we obtain our results by applying the Kac-Rice formula. The main contribution of this part for the two different cases is stated separately in the following theorems.

Theorem 3.1. *Let $D_n(\theta)$ be the random trigonometric polynomial given in* (1.1) *for which $A_k = \Delta_0^{(1)} + \Delta_1^{(1)} + \cdots + \Delta_k^{(1)}$, $B_k = \Delta_0^{(2)} + \Delta_1^{(2)} + \cdots + \Delta_k^{(2)}$, $k = 0, 1, \ldots, n$, where $\Delta_k^{(i)}$, $k = 0, 1, \ldots, n$, $i = 1, 2$ are standard normal i.i.d random variables independent. We prove that for all sufficiently large n, the expected number of real roots of the equation $D_n(\theta) = 0$, satisfies*

$$EN(\varepsilon, \pi-\varepsilon) = EN(\pi+\varepsilon, 2\pi-\varepsilon) \simeq \frac{\sqrt{2}n}{\sqrt{3}}$$

Proof. In order to use the Kac-Rice formula, we first evaluate asymptotic value for each variable needed by using the error terms obtained in (2.4)–(2.7). Since $\mathrm{E}(A_k) = 0$ and $\mathrm{E}(B_k) = 0$ we have

$$\mathrm{E}(D_n(\theta)) = 0, \quad \mathrm{E}(D_n'(\theta)) = 0, \tag{3.1}$$

Now using (2.4)-(2.7) and (1.2) and using some trigonometric identities, we obtain the variance of $D_n(\theta)$ and $D_n'(\theta)$, as

$$\begin{aligned} A_D^2 = \mathrm{Var}(D_n(\theta)) &= \sum_{k=0}^{n} (a_{k1}^2(\theta) + b_{k1}^2(\theta)) \\ &= \sum_{k=0}^{n} \left[\left(\sum_{j=k}^{n} \cos j\theta \right)^2 + \left(\sum_{j=k}^{n} \sin j\theta \right)^2 \right] = \frac{n}{2\sin^2\frac{\theta}{2}} + O(\frac{1}{\varepsilon}), \end{aligned} \tag{3.2}$$

$$\begin{aligned} B_D^2 = \mathrm{Var}(D_n'(\theta)) &= \sum_{k=0}^{n} (c_{k1}^2(\theta) + d_{k1}^2(\theta)) \\ &= \sum_{k=0}^{n} \left[\left(-\sum_{j=k}^{n} j \sin j\theta \right)^2 + \left(\sum_{j=k}^{n} j \cos j\theta \right)^2 \right] \\ &= \sum_{k=0}^{n} \frac{4n^2 + 4k^2}{16\sin^2\frac{\theta}{2}} + O\left(\frac{n^2}{\varepsilon}\right) = \frac{n^3}{3\sin^2\frac{\theta}{2}} + O\left(\frac{n^2}{\varepsilon}\right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} C_D = \mathrm{Cov}(D_n(\theta), D_n'(\theta)) &= \sum_{k=0}^{n} (a_{k1}(\theta)c_{k1}(\theta) + b_{k1}(\theta)d_{k1}(\theta)) \\ &= \sum_{k=0}^{n} \left[\left(\sum_{j=k}^{n} \cos j\theta \right) \left(-\sum_{j=k}^{n} j \sin j\theta \right) + \left(\sum_{j=k}^{n} \sin j\theta \right) \left(\sum_{j=k}^{n} j \cos j\theta \right) \right] \\ &= O\left(\frac{n}{\varepsilon}\right), \end{aligned} \tag{3.4}$$

Then, finally from (3.2)-(3.4), we can obtain

$$\Delta^2 = A_D^2 B_D^2 - C_D^2 = \frac{n^4}{6\sin^4\frac{\theta}{2}} + O\left(\frac{n^3}{\varepsilon}\right), \tag{3.5}$$

The results of (3.2) and (3.5) into the Kac-Rice formula (2.1), we have

$$E(N(\varepsilon, \pi - \varepsilon)) = E(N(\pi + \varepsilon, 2\pi - \varepsilon)) \sim \frac{\sqrt{2}n}{\sqrt{3}}$$

The theorem is proved. □

Theorem 3.2. *Let $Q_n(\theta)$ be the random trigonometric polynomial given in* (1.3) *where $A_k = \Delta_0^{(1)} + \Delta_1^{(1)} + \cdots + \Delta_k^{(1)}$, $B_k = \Delta_0^{(2)} + \Delta_1^{(2)} + \cdots + \Delta_k^{(2)}$, $k = 0, 1, \ldots, \frac{n-1}{2}$, where $\Delta_k^{(i)}$, $k = 0, 1, \ldots, \frac{n-1}{2}$, $i = 1, 2$, are standard normal i.i.d random variables. We prove that for all sufficiently large n, the expected number of real roots of the equation $Q_n(\theta) = 0$, satisfies*

$$EN(\varepsilon, \pi - \varepsilon) = EN(\pi + \varepsilon, 2\pi - \varepsilon) \simeq \frac{n}{\sqrt{3}}$$

Proof. We obtain our results by applying the Kac-Rice formula. Since $\mathrm{E}(A_k) = 0$ and $\mathrm{E}(B_k) = 0$ we have

$$\mathrm{E}(Q_n(\theta)) = 0, \quad \mathrm{E}(Q_n'(\theta)) = 0$$

Now from (1.4) and (2.4)–(2.7) and making some trigonometric identities, we obtain

$$\begin{aligned} A_Q^2 = \mathrm{Var}(Q_n(\theta)) &= \sum_{k=0}^{\frac{n-1}{2}} (a_{k2}^2(\theta) + b_{k2}^2(\theta)) \\ &= \sum_{k=0}^{\frac{n-1}{2}} \left[\left(\sum_{j=k}^{\frac{n-1}{2}} (\cos j\theta + \cos{(n-j)\theta}) \right)^2 + \left(\sum_{j=k}^{\frac{n-1}{2}} (\sin j\theta + \sin{(n-j)\theta}) \right)^2 \right] \\ &= \frac{n}{4\sin^2\frac{\theta}{2}} + O(\frac{1}{\varepsilon}), \end{aligned} \tag{3.6}$$

$$\begin{aligned} B_Q^2 = \mathrm{Var}(Q_n'(\theta)) &= \sum_{k=0}^{\frac{n-1}{2}} (c_{k2}^2(\theta) + d_{k2}^2(\theta)) \\ &= \sum_{k=0}^{\frac{n-1}{2}} \left[\left(-\sum_{j=k}^{\frac{n-1}{2}} (j\sin j\theta + (n-j)\sin{(n-j)\theta}) \right)^2 \right. \\ &\quad \left. + \left(\sum_{j=k}^{\frac{n-1}{2}} (j\cos j\theta + (n-j)\cos{(n-j)\theta}) \right)^2 \right] \\ &= \sum_{k=0}^{\frac{n-1}{2}} \frac{4n^2 - 8k^2 - 8nk}{16\sin^2\frac{\theta}{2}} + O(\frac{n^2}{\varepsilon}) = \frac{n^3}{12\sin^2\frac{\theta}{2}} + O(\frac{n^2}{\varepsilon}), \end{aligned} \tag{3.7}$$

ani

$$C_Q = \mathrm{Cov}(Q_n(\theta), Q'_n(\theta)) = \sum_{k=0}^{\frac{n-1}{2}} (a_{k2}(\theta)c_{k2}(\theta) + b_{k2}(\theta)d_{k2}(\theta))$$

$$= \sum_{k=0}^{\frac{n-1}{2}} \left[\left(\sum_{j=k}^{\frac{n-1}{2}} (\cos j\theta + \cos(n-j)\theta) \right) \left(-\sum_{j=k}^{\frac{n-1}{2}} (j \sin j\theta + (n-j)\sin(n-j)\theta) \right) + \left(\sum_{j=k}^{\frac{n-1}{2}} (\sin j\theta + \sin(n-j)\theta) \right) \left(\sum_{j=k}^{\frac{n-1}{2}} (j\cos j\theta + (n-j)\cos(n-j)\theta) \right) \right]$$

(3.8)
$$= O\left(\frac{n}{\varepsilon}\right),$$

Then, finally from (3.6)–(3.8) we can get

$$\Delta^2 = A_Q^2 B_Q^2 - C_Q^2 = \frac{n^4}{48\sin^4\frac{\theta}{2}} + O\left(\frac{n^3}{\varepsilon}\right), \tag{3.9}$$

The results of (3.6) and (3.9) into the Kac-Rice formula (2.1), we can obtain

$$E(N(\varepsilon, \pi-\varepsilon)) = E(N(\pi+\varepsilon, 2\pi-\varepsilon)) \sim \frac{n}{\sqrt{3}}$$

□

3.2. **Results On the Intervals** $(0,\varepsilon)$**,** $(\pi-\varepsilon, \pi+\varepsilon)$**,** $(2\pi-\varepsilon, 2\pi)$**.** In this subsection, we are going to show the expected number of real roots in the intervals $(0,\varepsilon)$, $(\pi-\varepsilon, \pi+\varepsilon)$, $(2\pi-\varepsilon, 2\pi)$ is negligible. The period of $D_n(\theta)$ is 2π, and so the number of real roots in the interval $(0,\varepsilon)$ and $(2\pi-\varepsilon, 2\pi)$ is the same as the number in $(-\varepsilon,\varepsilon)$, the interval $(\pi-\varepsilon, \pi+\varepsilon)$ can be treated in the same way to give the same result. Here we deal only with $D_n(\theta)$, since the same method is applicable for the random trigonometric polynomial, $Q_n(\theta)$, and the results of $D_n(\theta)$ remain the same for $Q_n(\theta)$. We consider the function of the complex variable z,

$$D_n(z,\omega) = \sum_{k=0}^{n} (A_k(\omega)\cos kz + B_k(\omega)\sin kz)$$

We seek an upper bound to the number of real roots in the segment of the real axis joining the points $\pm\varepsilon$, and this certainly does not exceed the number of real roots in the circle $|z| < \varepsilon$.

Let $N(r) \equiv N(r,\omega)$ denote the number of real roots of $D_n(z,\omega) = 0$ in $|z| < \varepsilon$. We will modify the method based on the Jensen's theorem [12], which has been used by Dunnage [2], then By Jensen's theorem,

$$\int_{\varepsilon}^{2\varepsilon} r^{-1}N(r)dr \le \int_0^{2\varepsilon} r^{-1}N(r)dr = \frac{1}{2\pi}\int_0^{2\pi} \log|\frac{D_n(2\varepsilon e^{i\theta},\omega)}{D_n(0)}|d\theta$$

for which we have

$$N(\varepsilon)\log 2 \leq \frac{1}{2\pi}\int_0^{2\pi}\log|\frac{D_n(2\varepsilon e^{i\theta},\omega)}{D_n(0)}|d\theta, \tag{3.10}$$

Now since the distribution function of $D_n(0,\omega)=\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta_k^1(\omega)$ is

$$G(x)\sim N\left(0,\frac{(2n^3+9n^2+13n+6)}{6}\right)$$

We can see that for any positive υ,

$$\begin{aligned} &P(-e^{-\upsilon}\leq D_n(0,\omega)\leq e^{-\upsilon})\\ &=\sqrt{\frac{3}{\pi(2n^3+9n^2+13n+6)}}\int_{-e^{-\upsilon}}^{e^{-\upsilon}}\exp\left\{-\frac{3t^2}{2n^3+9n^2+13n+6}\right\}dt\\ &<\frac{2\sqrt{3}e^{-\upsilon}}{\sqrt{\pi(2n^3+9n^2+13n+6)}}, \end{aligned} \tag{3.11}$$

Also we have

$$\begin{aligned} |D_n(2\varepsilon e^{i\theta})| &= |\sum_{k=0}^{n}\left(\sum_{j=k}^{n}\cos(2j\varepsilon e^{i\theta})\right)\Delta_k^1+\sum_{k=0}^{n}\left(\sum_{j=k}^{n}\sin(2j\varepsilon e^{i\theta})\right)\Delta_k^2|\\ &\leq 2M(n+1)(n+2)e^{2n\varepsilon}, \end{aligned} \tag{3.12}$$

where $M=\mathrm{Max}_k(\max|\Delta_k^1|,\max|\Delta_k^2|)$. The distribution function of $|\Delta_k^1|$ and $|\Delta_k^2|$ is

$$F(x)=\begin{cases}\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}} & \text{if } x\geq 0\\ 0 & \text{if } x<0\end{cases}$$

For any positive υ and all sufficiently large n, the probability $M>ne^{\upsilon}$ is

$$\begin{aligned} P(M>ne^{\upsilon}) &\leq nP(|\Delta_1^1|>ne^{\upsilon})\\ &=n\frac{1}{\sqrt{2\pi}}\int_{ne^{\upsilon}}^{\infty}e^{-\frac{t^2}{2}}dt\simeq\sqrt{\frac{2}{\pi}}\exp\left\{-\upsilon-\frac{(ne^{\upsilon})^2}{2}\right\}, \end{aligned} \tag{3.13}$$

Therefore from (3.12) and (3.13), except for sample functions in an ω-set of measure not exceeding

$$\sqrt{\frac{2}{\pi}}\exp\left\{-\upsilon-\frac{(ne^{\upsilon})^2}{2}\right\}|D_n(2\varepsilon e^{i\theta})|<3n(n+1)(n+2)e^{2n\varepsilon+\upsilon}, \tag{3.14}$$

Hence from (3.11), (3.14) and since we obtain

$$|\frac{D_n(2\varepsilon e^{i\theta},\omega)}{D_n(0,\omega)}|\leq 3n(n+1)(n+2)e^{2n\varepsilon+2\upsilon}, \tag{3.15}$$

Except for sample function in an ω-set of measure not exceeding

$$\frac{2\sqrt{3}e^{-\upsilon}}{\sqrt{\pi(2n^3+9n^2+13n+6)}}+\sqrt{\frac{2}{\pi}}\exp\left\{-\upsilon-\frac{(ne^{\upsilon})^2}{2}\right\}$$

Therefore from (3.10) and (3.15) we can show that outside the exceptional set

$$N(\varepsilon) \leq \frac{\log 3 + \log n \log(n+1) + \log(n+2) + 2n\varepsilon + 2\upsilon}{\log 2}, \tag{3.16}$$

$\varepsilon = n^{-1/4}$, it follows from (3.16) and for any sufficiently large n that

$$P(N(\varepsilon) > 3n\varepsilon + 2\upsilon) \leq \frac{2\sqrt{3}e^{-\upsilon}}{\sqrt{\pi(2n^3+9n^2+13n+6)}} + \sqrt{\frac{2}{\pi}} \exp\left\{-\upsilon - \frac{(ne^{\upsilon})^2}{2}\right\}, \tag{3.17}$$

Let $n' = [3n^{3/4}]$ be the greatest integer less than equal to $3n^{3/4}$, then from (3.17) and for n large enough we obtain

$$\begin{aligned} EN(\varepsilon) &= \sum_{j>0} P(N(\varepsilon) \geq j) = \sum_{1 \leq j \leq n'} P(N(\varepsilon) > j) + \sum_{j \geq 1} P(N(\varepsilon) \geq n' + j) \\ &\leq n' + \sqrt{\frac{12}{\pi(2n^3+9n^2+13n+6)}} \sum_{j \geq 1} e^{-j/2} + \sqrt{\frac{2}{\pi}} \sum_{j \geq 1} \exp\left\{-\frac{j}{2} - \frac{(ne^j)^2}{2}\right\} \\ &= O(n^{3/4}), \end{aligned} \tag{3.18}$$

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