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NEW HIGHLY ACCURATE STABLE SCHEMES FOR THE SOLUTION OF TELEGRAPHIC EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we present two new three level implicit schemes to solve telegraphic equation with Neumann boundary conditions. The accuracy of the proposed schemes is of $O(k^2 + k^2h^2 + h^4)$ and $O(k^4 + k^4h^2)$, where $h > 0$ and $k > 0$ are the mesh sizes in the space and time directions respectively. The proposed schemes are shown to be solvable and unconditionally stable. Numerical experiments are presented to demonstrate the accuracy and efficiency of the new schemes.

Keywords. maximum absolute errors; Neumann boundary conditions; Richardson extrapolation; unconditionally stable scheme; telegraphic equation

AMS (MOS) Subject Classification. 39A10.

1. INTRODUCTION

We consider the following one-dimensional telegraphic equation

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + f(x,t), \quad 0 \le x \le 1,\ t > 0 \tag{1.1}$$

subject to the initial conditions

$$u(x,0) = \phi(x), u_t(x,0) = \psi(x), \quad 0 \le x \le 1 \tag{1.2}$$

and the Neumann boundary conditions

$$u_x(0,t) = \omega(t), u_x(1,t) = \rho(t), \quad t > 0 \tag{1.3}$$

where $\alpha > 0$, $\beta \ge 0$ are constants. We assume $\omega(t)$, $\rho(t)$ and their derivatives to be continuous functions of t. The telegraphic equation is employed to model wave fields taking into account energy dissipation and media stiffness. The telegraphic equation is more suitable than ordinary diffusion equation in modeling reaction-diffusion for many branches of sciences, for example, biologists use these equations to study the

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pulse rate, blood flow in arteries and in one-dimensional random motion of bugs along a hedge.

In recent past various numerical schemes have been developed to solve telegraphic equation [1]–[7] with Dirichlet boundary conditions. In [4], high order accurate three level implicit scheme has been discussed for solving hyperbolic equations. Mohanty et al. [5] have discussed a super stable scheme which behaves like a fourth order scheme when mesh size in time direction is directly proportional to the square of mesh size in space direction. In these cases, the telegraphic equation includes only Dirichlet boundary conditions. However, not many researchers have considered telegraphic equation subject to Neumann boundary conditions. Since, many physical applications involve Neumann boundary conditions, it is important to have a high order approximation of telegraphic equation with Neumann boundary conditions. Recently, Liu and Liu [8] proposed unconditionally stable two level compact difference schemes based on the generalized trapezoidal formula for solving one dimensional telegraphic equation with Neumann boundary conditions which are $O(k^2 + h^3)$ and $O(k^3 + h^3)$ accurate. Motivated by Liao et al. [9], we present a fourth order approximation of Neumann boundary conditions which maintains the tri-diagonal structure of the coefficient matrices. In this paper, we introduce two new higher order accurate three level implicit schemes. Based on Numerov type discretization [4], we first introduce a scheme of $O(k^2 + k^2h^2 + h^4)$. Coupled with Richardson extrapolation [11] the accuracy of the scheme has been improved to $O(k^4+k^4h^2)$. The unconditional stability of the schemes has been discussed by Matrix method. The first scheme behaves like a fourth order scheme when mesh size in time is directly proportional to mesh size in space direction and the second scheme behaves like an eighth order scheme when mesh size in time is directly proportional to the square of mesh size in space direction.

The organization of the paper is as follows.

In section 2, we present a high order three level implicit scheme. In section 3, a high order approximation for Neumann boundary conditions is presented. The solvability and stability analysis is given in section 4. In section 5, we present numerical experiments to verify the efficiency and accuracy of the new algorithm. Finally, concluding remarks are given in section 6.

2. REVIEW OF HIGH ORDER THREE LEVEL IMPLICIT SCHEME

In this section, we first briefly introduce the three level implicit Numerov type discretization with accuracy of order two in time and four in space for the solution of telegraphic equation (1.1).

We discretize the region $\{(x,t) : 0 \le x \le 1, t \ge 0\}$ into N subintervals in space direction with spacing of $h > 0$ such that Nh =1 and J sub-intervals in time direction with spacing of $k > 0$ where N and J are positive integers. For l =0(1)N and $0 < j < J$, the grid point $(x_l, t_j) = (lh, jk)$ is denoted by (l, j). Let exact solution of (1.1) at grid point (l, j) be denoted by u_l^j. Similarly, we denote u_x at (l, j) by u_{xl}^j, u_{xx} at (l, j) by u_{xxl}^j and so on. Let $p = \frac{k}{h} > 0$ be the mesh ratio parameter.

A Numerov type discretization [4] with accuracy of $O(k^2 + k^2h^2 + h^4)$ for the solution of (1.1) for l =1(1)N−1, 1≤ $j \leq J$ may be written as

$$\delta_t^2 u_l^j + \alpha k(2\mu_t\delta_t)u_l^j + \frac{\alpha k}{12}(\delta_x^2(2\mu_t\delta t))u_l^j + \left(\frac{\beta^2k^2}{12} - p^2\right)\delta_x^2 u_l^j + \beta^2k^2u_l^j + \frac{\delta_x^2\delta_t^2}{12}u_l^j = \frac{k^2}{12}\left(f_{l+1}^j + f_{l-1}^j + 10f_l^j\right) + O(k^4 + k^4h^2 + k^2h^4) \tag{2.1}$$

where

$$\delta_t^2 u_l^j = u_l^{j+1} - 2u_l^j + u_l^{j-1} \tag{2.2}$$

$$\delta_x^2 u_l^j = u_{l+1}^j - 2u_l^j + u_{l-1}^j \tag{2.3}$$

$$(2\mu_t\delta_t)u_l^j = u_l^{j+1} - u_l^{j-1} \tag{2.4}$$

Above scheme has a temporal truncation error of $O(k^4 + k^4h^2 + k^6)$. In order to improve the temporal accuracy of the scheme we apply Richardson extrapolation technique, such that, we have new truncation error representation as follows,

$$u = \frac{4u^{\frac{k}{2}} - u^k}{3} + O(k^6 + k^6h^2) \tag{2.5}$$

where $a = \alpha^2k^2$, $b = \beta^2k^2$ and γ and η are free parameters to be determined. The additional terms are of high order and do not affect the accuracy of the scheme.

For stability, we consider the homogeneous part of the scheme (2.5), which in matrix form together with Neumann boundary approximations can be written as:

$$\mathbf{Z}\mathbf{u}^{j+1} + \mathbf{X}\mathbf{u}^j + \mathbf{Y}\mathbf{u}^{j-1} = \mathbf{C} \tag{2.6}$$

where,

$$\mathbf{Z} = (1 + \eta b^2 + \sqrt{a})\mathbf{D} - 12\gamma p^2\mathbf{P},$$
$$\mathbf{X} = (-2 - 2\eta b^2 + b)\mathbf{D} + 12(2\gamma - 1)p^2\mathbf{P},$$
$$\mathbf{Y} = (1 + \eta b^2 - \sqrt{a})\mathbf{D} - 12\gamma p^2\mathbf{P},$$

$$\mathbf{D} = \begin{bmatrix} 5 & 1 & & & & \\ 1 & 10 & 1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 10 & 1 \\ & & & & 1 & 5 \end{bmatrix}_{N+1,N+1},$$

$$\mathbf{P} = \begin{bmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{bmatrix}_{N+1,N+1},$$

$\mathbf{C}$ is the column vector corresponding to the boundary conditions and

$$\mathbf{u}^j = \begin{bmatrix} u_0^j & u_1^j & \dots & u_N^j \end{bmatrix}'$$

is the solution vector.

The matrices $\mathbf{D}$ and $\mathbf{Z}$ being strictly diagonally dominant are invertible. Equation (2.6) can be written as

$$\begin{bmatrix} \mathbf{u}^{j+1} \\ \dots \\ \mathbf{u}^j \end{bmatrix} = \begin{bmatrix} -\mathbf{Z}^{-1}\mathbf{X} & \vdots & -\mathbf{Z}^{-1}\mathbf{Y} \\ \dots & \dots & \dots \\ \mathbf{I}_{N+1} & \vdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^j \\ \dots \\ \mathbf{u}^{j-1} \end{bmatrix} + \begin{bmatrix} \mathbf{Z}^{-1}\mathbf{C} \\ \dots \\ \mathbf{0} \end{bmatrix} \tag{2.7}$$

which may be rewritten as

$$\mathbf{v}^{j+1} = \mathbf{B}\mathbf{v}^j + \mathbf{G}, \tag{2.8}$$

where

$$\mathbf{v}^{j+1} = \begin{bmatrix} \mathbf{u}^{j+1} & \vdots & \mathbf{u}^j \end{bmatrix}',$$

$$\mathbf{B} = \begin{bmatrix} -\mathbf{Z}^{-1}\mathbf{X} & \vdots & -\mathbf{Z}^{-1}\mathbf{Y} \\ \dots & \dots & \dots \\ \mathbf{I}_{N+1} & \vdots & \mathbf{0} \end{bmatrix}$$

and

$$\mathbf{G} = \begin{bmatrix} \mathbf{Z}^{-1}\mathbf{C} \\ \dots \\ \mathbf{0} \end{bmatrix}.$$

Thus the difference scheme is solvable. For stability, we require the following lemma.

Lemma: If λ is an eigenvalue of the matrix $\mathbf{D}^{-1}\mathbf{P}$, then $\lambda \leq 0$.

Proof: Let x be an eigenvector of $\mathbf{D}^{-1}\mathbf{P}$ corresponding to the eigenvalue λ, then

$$\mathbf{D}^{-1}\mathbf{P}x = \lambda x$$

$$x^t\mathbf{P}x = \lambda x^t\mathbf{D}x \tag{2.9}$$

Since,

$$x^t\mathbf{P}x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ x_{N+1} \end{bmatrix}^t \begin{bmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ x_{N+1} \end{bmatrix}$$

$$= -x_1^2 - 2x_2^2 - 2x_3^2 - \cdots - 2x_N^2 - x_{N+1}^2 + 2x_1x_2 + 2x_2x_3 + \cdots + 2x_Nx_{N+1}$$

$$= -\left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_N - x_{N+1})^2\right]$$

$$\leq 0$$

and

$$x^t\mathbf{D}x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ x_{N+1} \end{bmatrix}^t \begin{bmatrix} 5 & 1 & & & & \\ 1 & 10 & 1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 10 & 1 \\ & & & & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \\ x_{N+1} \end{bmatrix}$$

$$= 5x_1^2 + 10x_2^2 + 10x_3^2 + \cdots + 10x_N^2 + 5x_{N+1}^2 + 2x_1x_2 + 2x_2x_3 + \cdots + 2x_Nx_{N+1}$$

$$= (4x_1^2 + 8x_2^2 + 8x_3^2 + \cdots + 8x_N^2 + 4x_{N+1}^2) + (x_1 + x_2)^2 + (x_2 + x_3)^2 + \cdots$$

$$+ (x_N + x_{N+1})^2$$

$$> 0$$

Hence, from (2.9) we get, $\lambda \leq 0$.

Now, λ is eigenvalue of $\mathbf{D}^{-1}\mathbf{P}$, therefore, $\frac{(-2-2\eta b^2+b)+12(2\gamma-1)p^2\lambda}{(1+\eta b^2+\sqrt{a})-12\gamma p^2\lambda}$ and $\frac{1+\eta b^2-\sqrt{a}-12\gamma p^2\lambda}{1+\eta b^2+\sqrt{a}-12\gamma p^2\lambda}$ are the eigenvalues of $\mathbf{Z}^{-1}\mathbf{X}$ and $\mathbf{Z}^{-1}\mathbf{Y}$ respectively, having the same corresponding eigen vectors. Now, the eigenvalues Λ of $\mathbf{B}$, are the eigenvalues of the matrix (2.10) (see [12]),

$$\begin{bmatrix} \dfrac{2+2\eta b^2 - b - 12(2\gamma-1)p^2\lambda}{1+\eta b^2+\sqrt{a}-12\gamma p^2\lambda} & \dfrac{-1-\eta b^2+\sqrt{a}+12\gamma p^2\lambda}{1+\eta b^2+\sqrt{a}-12\gamma p^2\lambda} \\ \\ 1 & 0 \end{bmatrix} \tag{2.10}$$

The characteristic equation of which is

$$\Lambda^2 + M\Lambda + N = 0 \tag{2.11}$$

where

$$M = \frac{(-2-2\eta b^2+b)+12(2\gamma-1)p^2\lambda}{(1+\eta b^2+\sqrt{a})-12\gamma p^2\lambda}$$

and

$$N = \frac{1+\eta b^2-\sqrt{a}-12\gamma p^2\lambda}{1+\eta b^2+\sqrt{a}-12\gamma p^2\lambda}.$$

Using the transformation $\Lambda = \frac{1+z}{1-z}$, the characteristic equation (2.11) reduces to

$$(1 - M + N)z^2 + 2(1 - N)z + (1 + M + N) = 0$$

We have,

$$1 + M + N = \frac{b - 12p^2\lambda}{1 + \eta b^2 + \sqrt{a} - 12\gamma p^2\lambda} > 0,$$

$$1 - N = \frac{2\sqrt{a}}{1 + \eta b^2 + \sqrt{a} - 12\gamma p^2\lambda} > 0$$

and

$$1 - M + N = \frac{4 + 4\eta b^2 - b - 12(4\gamma - 1)p^2\lambda}{1 + \eta b^2 + \sqrt{a} - 12\gamma p^2\lambda} > 0$$

if $\eta \geq \frac{1}{64}$, $\gamma \geq \frac{1}{4}$. Therefore, we obtain $|\Lambda| \leq 1$.

Thus, for the choice $\eta \geq \frac{1}{64}$, $\gamma \geq \frac{1}{4}$ the scheme is unconditionally stable. Moreover, since Richardson extrapolation is applied only once, therefore, unconditional stability of the scheme is preserved.

3. NUMERICAL EXPERIMENTS

In this section, we present numerical examples to demonstrate the efficiency and accuracy of the proposed schemes, viz., the scheme (2.6) and the scheme obtained by applying Richardson extrapolation. Let's refer these schemes as scheme I and scheme II respectively. We compare numerical results with analytical solution. The errors presented in the tables below are obtained by the following formula,

$$L_\infty = \max \left| (u_l^j)^{analytic} - (u_l^j)^{num} \right|$$

In the tables, e_1 and e_2 represent the errors obtained with schemes I and II respectively.

The results are also compared with the scheme of $O(k^2 + h^2)$ for the solution of telegraphic equation (1.1) given as

$$(1 - \gamma p^2\delta_x^2)\delta_t^2 u_l^j - p^2\delta_x^2 u_l^j + \sqrt{a}(2\mu_t\delta_t)u_l^j + bu_l^j = k^2 f_l^j, l = 0(1)N, j = 1(1)J \tag{3.1}$$

with Neumann boundary approximations

$$u_{-1}^j = u_1^j - 2h\omega^j \tag{3.2}$$

$$u_{N+1}^j = u_{N-1}^j + 2h\rho^j \tag{3.3}$$

We denote error obtained using this low order scheme by e_0. The error graph shown in figures reflects the comparison between error obtained using this scheme and the proposed schemes.

Example 1.

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + (3 - 4\alpha + \beta^2)e^{-2t}\sinh(x), \quad 0 \le x \le 1, t > 0$$

subject to the initial conditions

$$u(x,0) = \sinh(x), \quad u_t(x,0) = -2\sinh(x), \quad 0 \le x \le 1$$

and the Neumann boundary conditions

$$u_x(0,t) = e^{-2t}, \quad u_x(1,t) = e^{-2t}\left(\frac{e + e^{-1}}{2}\right), \quad t > 0$$

The analytical solution is given by $u = e^{-2t}\sinh(x)$. Since, our scheme is a linear tri-diagonal system, we can solve this system using tri-diagonal solver method. The data given in Tables 1–3 shows the L_∞ error between the numerical solution and the analytical solution at time t =1 for $\alpha = 10$, $\beta = 5$, $\eta = 0.5$, $\gamma = 1$.

TABLE 1. Error obtained for Example 1 for fixed mesh size $h = 0.001$

k	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
e_1	9.0098e-04	2.6390e-04	6.8816e-05	1.7426e-05
order	-	1.77	1.94	1.98
e_2	5.1645e-05	3.7630e-06	2.9351e-07	2.5850e-08
order	-	3.78	3.67	3.50

The data in Table 1 shows that for a fixed $h = 0.001$, when k is reduced by a factor of 2, e_1 shows a second order decrease and e_2 shows a fourth order decrease in error, which indicates the second and fourth order accuracy of the schemes I and II respectively in time. Now, the data given in Table 2 demonstrates that for the same

TABLE 2. Error obtained for Example 1 when $k \propto h$

$h = k$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
e_1	8.9389e-04	2.6353e-04	6.8797e-05	1.7427e-05
order	-	1.76	1.94	1.98
e_2	5.1357e-05	3.7558e-06	2.9534e-07	2.5791e-08
order	-	3.77	3.67	3.50

mesh sizes along space and time directions, schemes I and II behave as second order and fourth order accurate respectively in time. The data in Table 3 shows that for a

TABLE 3. Error obtained for Example 1 when $k \propto h^2$

h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
k	$\frac{1}{5}$	$\frac{1}{20}$	$\frac{1}{80}$	$\frac{1}{320}$
e_1	6.4000e-03	6.0376e-04	4.3960e-05	2.7972e-06
order	-	3.41	3.78	3.97
e_2	4.0000e-03	2.1338e-05	1.2413e-07	8.3488e-10
order	-	7.55	7.43	7.22

fixed parameter value $\frac{k}{h^2} = 3.2$, scheme I behaves like a fourth order accurate scheme in space and scheme II behaves like an eighth order accurate scheme in space.

Also, to achieve similar accuracy as in scheme II, scheme I requires much finer mesh sizes. For example, from Table 3, we observe that in the case of $(h, k) = \left(\frac{1}{16}, \frac{1}{80}\right)$, we achieve an accuracy of $4.3960e-05$ with scheme I whereas for $(h, k) = \left(\frac{1}{8}, \frac{1}{20}\right)$, we achieve an accuracy of $2.1338e-05$ with scheme II.

Also, it can be observed from Table 2 that for $(h, k) = \left(\frac{1}{32}, \frac{1}{32}\right)$, e_2 is $3.7558e-06$ and from Table 3 for $(h, k) = \left(\frac{1}{32}, \frac{1}{320}\right)$, e_1 is $2.7972e-06$.

Comparison between the errors obtained by the proposed schemes and low order scheme is shown in Figure 1 for various mesh sizes.

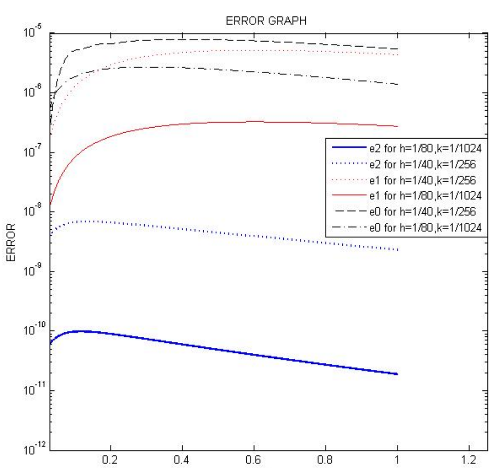

FIGURE 1. Error graph obtained by the schemes I, II, low order scheme for Example 1 at $t = 1$ for $\alpha = 10$, $\beta = 5$, $\eta = 0.5$, $\gamma = \frac{1}{3}$.

Example 2.

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + \alpha\left(1 + \tan^2\left(\frac{x+t}{2}\right)\right) + \beta^2 \tan\left(\frac{x+t}{2}\right),$$
$$0 \le x \le 1, t > 0$$

subject to the initial conditions

$$u(x,0) = \tan\left(\frac{x}{2}\right), \quad u_t(x,0) = \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right), \quad 0 \le x \le 1$$

and the Neumann boundary conditions

$$u_x(0,t) = \frac{1}{2}\left(1 + \tan^2\left(\frac{t}{2}\right)\right), \quad u_x(1,t) = \frac{1}{2}\left(1 + \tan^2\left(\frac{1+t}{2}\right)\right), \quad t > 0$$

The analytical solution is $u = \tan\left(\frac{x+t}{2}\right)$. The data given in Tables 4, 5 shows the L_∞ error between the numerical solution and the analytical solution at time t =1 for $\alpha = 0.07$, $\beta = 1$, $\eta = 2$, $\gamma = 1$.

Table 4. Error obtained for Example 2 for fixed mesh size $h = 0.001$

k	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
e_1	1.2400e-02	3.3000e-03	8.2009e-04	2.0722e-04
order	-	1.91	2.01	1.99
e_2	1.9460e-04	1.5728e-05	1.2715e-06	8.7029e-08
order	-	3.63	3.63	3.87

The data in Table 4 shows that schemes I and II behave as second and fourth order accurate respectively for fixed $h = 0.001$. The data in Table 5 shows that when both h and k are reduced by a factor of 2, scheme I behaves as second order accurate whereas scheme II behaves as fourth order accurate.

Figure 2 shows the comparison between the errors obtained by scheme I and II for mesh sizes $(h,k) = \left(\frac{1}{100}, \frac{1}{8}\right)$ and $\left(\frac{1}{200}, \frac{1}{32}\right)$. Figure 3 shows the comparison between the errors obtained by scheme I and low order scheme for mesh sizes $(h,k) = \left(\frac{1}{32}, \frac{1}{64}\right)$ and $\left(\frac{1}{64}, \frac{1}{256}\right)$.

Example 3.

$$u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + e^t \sin(\pi x)(1 + \pi^2 + 2\alpha + \beta^2), \quad 0 \le x \le 1,\ t > 0$$

subject to the initial conditions

$$u(x,0) = \sin(\pi x), \quad u_t(x,0) = \sin(\pi x), \quad 0 \le x \le 1$$

and the Neumann boundary conditions

$$u_x(0,t) = \pi e^t, \quad u_x(1,t) = -\pi e^t, \quad t > 0$$

TABLE 5. Error obtained for Example 2 when $k \propto h$

h	$\frac{1}{24}$	$\frac{1}{48}$	$\frac{1}{96}$	$\frac{1}{192}$
k	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
e_1	4.2500e-02	1.2400e-02	3.2000e-03	8.2325e-04
order	-	1.78	1.95	1.96
e_2	2.2000e-03	1.6629e-04	9.2480e-06	6.4006e-07
order	-	3.72	4.17	3.85

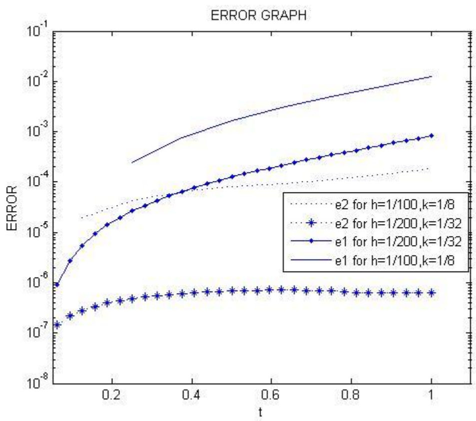

FIGURE 2. Error graph obtained by the schemes I, II for Example 2 at $t = 1$ for $\alpha = 0.07$, $\beta = 1$, $\eta = 2$, $\gamma = 1$.

The analytical solution is $u = e^t \sin(\pi x)$. The data given in Table 6 shows the L_∞ error between the numerical solution and the analytical solution at time t =1 for $\alpha = 0.5$, $\beta = 0$, $\eta = 0$, $\gamma = \frac{1}{3}$ using schemes I, II. Scheme I shows second order accuracy for $k \propto h$ whereas scheme II shows fourth order accuracy. Figure 4 shows the comparison between the errors obtained by the schemes I, II and low order scheme for mesh sizes $(h, k) = \left(\frac{1}{40}, \frac{1}{256}\right)$ and $\left(\frac{1}{80}, \frac{1}{1024}\right)$.

4. CONCLUSION

The available schemes for the solution of one-dimensional telegraphic equation are of low order. In this paper, we have presented two new three level implicit schemes of $O(k^2 + k^2h^2 + h^4)$ and $O(k^4 + k^4h^2)$ for solving telegraphic equation with Neumann boundary conditions. The scheme I behaves like a fourth order scheme for $k \propto h^2$, whereas the scheme II behaves like a fourth order scheme for $k \propto h$ and it behaves like an eighth order scheme for $k \propto h^2$ which is exhibited by the numerical results. The proposed scheme II allows much larger mesh-size in time direction. The schemes discussed in [4], involve parameters η and γ which are dependent on the choice of mesh

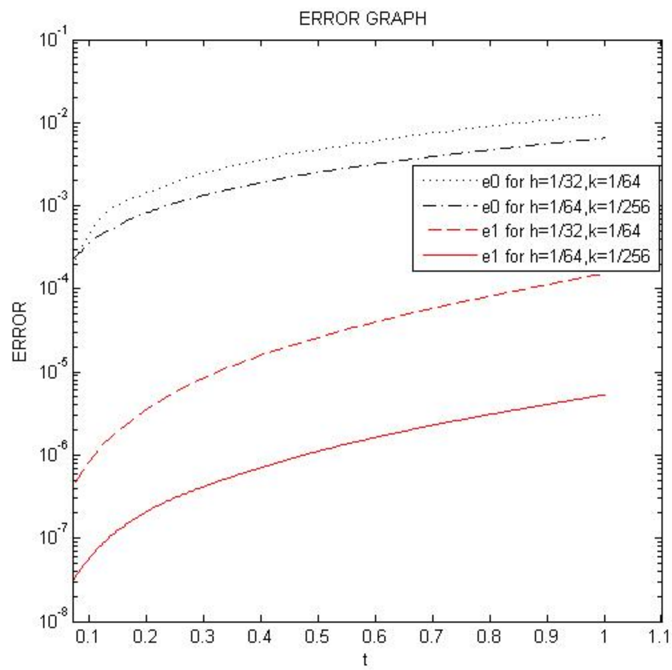

Figure 3. Error graph obtained by the schemes I and the low order scheme for Example 2 at $t = 1$ for $\alpha = 0.07$, $\beta = 1$, $\eta = 2$, $\gamma = 1$.

Table 6. Error obtained for Example 3 for $k \propto h$

h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
k	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$
e_1	0.0325	0.0111	0.0031	0.7948e-05
order	-	1.54	1.84	1.96
e_2	0.0135	8.7653e-04	5.3224e-05	2.5881e-06
order	-	3.94	4.04	4.36

sizes and mesh ratio parameter whereas the schemes discussed in the present paper involve parameters which are independent of the mesh sizes. The Matrix stability analysis shows that the scheme is unconditionally stable for the telegraphic equation with Neumann boundary conditions.

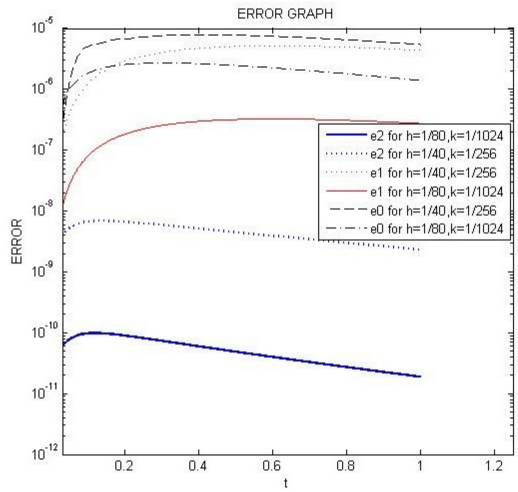

FIGURE 4. Error graph obtained by the schemes I, II and low order scheme for Example 3 at $t = 1$ for $\alpha = 0.5$, $\beta = 0$, $\eta = 0$, $\gamma = \frac{1}{3}$.

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