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# CHEBYSHEV POLYNOMIALS OF THE FIFTH KIND AND THEIR APPLICATION TO STUDY THE DYNAMICS OF LIENARD–TYPE EQUATIONS

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**ABSTRACT:** In this article a model with Chebyshev polynomials of the fifth kind $C_n(x) = xC_{n-1}(x) - \frac{(n-1)^2+n+(-1)^n(2n-1)}{4n(n-1)}C_{n-2}(x)$ with $C_0(x) = 1; C_1(x) = x$ as corrections in the Lienard differential system is presented. The type of limit cycles in the light of Melnikov's consideration and level curves are studied. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

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**Key Words:** Lienard system, Melnikov's approach, "corrections of fifth kind Chebyshev's polynomial $C_n(x)$, level curves

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## 1. INTRODUCTION

The first, second, third, and four kinds Chebyshev polynomials $(T(x), U(x), V(x), W(x))$ can be generating by the following recurrence

$$F_i(x) = 2xF_{i-1}(x) - F_{i-2}(x), \quad i \geq 2$$

with the initial values

$$\begin{aligned} &T_0(x) = 1;\ T_1(x) = x;\\ &U_0(x) = 1;\ U_1(x) = 2x;\\ &V_0(x) = 1;\ V_1(x) = 2x - 1;\\ &W_0(x) = 1;\ W_1(x) = 2x + 1. \end{aligned}$$

They have their roles especially in the scope of solving different types of differential equations.

The fifth–kind Chebyshev polynomials $C_i(x)$ can be written in the following form ([1], [2]):

$$C_n(x) = xC_{n-1}(x) - \frac{(n-1)^2 + n + (-1)^n(2n-1)}{4n(n-1)}C_{n-2}(x) \tag{1}$$

with initial values: $C_0(x) = 1$; $C_1(x) = x$.

The Melnikov polynomial [3] for the system

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon\left(a_1x + a_2x^2 + \cdots + a_{2n+1}x^{2n+1}\right) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{2}$$

is defined as

$$P(r^2, n) = \frac{a_1}{2} + \frac{3}{8}a_3\alpha^2 + \cdots + \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix} \frac{a_{2n+1}}{2^{2n+2}}r^{2n}. \tag{3}$$

It is known [5]–[6] that the system for sufficiently small $\epsilon \neq 0$ has at most $n$ limit cycles asymptotic to circles of radii $r_j$, $j = 1, 2, \ldots, n$ as $\epsilon \to 0$ if and only if the $n$th degree polynomial $P(r^2, n)$ has $n$ positive roots $r^2 = r_j^2$, $j = 1, 2, \ldots, n$.

In this paper we consider a new extended Lienard–type planar system with the polynomial $C_n(x)$. The type of limit cycles in the light of Melnikov's consideration and level curves are studied. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

## 2. MAIN RESULTS. SIMULATIONS

### 2.1. EXTENDED LIENARD–TYPE PLANAR SYSTEM

In this Section we consider the following model:

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon C_n(x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{4}$$

where $\epsilon > 0$ and $C_n(x)$ for $n = 3, 5, 7, ...$ is the Chebyshev's polynomial of the fifth kind.

For example for $n = 3, 5, 7, ...$ we have (see Fig. 1)

$$\begin{aligned} C_3(x) &= x^3 - \tfrac{5}{6}x \\ C_5(x) &= x^5 - \tfrac{7}{5}x^3 + \tfrac{7}{16}x \\ C_7(x) &= x^7 - \tfrac{27}{14}x^5 + \tfrac{9}{8}x^3 - \tfrac{3}{16}x \\ C_9(x) &= x^9 - \tfrac{22}{9}x^7 + \tfrac{33}{16}x^5 - \tfrac{11}{16}x^3 + \tfrac{55}{768}x \\ C_{11}(x) &= x^{11} - \tfrac{65}{22}x^9 + \tfrac{13}{4}x^7 - \tfrac{13}{8}x^5 + \tfrac{91}{256}x^3 - \tfrac{13}{512}x \\ C_{13}(x) &= x^{13} - \tfrac{45}{13}x^{11} + \tfrac{75}{16}x^9 - \tfrac{25}{8}x^7 + \tfrac{135}{128}x^5 - \tfrac{21}{128}x^3 + \tfrac{35}{4096}x. \end{aligned}$$

### 2.2. THE NEW MODEL IN THE LIGHT OF MELNIKOV'S CONSIDERATIONS.

*The case* $n = 7$. Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(x^7 - \tfrac{27}{14}x^5 + \tfrac{9}{8}x^3 - \mu x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{5}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 1. The Lienard–type system (5) (for $n = 7$), and for all sufficiently small $\epsilon \neq 0$ for

a) $\mu = 0.1875...$ has three simple cycles 0.689922, 0.837828, 1.01298;

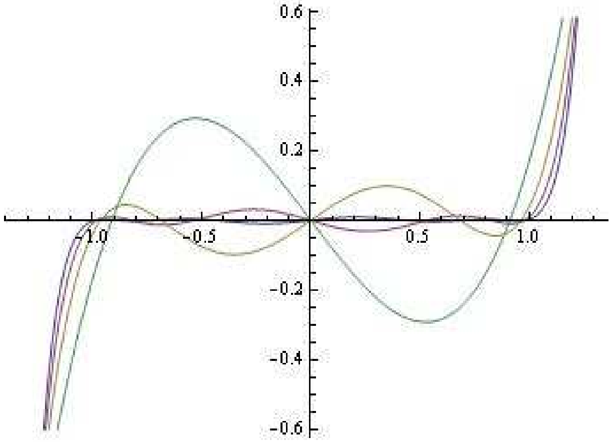


Figure 1: The polynomials $C_n(x)$ for $n = 3$, $n = 5$, $n = 7$ and $n = 9$.

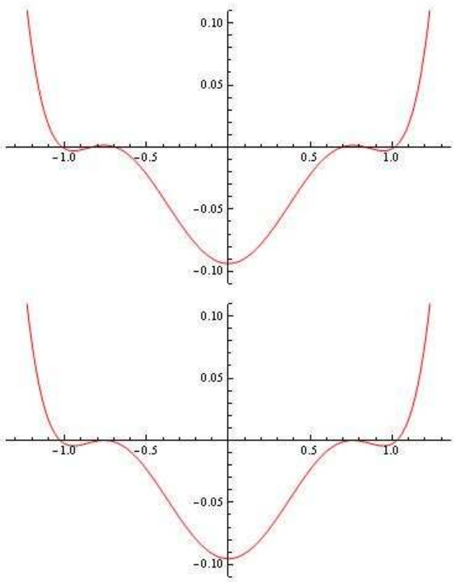


Figure 2: The Melnikov polynomial $P(r^2, 3)$ for $n = 7$ and a) $\mu = 0.1875...$; b) $\mu = 0.19061....$

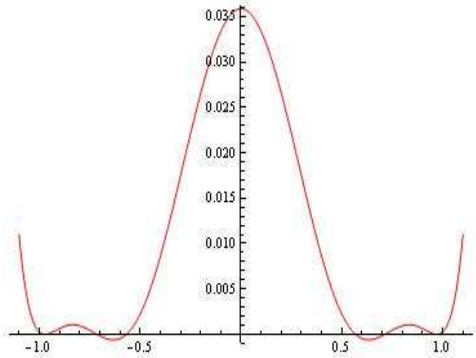


Figure 3: The Melnikov polynomial $P(r^2, 4)$ for $n = 9$ and $\mu = 0.07175....$

b) for $\mu = 0.19061...$ has a simple limit cycle 1.02668 and limit cycle 0.7583 with multiplicity – two.

Evidently for the Melnikov polynomial in $r^2$ (see Fig. 2) we have:

$$P(r^2, 3) = -\frac{\mu}{2} + \frac{27}{64}r^2 - \frac{135}{224}r^4 + \frac{35}{128}r^6. \tag{6}$$

*The case* $n = 9$. Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(x^9 - \frac{22}{9}x^7 + \frac{33}{16}x^5 - \frac{11}{16}x^3 + \mu x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{7}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Proposition 2. The Lienard–type system (7) (for $n = 9$), and for all sufficiently small $\epsilon \neq 0$ for $\mu = 0.07175...$ has simple limit cycles 0.568304, 0.712708 and limit cycle 0.970881 with multiplicity – two.

Evidently for the Melnikov polynomial in $r^2$ (see Fig. 3) we have:

$$P(r^2, 4) = \frac{\mu}{2} - \frac{33}{128}r^2 + \frac{165}{256}r^4 - \frac{385}{576}r^6 + \frac{63}{256}r^8. \tag{8}$$

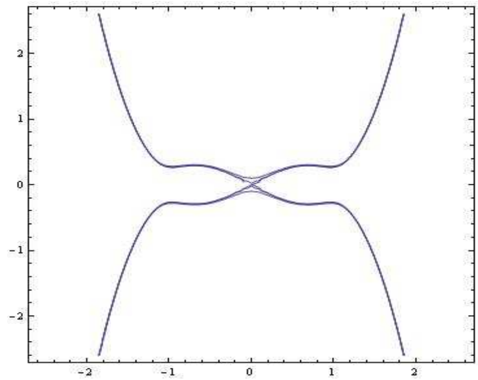


Figure 4: Level curves (the case 1).

## 2.3. THE LEVEL CURVES

Consider the model

$$\begin{cases} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & Poly_i(x) + \epsilon f_i(x) y \end{cases} \tag{9}$$

where $0 \leq \epsilon < 1$; $f_i(x)$ are specially chosen polynomials, and $Poly_i(x)$ are some of the Chebyshev polynomial $C_n(x)$. Without going into details, we will note some more interesting level curves:

*The case 1): $n = 5$.*

Let $Poly_i(x)$ (in (9) coincides with Chebyshev polynomial $C_5(x)$. The Hamiltonian of system (9) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} - \frac{1}{6}x^6 + \frac{7}{20}x^4 - \frac{7}{32}x^2.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ are depicted at Fig. 4.

*The case 2): $n = 13$.*

The Hamiltonian of system (9) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} - \frac{1}{14}x^{14} + \frac{45}{156}x^{12} - \frac{75}{160}x^{10} + \frac{25}{64}x^8 - \frac{135}{768}x^6 + \frac{21}{512}x^4 - \frac{35}{8192}x^2.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ are depicted at Fig. 5.

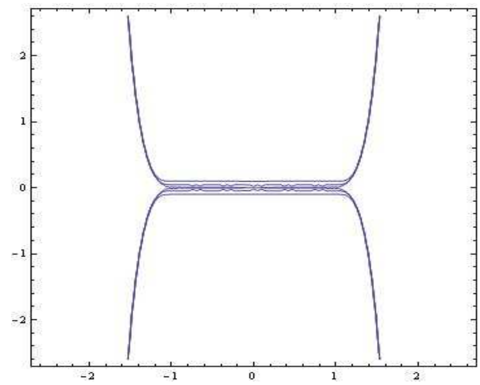


Figure 5: Level curves (the case 2).

## 2.4. SOME SIMULATIONS

The simulation on the Lienard–type system:

$$\begin{cases} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & -C_7(x) + \epsilon f(x) y \end{cases} \tag{10}$$

where $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$ is depicted on Fig. 6.

The simulation on the Lienard–type system:

$$\begin{cases} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & -C_5(x) + \epsilon f(x) y \end{cases} \tag{11}$$

where $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$ is depicted on Fig. 7.

## 2.5. POSSIBLE APPLICATIONS

It is easy to take into account that the change of the variable $t$ with $t = b\cos\theta + c$ ($\theta$ is the azimuthal angle and $c$ is the phase difference) in the $y(t)$-component of the solution of the systems (10) and (11) leads to radiation diagrams [7]–[8].

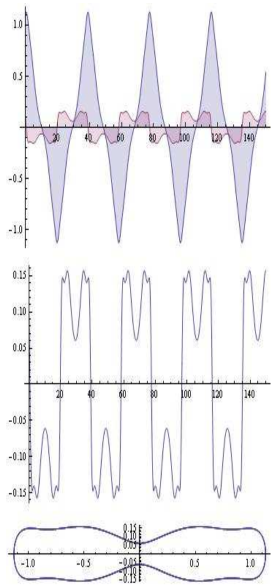

Figure 6: The simulations (system (10)) for $x_0 = 1.1; y_0 = 0.1; \epsilon = 0.0001$).

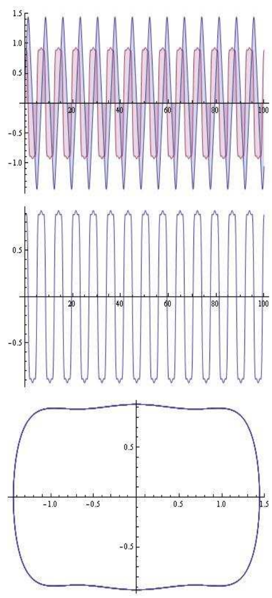

Figure 7: The simulations (system (11)) for $x_0 = 0.4; y_0 = 0.9; \epsilon = 0.0001$).

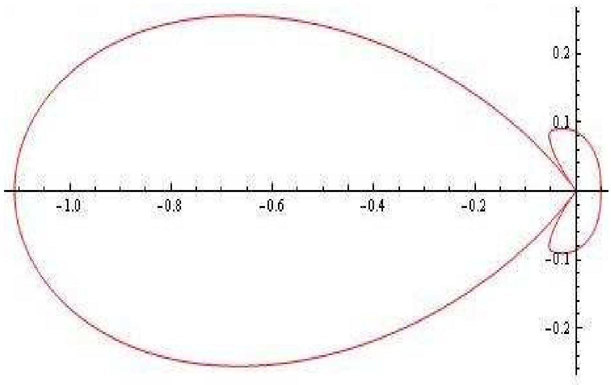


Figure 8: A typical radiation diagram.

The simulation on the $y(t)$-component of the solution of the system (10) (for $x_0 = 1.1; y_0 = 0.1; b = 0.213; c = 0.098$ in interval $(-\pi, \pi)$) is depicted on Fig. 8.

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