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LEGENDRE POLYNOMIALS AS "CORRECTION FACTORS" IN THE LIENARD DIFFERENTIAL SYSTEM. SIMULATIONS

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ABSTRACT: In this article we consider a new extended Lienard–type differential system with "corrections" of the Legendre polynomials L_n.

The number and type of limit cycles in the light of Melnikov's consideration are also studied.

Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

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Key Words: Lienard system, Melnikov's approach, Legendre polynomial, type of limit cycles, level curves

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1. INTRODUCTION

The Melnikov function [1] for the Lienard system [2]

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon \left(a_1 x + a_2 x^2 + \cdots + a_{2n+1} x^{2n+1}\right) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{1}$$

is defined as

$$M(\alpha, \mu) = -2\pi\alpha^2 \left(\frac{a_1}{2} + \frac{3}{8} a_3 \alpha^2 + \cdots + \left(\begin{array}{c} 2n+2 \\ n+1 \end{array} \right) \frac{a_{2n+1}}{2^{2n+2}} \alpha^{2n} \right) \tag{2}$$

The *Melnikov polynomial* is defined as

$$P(r^2, n) = -\frac{1}{2\pi r^2} M(r, \mu). \tag{3}$$

The following result provides the necessary information about the number of limit cycles and their radii

Theorem [3]–[4]. The Lienard system (1) for sufficiently small $\epsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii r_j, $j = 1, 2, \ldots, n$ as $\epsilon \to 0$ if and only if the nth degree polynomial in r^2,

$$P(r^2, n) = \frac{a_1}{2} + \frac{3}{8} a_3 r^2 + \cdots + \left(\begin{array}{c} 2n+2 \\ n+1 \end{array} \right) \frac{a_{2n+1}}{2^{2n+2}} r^{2n} \tag{4}$$

has n positive roots $r^2 = r_j^2$, $j = 1, 2, \ldots, n$.

Denote by L_n the Legendre polynomials. In this paper we consider a new extended Lienard–type planar system with the polynomial L_n. The number and type of limit cycles is also studied. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

2. MAIN RESULTS. SIMULATIONS

2.1. EXTENDED LIENARD–TYPE PLANAR SYSTEM

In this Section we consider the following model of the type:

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon L_n(x) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{5}$$

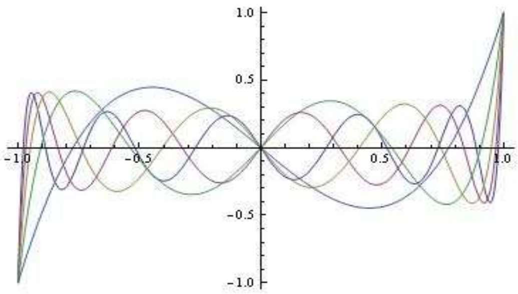

Figure 1: The polynomials $L_n(x)$ for $n = 3, 5, 7, 9, 11$.

where $\epsilon > 0$ and $L_n(x)$ for $n = 3, 5, 7, 9, ...$ are the Legandre polynomials.

For example we have (see Fig. 1)

$$\begin{aligned}
L_3(x) &= \tfrac{1}{2}\left(5x^3 - 3x\right) \\
L_5(x) &= \tfrac{1}{8}\left(63x^5 - 70x^3 + 15x\right) \\
L_7(x) &= \tfrac{1}{16}\left(429x^7 - 693x^5 + 315x^3 - 35x\right) \\
L_9(x) &= \tfrac{1}{128}\left(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x\right) \\
L_{11}(x) &= \tfrac{1}{256}\left(88179x^{11} - 230945x^9 + 21879x^7 - 90090x^5 + 15015x^3 - 693x\right)
\end{aligned}$$

The solutions of the system

$$\left\{\begin{aligned} \frac{dx}{dt} &= y - \epsilon(L_5(x)) \\ \frac{dy}{dt} &= -x \end{aligned}\right. \tag{6}$$

for $\epsilon = 0.001; x_0 = 0.2, y_0 = 0.3$ is depicted on Fig. 2.

For the model

$$\left\{\begin{aligned} \frac{dx}{dt} &= y - \epsilon(L_7(x)) \\ \frac{dy}{dt} &= -x \end{aligned}\right. \tag{7}$$

for $\epsilon = 0.001; x_0 = 0.9, y_0 = 0.1$ see Fig. 3.

The catastrophe surfaces for $n = 7$ and $n = 9$ $(x, y, p_1) = -p_1x + 315x^3 - 693x^5 + 429x^7 - y$; $(x, y, p_2) = p_2x - 4620x^3 + 18018x^5 - 25740x^7 + 12155x^9 - y$ for the model are shown on Fig. 4–5.

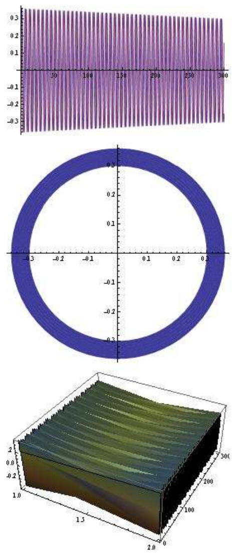

Figure 2: The solutions of the differential system (6). The portrait of the planar system.

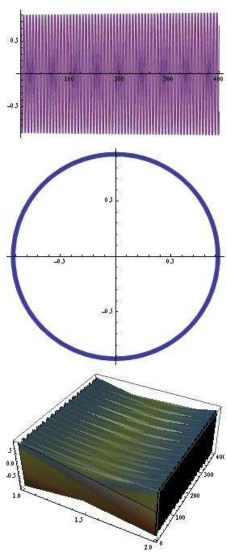

Figure 3: The solutions of the differential system (7). The portrait of the planar system.

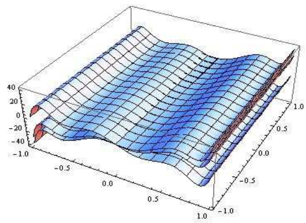

Figure 4: The catastrophe surface (x, y, p_1) for the following values of $p_1 = 10; 20; 50$.

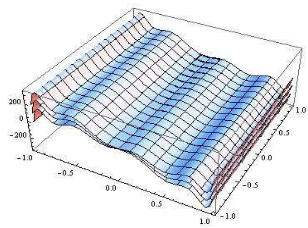

Figure 5: The catastrophe surface (x, y, p_2) for the following values of $p_2 = 5; 100; 200$.

2.2. THE NEW MODEL IN THE LIGHT OF MELNIKOV'S CONSIDERATIONS.

The case $n = 7$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon(-\mu x + \frac{315}{16}x^3 - \frac{693}{16}x^5 + \frac{429}{16}x^7) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{8}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Theorem. The Lienard–type system for $n = 7$, and for all sufficiently small $\epsilon \neq 0$

a) for $\mu \in (\mu_0, 2.514019917)$ has three hyperbolic limit cycles with radii r_1, r_2 and r_3.

b) for $\mu = 2.514019917$ has a simple limit cycle 1.01503 and limit cycle

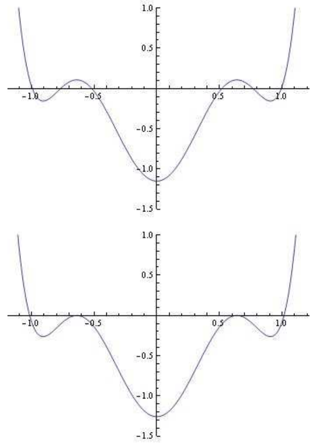

Figure 6: a) The Melnikov polynomial $P(r^2, 3)$ for $n = 7$ and $\mu = 2.4$ (three limit cycles); b) The Melnikov polynomial $P(r^2, 3)$ for $n = 7$ and $\mu = 2.514019917$ (simple limit cycle 1.01503 and limit cycle 0.638701 with multiplicity – two).

0.638701 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 6) we have:

$$P(r^2, 3) = -\frac{\mu}{2} + \frac{945}{128}r^2 - \frac{3465}{256}r^4 + \frac{15015}{2048}r^6. \tag{9}$$

Evidently, for example $\mu = 2.514019917$ we have a simple limit cycle and cycle with multiplicity – two.

The case $n = 9$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} = y - \epsilon(\mu x - \frac{4620}{128}x^3 + \frac{18018}{128}x^5 - \frac{25740}{128}x^7 + \frac{12155}{128}x^9) \\ \dfrac{dy}{dt} = -x \end{cases} \tag{10}$$

where $\mu > 0$, $\epsilon > 0$.

The following is valid

Theorem. The Lienard–type system for $n = 9$, and for all sufficiently small $\epsilon \neq 0$

a) for $\mu \in (\mu_0, 2.55232023)$ has four hyperbolic limit cycles with radii r_1, r_2, r_3 and r_4.

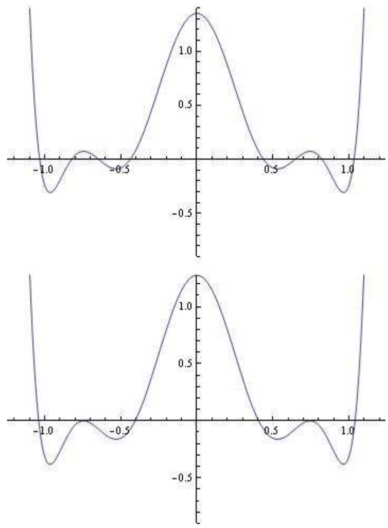

Figure 7: a) The Melnikov polynomial $P(r^2, 4)$ for $n = 9$ and $\mu = 2.55$ (four limit cycles); b) The Melnikov polynomial $P(r^2, 4)$ for $n = 9$ and $\mu = 2.55232023$ (two simple limit cycles: 0.405897, 1.03944 and limit cycle 0.7442 with multiplicity – two).

b) for $\mu = 2.55232023$ has two simple limit cycles: 0.405897, 1.03944 and limit cycle 0.7442 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 7) we have:

$$P(r^2, 4) = \frac{\mu}{2} - \frac{3465}{256}r^2 + \frac{45045}{1024}r^4 - \frac{225225}{4096}r^6 + \frac{765765}{32768}r^8. \tag{11}$$

Evidently, for example $\mu = 2.55232023$ we have two simple limit cycles and cycle with multiplicity – two.

The case $n = 11$.

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(-\mu x + \frac{15015}{256}x^3 - \frac{90090}{256}x^5 + \frac{218790}{256}x^7 - \frac{230945}{256}x^9 + \frac{88179}{256}x^{11}) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{12}$$

where $\mu > 0$, $\epsilon > 0$.

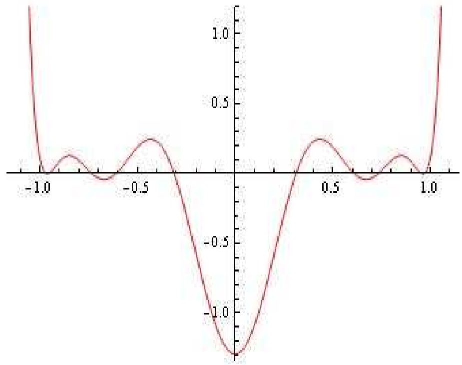

Figure 8: The Melnikov polynomial $P(r^2, 5)$ for $n = 11$ and $\mu = 2.589392$ (three simple limit cycles: 0.311859, 0.61074, 0.732595 and limit cycle 0.961818 with multiplicity – two).

The following is valid

Theorem. The Lienard–type system for $n = 11$, and for all sufficiently small $\epsilon \neq 0$ for $\mu = 2.589392$ has three simple limit cycles: 0.311859, 0.61074, 0.732595 and limit cycle 0.961818 with multiplicity – two.

Proof. For the Melnikov polynomial in r^2 (see Fig. 8) we have:

$$P(r^2, 5) = -\frac{\mu}{2} + \frac{45045}{2048}r^2 - \frac{225225}{2048}r + \frac{3828825}{16384}r^6 - \frac{14549535}{65536}r^8 + \frac{20369349}{262144}r^{10}. \tag{13}$$

Evidently, for example $\mu = 2.589392$ we have three simple limit cycles and cycle with multiplicity – two.

Consider a Lienard system of type

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -g(x) + \epsilon f(x) y \end{cases} \tag{14}$$

where $0 \leq \epsilon \leq 1$.

The solution of the system (14) for $\epsilon = 0.0001$, $g(x) = L_9(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$ (see oscillator model considered in [9]) is visualized on Fig. 9.

For other results see [9]–[18].

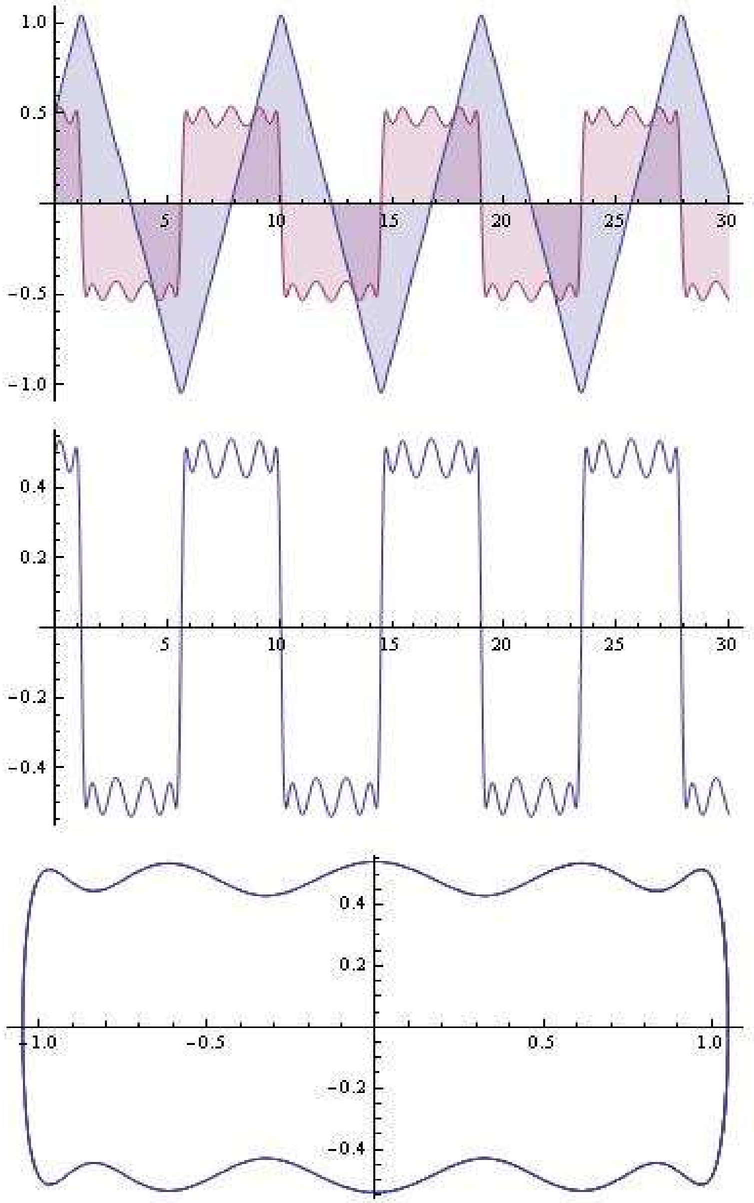

Figure 9: a) The solutions of the planar system (12) for $\epsilon = 0.001$, $\epsilon = 0.0001$, $g(x) = L_9(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$; b) the y-component of the solution; c) the portrait of the planar system.

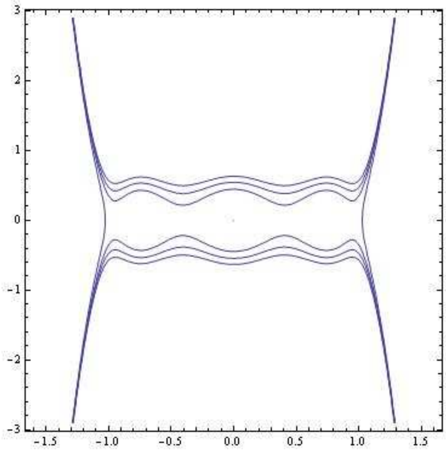

Figure 10: Level curves (the case f).

2.3. THE LEVEL CURVES

For more details of existing important results on the topic: Limit cycles bifurcations of some generalized polynomial Lienard system see [19]–[34]. Consider the class of Lienard polynomial systems of the type

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= L_7(x) + \epsilon(ax + bx^2 + cx^4 + dx^6)y \end{cases} \tag{15}$$

where $0 \leq \epsilon < 1$; $L_7(x) = \frac{1}{16}\left(429x^7 - 693x^5 + 315x^3 - 35x\right)$ is the Legendre polynomial and a, b, c, d are bounded parameters. Without going into details, we will note some interesting level curves.

The Hamiltonian of system (15) ($\epsilon = 0$) is

$$H(x,y) = \frac{y^2}{2} + \frac{35x^2}{32} - \frac{315x^4}{64} + \frac{231x^6}{32} - \frac{429x^8}{128}.$$

The level curves $L_{h_i} = \{H(x,y) = h_i\}$ are depicted in Fig. 10

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