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A TECHNIQUE FOR SIMULATING THE DYNAMICS OF SOME EXTENDED RELAXATION OSCILLATOR MODELS. II

VESSELIN KYURKCHIEV[1], ANTON ILIEV[1], ASEN RAHNEV[1]
AND NIKOLAY KYURKCHIEV[1]

[1]Faculty of Mathematics and Informatics
University of Plovdiv Paisii Hilendarski
24, Tzar Asen Str., 4000 Plovdiv, BULGARIA

ABSTRACT: Based on detailed research in [6] new extended oscillator model, in this article we offer a natural summary of this dynamic model with a polynomial type correction factor $c(t)$. We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered model. The considered methodological aspects can be successfully applied to study the dynamics of some nonlinear models.

AMS Subject Classification: 65L07, 34A34

Key Words: Van der Pol equation, Lienard system, Melnikov's approach, extended generalized relaxation oscillator model with polynomial type correction factor

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1. INTRODUCTION

In [6] the authors consider a new **"extended generalized relaxation oscillator"** model of the form:

$$\begin{cases} \dfrac{dx}{dt} &= c(f_n(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{1}$$

where $c > 0$ and

$$f_n(x) = \sum_{i=1}^{\left[\frac{n}{2}\right]} (-1)^{i+1} x^{n-2i} - \frac{x^n}{n}$$

for $n = 3, 7, 11, 15, 19, \ldots,$. In the case $n = 3$ the model coincides with the "classical" Van der Pol's model [1]–[5]. The proof of existence of limit cycle is based on the verification of the conditions in Lienard's theorem. The number of hyperbolic limit cycles with radii r_i in the light of Melnikov's approach is also studied.

The much more general nonlinear model (with $c = c(t)$) discussed in this paper is also of legitimate interest. We offer a software tool for simulating the dynamics of the new family.

2. MAIN RESULTS

Consider the following extended nonlinear model:

$$\begin{cases} \dfrac{dx}{dt} &= c_r(t)(f_n(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c_r(t)}x \end{cases} \tag{2}$$

where

$$c_r(t) = \sum_{i=0}^{r} c_i t^i$$

and

$$f_n(x) = \sum_{i=1}^{\left[\frac{n}{2}\right]} (-1)^{i+1} x^{n-2i} - \frac{x^n}{n}$$

for $n = 3, 7, 11, 15, 19, \ldots,$.

```
Print["A technique for simulating the dynamics of the
''extended generalized relaxation oscilator_model"];

Print["Input the intervention polynomial factor c of degree r"];

r = Input["Input degree of polynomial - r"]; (* 3 *)
Print["degree of polynomial r = ", r];

Print["c[t]=0.0042+0.0012*t+0.0001*t^2-0.0000045*t^3"];

Print["x'[t] ==(0.0042+0.0012*t+0.0001*t^2-0.0000045*t^3)*
(x[t]-x[t]^3+x[t]^5-1/7*x[t]^7-y[t])"];

Print["y'[t] ==(1/(0.0042+0.0012*t+0.0001*t^2-0.0000045*t^3))*x[t]"];

x0 = Input["Input initial condition - x[0]"]; (* 0 *)
Print["Initial condition x0 = ", x0];
y0 = Input["Input initial condition - y[0]"]; (* 1 *)
Print["Initial condition y0 = ", y0];
t0 = Input["Input t0"];
Print["t0 = ", t0];
t1 = Input["Input t1"];
Print["t1 = ", t1];
Print["Graphics of the solutions of the system of differential equations
as functions of the time t"];

NDSolve[{x'[t] == (0.0042 + 0.0012 * t + 0.0001 * t ^ 2 - 0.0000045 * t ^ 3) *
      (x[t] - x[t] ^ 3 + x[t] ^ 5 - 1 / 7 * x[t] ^ 7 - y[t]),
   y'[t] == (1 / (0.0042 + 0.0012 * t + 0.0001 * t ^ 2 - 0.0000045 * t ^ 3)) * x[t],
   x[0] == x0, y[0] == y0}, {x, y}, {t, t0, t1}];

Plot[Evaluate[{x[t], y[t]} /. First[%]], {t, t0, t1}, Filling → Axis]
exactsol2 = NDSolve[{x'[t] == (0.0042 + 0.0012 * t + 0.0001 * t ^ 2 - 0.0000045 * t ^ 3) *
        (x[t] - x[t] ^ 3 + x[t] ^ 5 - 1 / 7 * x[t] ^ 7 - y[t]),
     y'[t] == (1 / (0.0042 + 0.0012 * t + 0.0001 * t ^ 2 - 0.0000045 * t ^ 3)) * x[t],
     x[0] == x0, y[0] == y0}, {x, y}, {t, t0, t1}];

data = Table[{x[t], y[t]} /. exactsol2[[1]], {t, t0, t1}];
ListPlot[data, Joined → True, InterpolationOrder → 3, Mesh → Full,
 MeshStyle → Directive[PointSize[Large], Red]]
ListPlot3D[data, Mesh → None, InterpolationOrder → 3, ColorFunction → "SouthwestColors"]
```

Figure 1: The module in *CAS Mathematica*.

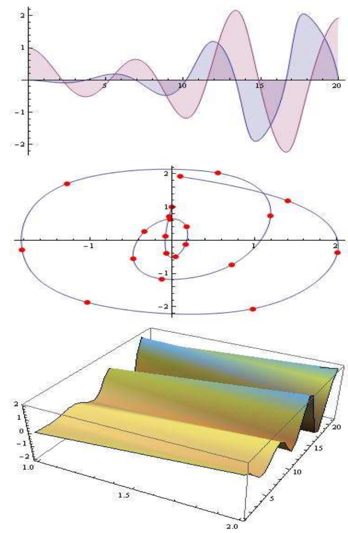

Figure 2: The solutions of the system of differential equations (2) (Example 1).

3. SIMULATIONS

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered model (2) (Fig. 1).

We consider the following examples

3.1. THE CASE $N = 3$

Example 1. Let $r = 1; n = 3$.

A simulation for user-selected polynomial

$$c_1(t) = 0.01 + 0.006t$$

for the model (2) is shown in Fig. 2

Example 2. Let $r = 1; n = 3$.

A simulation for user-selected polynomial

$$c_1(t) = 0.02 + 0.6t$$

for the model (2) is shown in Fig. 3.

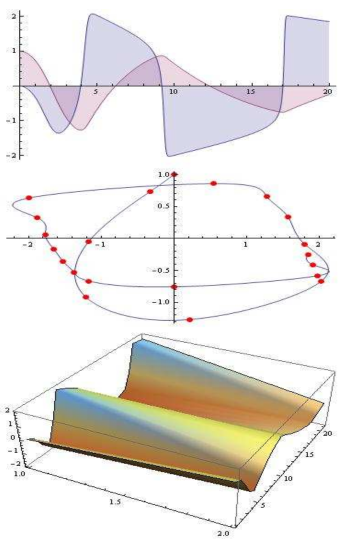

Figure 3: The solutions of the system of differential equations (2) (Example 2).

3.2. THE CASE $N = 7$

Example 3. Let $r = 2; n = 7$.

A simulation for user-selected polynomial

$$c_2(t) = 0.01 + 0.006t - 0.00015t^2$$

for the model (2) is shown in Fig. 4.

Example 4. Let $r = 3; n = 7$.

A simulation for user-selected polynomial

$$c_3(t) = 0.0042 + 0.0012t + 0.0001t^2 - 0.0000045t^3$$

for the model (2) is shown in Fig. 5.

4. THE EXTENDED LIENARD'S EQUATION

Using the notation in this article, we will formally consider the following generalized equation:

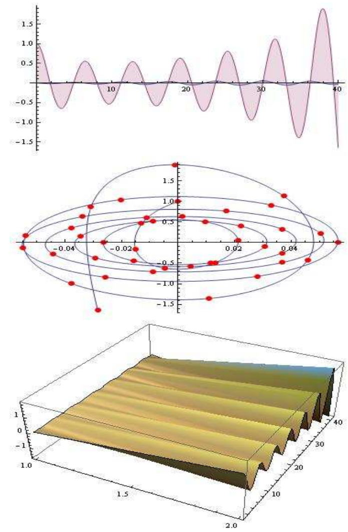

Figure 4: The solutions of the system of differential equations (2) (Example 3).

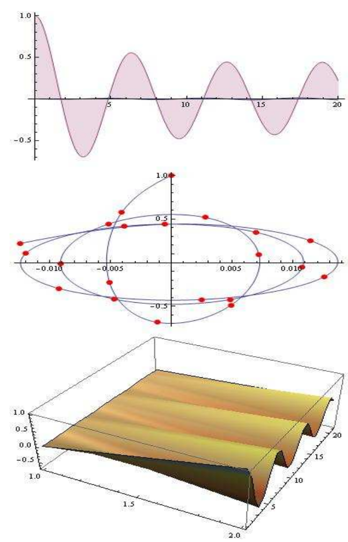

Figure 5: The solutions of the system of differential equations (2) (Example 4).

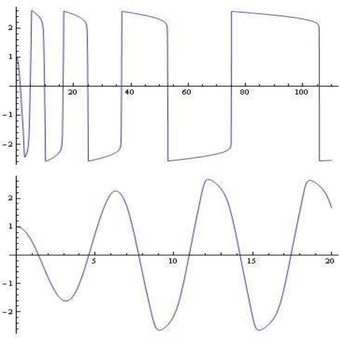

Figure 6: The solutions of the differential equation (3); a); b) ;(Example 5).

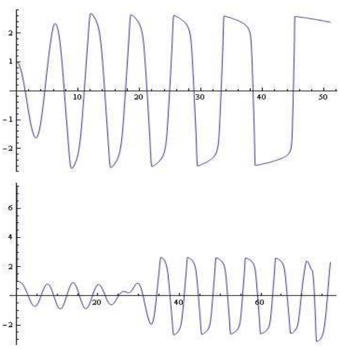

Figure 7: The solutions of the differential equation (3); c); d); (Example 5).

$$x'' - F(x;t)x' + x = 0 \tag{3}$$

where

$$F(x;t) = c_r(t)f_n'(x) + \frac{c_r'(t)}{c_r(t)}.$$

4.1. SIMULATIONS OF THE SOLUTION OF THE DIFFERENTIAL EQUATION (3)

Example 5. Let $n = 7$.

Simulation for

a) $r = 1$ and intervention polynomial

$$c_1(t) = 0.1 + 0.05t$$

b) $r = 2$ and intervention polynomial

$$c_2(t) = 0.01 + 0.005t - 0.0001t^2$$

c) $r = 3$ and intervention polynomial

$$c_3(t) = 0.01 + 0.005t - 0.0001t^2 + 0.00001t^3$$

d) $r = 4$ and intervention polynomial

$$c_4(t) = 0.01 + 0.001t - 0.0002t^2 + 0.00001t^3 - 0.0000001t^4$$

for the model (3) are shown in Fig. 6 – 7.

4.2. CONCLUDING REMARKS.

In this article, we propose an algorithm for generating dynamic models using a technique for embedding polynomial-type correction factors.

Analyzing the simulations using model (2) (resp. (3), we conclude that the new extended model is very sensitive to the coefficients of the input polynomial.

This makes it attractive for conducting computer simulations.

The considered methodological aspects can be successfully applied to study the dynamics of some nonlinear models.

For other results see [7]–[13].

An interesting nonlinear oscillator, which is described by a Lienard-type equation, and which has a number of interesting characteristics is

$$x'' + kxx' + \frac{k^2}{9}x^3 + \lambda x = 0 \tag{4}$$

where k, λ are constants.

For the solution $x(t)$ for $k = 11.5$; $\lambda = 1.6$ with $x(0) = 1$; $x'(0) = 0$ in interval $(0, 10)$ see Fig. 8 a).

The model (4) for $\lambda = 1.6$ and polynomial intervention factor

$$k = k(t) = 11.5 - 0.2t - 0.05t^2 - 0.001t^3.$$

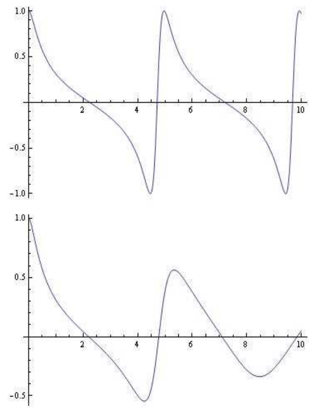

Figure 8: The solutions of the differential equation (4); a); b).

is visualized in Fig. 8 b).

Chen, Tang and Xiao [12] deal with a quintic Lienard system of the form

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon\left(a_1 x + a_2 x^3 + a_3 x^5\right) \\ \dfrac{dy}{dt} &= b_1 x + b_2 x^3 \end{cases} \tag{5}$$

Example 6. The solutions of the Lienard system (5) for

$$\epsilon = 1, a_1 = 0.2, a_2 = -0.1, a_3 = 0.05, b_1 = -0.3, b - 2 = -0.01$$

are shown in Fig. 9.

Consider the following extended modification of the model (5):

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(t)\left(a_1 x + a_2 x^3 + a_3 x^5\right) \\ \dfrac{dy}{dt} &= b_1 x + b_2 x^3 \end{cases} \tag{6}$$

where

$$\epsilon(t) = \sum_{i=0}^{r} \epsilon_i t^i.$$

Example 7. The solutions of the Lienard system (6) for

$$a_1 = 0.2, a_2 = -0.1, a_3 = 0.05, b_1 = -0.3, b - 2 = -0.01$$

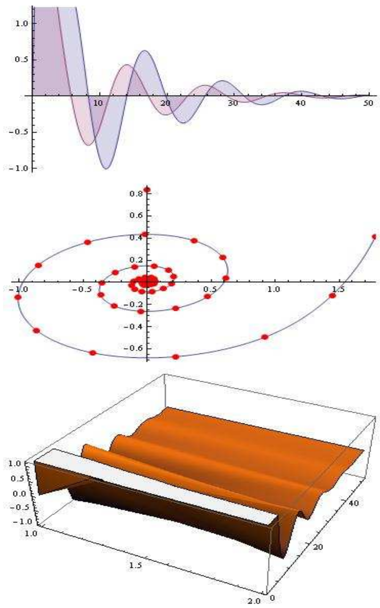

Figure 9: The solutions of the Lienard system (5); (Example 6).

$$\epsilon(t) = 0.1 + 0.001t$$

are shown in Fig. 10.

The equation was studied by Lloyd [13] is of the type

$$x'' + \epsilon\left(x^4 - x^2(k+1) + k\right)x' + x = 0 \tag{7}$$

He showed that it has no periodic solutions if $\frac{1}{5} < k < 5$. The corresponding Lienard's system is of the form

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon\left(\frac{1}{5}x^5 - \frac{1}{3}x^3(k+1) + kx\right) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{8}$$

Consider the following modification of Lloyd's model

$$x'' + \epsilon(t)\left(x^4 - x^2(k+1) + k\right)x' + x = 0 \tag{9}$$

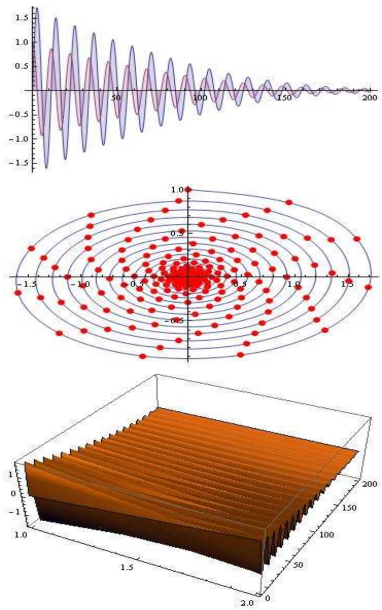

Figure 10: The solutions of the modified Lienard system (6); (Example 7).

or

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon(t)\left(\frac{1}{5}x^5 - \frac{1}{3}x^3(k+1) + kx\right) \\ \dfrac{dy}{dt} & = & -x \end{cases} \qquad (10)$$

where

$$\epsilon(t) = \sum_{i=0}^{r} \epsilon_i t^i.$$

Example 8. The solution of the equation (7) for $\epsilon = 2;\ k = 0.25$ is shown in Fig. 11 a).

The solution of the equation (9) for

$$\epsilon(t) = 0.91 + 0.05t + 0.001t^2 - 9.0001t^3;\ \ k = 0.25$$

is shown in Fig. 11 b).

Dumortier, Panazzolo and Roussarie [14] consider a general slow–fast Lien-

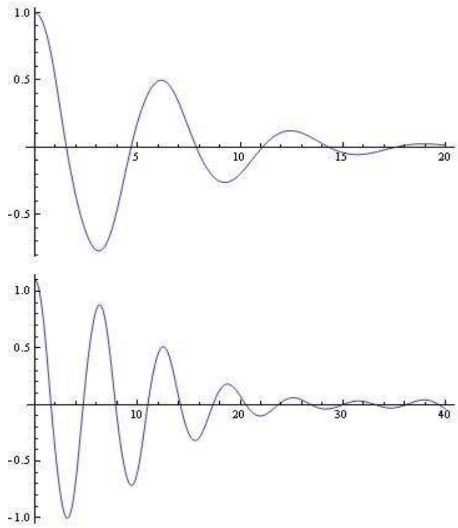

Figure 11: The solutions of the differential equations (7) and (9); (Example 8).

ard system

$$\begin{cases} x' = y - (x^2-1)^2(cex+1)(x^2+ex+\frac{1}{8}) - ax \\ y' = -\epsilon(x-\lambda) \end{cases}$$

as a function of the parameters $(c, e, \lambda, a, \epsilon) \in R^+ \times R^+ \times R \times R \times R^+$ near $(0, 0, 0, 0, 0)$.

Consider the following modification of the Lienard system:

$$\begin{cases} \frac{dx}{dt} & = & y - \epsilon(t)H(x) \\ \frac{dy}{dt} & = & -x \end{cases} \qquad (11)$$

where

$$\epsilon(t) = \sum_{i=0}^{r} \epsilon_i t^i,$$

$$H(x) = x^7 + 2x^6 - \frac{7}{8}x^5 - \frac{31}{8}x^4 - \frac{10}{8}x^3 + \frac{7}{4}x^2 + \frac{6}{8}x$$

(see Fig. 12). Here we study the corresponding generalized Van de Pol equation

$$x'' + \epsilon(t)H'(x)x' + x = 0.$$

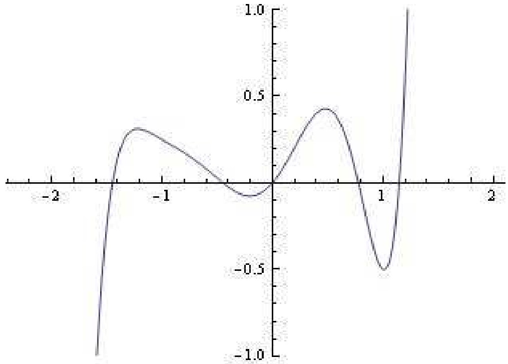

Figure 12: The polynomial H.

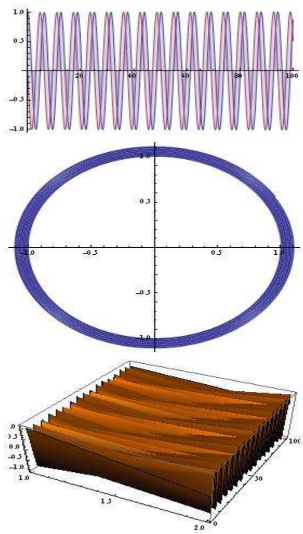

Figure 13: The solutions $x(t), y(t)$ of the Lienard system (11) for $\epsilon = 0.01$.

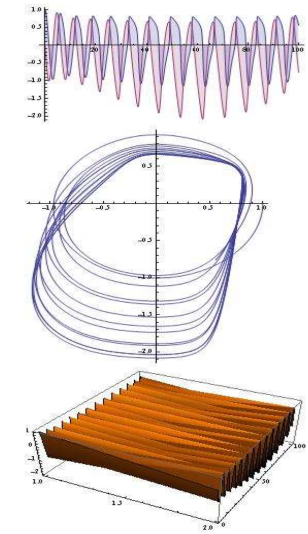

Figure 14: The solutions $x(t), y(t)$ of the Lienard system (11) for $\epsilon(t) = 0.01 - 0.09t - 0.0001t^2 + 0.000009t^3$.

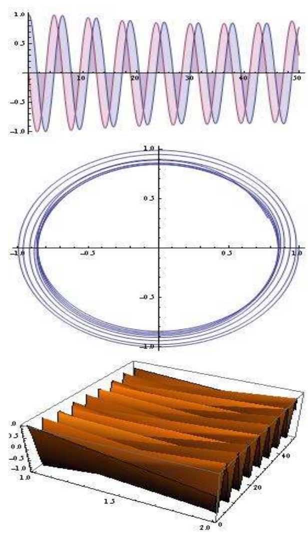

Figure 15: The solutions $x(t), y(t)$ of the Lienard system (11) for $\epsilon(t) = 0.0001 - 0.01t - 0.0001t^2 + 0.000009t^3 - 0.00000001t^4$.

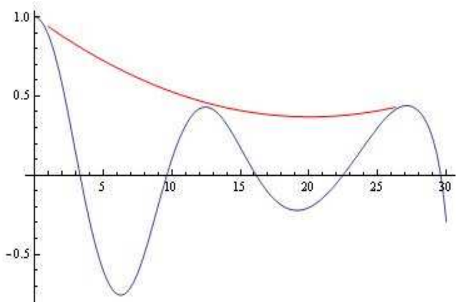

Figure 16: The solution $x(t)$ of the equation (13).

The solutions $x(t), y(t)$ of the Lienard system (11) for

a) $\epsilon = 0.01$;

b) $\epsilon(t) = 0.01 - 0.09t - 0.0001t^2 + 0.000009t^3$

c) $\epsilon(t) = 0.0001 - 0.01t - 0.0001t^2 + 0.000009t^3 - 0.00000001t^4$

are visualized on Fig. 13–15.

Finally, we will consider the following modification, which allows control of the amplitude distribution

$$\begin{cases} \dfrac{dx}{dt} = y - \epsilon(t)H_n(x) \\ \dfrac{dy}{dt} = -g(t)x \end{cases} \tag{12}$$

or

$$x'' + \epsilon(t)H_n'(x)x' + g(t)x = 0 \tag{13}$$

where

$$\epsilon(t) = \sum_{i=0}^{r} \epsilon_i t^i; \;\; g(t) = \sum_{i=0}^{r} g_i t^i$$

and

$$H_n(x) = \sum_{i=1}^{\left[\frac{n}{2}\right]} (-1)^{i+1} x^{n-2i} - \frac{x^n}{n}$$

for $n = 3, 7, 11, 15, 19, \ldots,$.

For example the solution $x(t)$ of (13) for $n = 15$; $\epsilon(t) = 0.1 + 0.0029t^2 - 0.00015t^3$ and $g(t) = 0.26 - 0.0001t^2 + 0.00000005t^3$ is visualized on Fig. 16.

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