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ON AN EXTENDED RELAXATION OSCILLATOR MODEL: NUMBER OF LIMIT CYCLES, SIMULATIONS. I

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ABSTRACT: In this paper we will consider a new **"extended generalized relaxation oscillator"** model. In the case $n = 3$ the model coincides with the "classical" Van der Pol's model. Although the motivation for such a modification is plausible, the effects of these changes are by no means obvious. Under certain conditions it can be shown that Lienard's–type equation has a limit cycle. The proof of existence of limit cycle is based on the verification of the conditions in Lienard's theorem. In section 2.3 we will study the new extended relaxation oscillation model for $n = 7, 11, 15$ in the light of Melnikov's approach. For example, we prove that the Lienard–type system for $n = 15$, and for all sufficiently small $\epsilon \neq 0$: a) for $0 < \mu < 0.322625$ has three hyperbolic limit cycles with radii r_1, r_2 and r_3; b) for $\mu = 0.322625$ has a simple limit cycle and limit cycle with multiplicity - two. The catastrophe surfaces (x, y, p); $n = 3, 7, 11, 19$ for the new model are studied. We offer a software tool for simulating the dynamics of the new family. We hope that the proposed module, implemented in *CAS Mathematica*, will support the work of researchers working in this scientific field. In many cases, the numerical calculation of the zeros of the polynomial $F(x)$ (appearing in the planar system of Lienard) and the polynomial of Melnikov $P(r^2, n)$ at a sufficiently high degree is very difficult. In appendix, we provide researchers with bilateral local

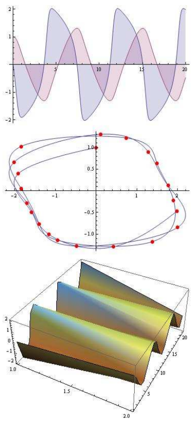

Figure 1: The solutions of the system of differential equations (1) for $c = 2$.

approximations for all positive roots (simple or multiple) of these algebraic equations.

AMS Subject Classification: 65L07, 34A34

Key Words: Van der Pol equation, Zeeman's model, Lienard system, Melnikov's approach, extended generalized relaxation oscillator model, catastrophe surface, number of limit cycles

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1. THE VAN DER POL'S EQUATION AND OSCILLATOR

The Van der Pol [1] equation, which arises in the study of nonlinear damping, is the nonlinear second-order equation of the type:

$$\frac{d^2x}{dt^2} + c(x^2 - 1)\frac{dx}{dt} + x = 0$$

where c is a positive constant. The Van der Pol equation has a long history of being used in both the physical and biological sciences. The Van der Pol

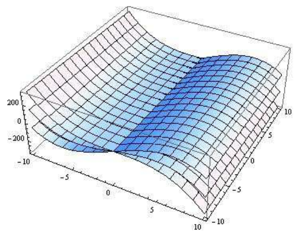

Figure 2: The catastrophe surface $(x, y, p) := px - \frac{x^3}{3} - y$ for the Zeeman's model (2) for the following values of $p = 1.1, 20, 40$.

"relaxation oscillator" can be written in its two–dimensional form:

$$\begin{cases} \frac{dx}{dt} &= c(f(x) - y) \\ \frac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{1}$$

with $c > 0$ and $f(x) = x - \frac{1}{3}x^3$. The simulation for user-selected coefficient $c = 2$ for the classical model (1) is shown in Fig. 1.

1.1. THE ZEEMAN'S APPROACH

If the simultaneous differential equations are solved for y and x for a range of time values, we find that y and x oscillate in a manner representing the fluctuations in the heart fiber length and stimulus. However, Zeeman [2] proposed the introduction into this model of a tension factor p, where $p > 0$ in an attempt to account for the effects of increased tension on the heart fiber. The model he suggested has the form:

$$\begin{cases} \frac{dx}{dt} &= c(f(x) - y) \\ \frac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{2}$$

with $c > 0, p > 0$ and $f(x) = px - \frac{1}{3}x^3$. The interrelation of the three parameters x, y and p in system (2) can be represented by a three dimensional surface called the catastrophe surface (see Beltrami [3]). The catastrophe surface $(x, y, p) := px - \frac{x^3}{3} - y$ for the Zeeman's model (2) is shown in Fig. 2.

Catastrophe theory is a branch of bifurcation theory in the study of dynamical systems. Arnold [4] gave the catastrophes the ADE classification, due to a deep connection with Lie groups.

1.2. THE LIENARD'S APPROACH

A great number of mathematical models of physical systems give rise to differential equation of the type

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0.$$

This is known as Lienard's equation [5]. The equation can be written as planar system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -g(x) - f(x)y. \end{cases}$$

Under certain conditions of f and g it can be shown that Lienard's equation has a limit cycle. This result is known as Lienard's theorem

Theorem A. Suppose that f and g satisfy the conditions

(i) f and g are continuously differentiable;

(ii) g is an odd function;

(iii) $g(x) > 0$ for $x > 0$;

(iv) f is an even function;

(v) let $F(x) = \int_0^x f(\xi)d\xi$ and $F(x) < 0$ for $0 < x < a$,

then Lienard's equation has a unique stable limit cycle surrounding the origin.

1.3. ANOTHER CONSIDERATIONS AND PROBLEMS

Consider the two–dimensional systems of the form:

$$\begin{cases} x' &= P_n(x,y) \\ y' &= Q_n(x,y) \end{cases} \tag{3}$$

where P_n and Q_n are polynomials of degree n with real coefficients. Although it has been proved that the number of limit cycles in system (3) is finite [6], [7], the determination of the maximum H_n of limits cycles is still far away of being know. Consider the equation $x'' + \epsilon f(x)x' + x = 0 = 0$. When f is a polynomial of degree $N = 2n+1$ or $2n$ this equation is of the form (3) with $P_{N+1}(x, y) = y$; $Q_{N+1}(x, y) = -\epsilon f(x)y - x$. It has been conjectured by Lins, Melo and Pugh [8] that the maximum number of limit cycles allowed is just n. It is true if $N = 2$, or $N = 3$ or if $f(x)$ is even and $N = 4$ [10], [9]. There are no general results about the limit cycles when $f(x)$ is a polynomial of degree greater than 5 neither, in general, when $f(x)$ is an arbitrary real function [10], [11].

1.4. THE RYCHKOV SYSTEM

In 1975 Rychkov [9] proved that system

$$\begin{cases} x' = y - (x^5 - \mu x^3 + \delta x) \\ y' = -x \end{cases}$$

where $\delta, \mu \in R$ has at most 2 limit cycles. Moreover, it is known that it has 2 limit cycles if and only if $\delta > 0$ and $0 < \delta < \Delta(\mu)$, for some unknown function Δ. For the value $\delta = \Delta(\mu)$ the system has a double limit cycle and, varying δ, it presents a saddle–node bifurcation of limit cycles. This system is also studied by Alsholm [13] and Odani [14]. In particular Odani proved that $\Delta(\mu) > \frac{\mu^2}{5}$. In [15] Gasull, Giacomini and Grau fix our attention on $\delta^* := \Delta(1)$. Notice that Odani's result implies that $\delta^* > \frac{1}{5} = 0.2$. For more details see [15]. In [15] the authors prove the following theorem

Theorem B [15]. Let $\delta = \delta^*$ be the value for which the Rychkov system

$$\begin{cases} x' = y - (x^5 - x^3 + \delta x) \\ y' = -x \end{cases}$$

has a semi–stable limit cycle. Then $0.224 < \delta^* < 9.224965$.

In 1984, Blows and Lloyd (Math. Proc. Cambridge Phil. Soc. 95 (1984)) proved the following interesting theorem concerning the number of limit cycles of the Lienard system

$$\begin{cases} \dfrac{dx}{dt} = y - \epsilon F(x) \\ \dfrac{dy}{dt} = -x \end{cases}$$

Theorem (Blows and Lloyd). The Lienard system with

$$F(x) = a_1 x + a_2 x^2 + \cdots + a_{2m+1} x^{2m+1}$$

has at most m local limit cycles and there are coefficients with $a_1.a_3, \dots, a_{2m+1}$ alternating in sign such that Lienard system has m local limit cycles.

In [16] the authors consider a general slow–fast Lienard system

$$\begin{cases} x' = y - F(x) \\ y' = -\epsilon(x - \lambda) \end{cases}$$

F is a Morse function. The authors prove that for a well-chosen polynomial f of degree 6, the equation $x'' + f(x)x' + x = 0$ exhibits 4 limit cycles. It induces that for $n \geq 3$ there exists polynomial of degree $2n$ such that the related equations exhibit more than n limit cycles. This contradicts the conjecture of Lins, de Melo and Pugh starting that for Lienard equations as above, with f of degree $2n$, the maximum number of limit cycles is n.

1.5. MELNIKOV APPROACH APPLIED TO LIENARD SYSTEMS. NUMBER OF LIMIT CYCLES AND THEIR RADII

The Lienard system for $F(x)$ of odd degree, containing all powers in x, may be written as

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon\left(a_1 x + a_2 x^2 + \cdots + a_{2n+1} x^{2n+1}\right) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{4}$$

which for $\mu = (a_1, a_2, \dots, a_{2n+1})$ has the form of system $x' = f(x) + \epsilon g(x, \epsilon, \mu)$. The Melnikov function for this system is [17]

$$M(\alpha, \mu) = -2\pi\alpha^2 \left(\frac{a_1}{2} + \frac{3}{8} a_3 \alpha^2 + \cdots + \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix} \frac{a_{2n+1}}{2^{2n+2}} \alpha^{2n} \right) \tag{5}$$

The *Melnikov polynomial* is defined as

$$P(r^2, n) = -\frac{1}{2\pi r^2} M(r, \mu). \tag{6}$$

The following result due to Perko and co–workers [18]–[19] provides the necessary information about the number of limit cycles and their radii

Theorem C. The Lienard system (4) for sufficiently small $\epsilon \neq 0$ has at most n limit cycles asymptotic to circles of radii $r_j,\ j = 1, 2, \ldots, n$ as $\epsilon \to 0$ if and only if the nth degree polynomial in r^2,

$$P(r^2, n) = \frac{a_1}{2} + \frac{3}{8}a_3 r^2 + \cdots + \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix} \frac{a_{2n+1}}{2^{2n+2}} r^{2n} \tag{7}$$

has n positive roots $r^2 = r_j^2,\ j = 1, 2, \ldots, n$. The theorem is proved using Melnikov's method.

2. MAIN RESULTS. SIMULATIONS

2.1. EXTENDED GENERALIZED RELAXATION OSCILLATOR MODEL

In this Section we consider the following **"extended generalized Relaxation oscillator"** model of the type:

$$\begin{cases} \dfrac{dx}{dt} &= c(f_n(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{8}$$

where $c > 0$ and

$$f_n(x) = \sum_{i=1}^{[\frac{n}{2}]} (-1)^{i+1} x^{n-2i} - \frac{x^n}{n}$$

for $n = 3, 7, 11, 15, 19, \ldots$, i.e.

$$\begin{aligned} f_3(x) &= x - \tfrac{x^3}{3}; \\ f_7(x) &= x - x^3 + x^5 - \tfrac{x^7}{7}; \\ \ldots \\ f_{19}(x) &= x - x^3 + x^5 - x^7 + x^9 - x^{11} + x^{13} - x^{15} + x^{17} - \tfrac{x^{19}}{19}; \\ \vdots \end{aligned}$$

Remark 1. For $n = 3$ the model (8) coincides with the "classical" model. Although the motivation for such a modification is plausible, the effects of these changes are by no means obvious. We propose a software module within

```
n = Input["Input  -  n"]; (* 7 *)
Print[" n = ", n];

"Input polynomial factor f_7[x]j: "
Print["f_7[x]==x-x^3+x^5-(1/7)*x^7 "];

Print["x'[t] ==c*(x[t]-x[t]^3+x[t]^5-1/7*x[t]^7-y[t])"];
Print["y'[t] ==(1/c)*x[t]"];

c = Input["Input  -  c"];(* 2 *)
Print[" c = ", c];

x0 = Input["Input initial condition - x[0]"]; (* 0 *)
Print["Initial condition x0 = ", x0];
y0 = Input["Input initial condition - y[0]"]; (* 1 *)
Print["Initial condition y0 = ", y0];
t0 = Input["Input t0"];
Print["t0 = ", t0];
t1 = Input["Input t1"];
Print["t1 = ", t1];
Print["Graphics of the solutions of the system of differential equations
as functions of the time t"];

NDSolve[{x'[t] == c * (x[t] - x[t]^3 + x[t]^5 - 1/7 * x[t]^7 - y[t]),
   y'[t] == (1/c) * x[t],
   x[0] == x0, y[0] == y0}, {x, y}, {t, t0, t1}];

Plot[Evaluate[{x[t], y[t]} /. First[%]], {t, t0, t1}, Filling → Axis]
exactsol2 = NDSolve[{x'[t] == c * (x[t] - x[t]^3 + x[t]^5 - 1/7 * x[t]^7 - y[t]), y'[t] == (1/c) * x[t],
    x[0] == x0, y[0] == y0}, {x, y}, {t, t0, t1}];

data = Table[{x[t], y[t]} /. exactsol2[[1]], {t, t0, t1}];
ListPlot[data, Joined → True, InterpolationOrder → 3, Mesh → Full,
 MeshStyle → Directive[PointSize[Large], Red]]
ListPlot3D[data, Mesh → None, InterpolationOrder → 3, ColorFunction → "SouthwestColors"]
```

Figure 3: The module in *CAS Mathematica*.

the programming environment *CAS Mathematica* for the analysis of the considered model (8) (Fig. 3). First we consider the following

Example 1. The simulations for user-selected coefficient a) $c = 0.5$; b) $c = 0.2$ and $n = 7$, with the "extended generalized Relaxation oscillator" model (8) are shown in Fig. 4–Fig. 5.

2.2. THE CATASTROPHE SURFACES

The new model (8) (in the context of Zeeman's considerations) can be written as follows:

$$\begin{cases} \dfrac{dx}{dt} &= c(f_n^*(x) - y) \\ \dfrac{dy}{dt} &= \frac{1}{c}x \end{cases} \tag{9}$$

where $p > 0$, $c > 0$ and

$$f_n^*(x) = px + \sum_{i=1}^{\left[\frac{n}{2}\right]-1} (-1)^{i+1} x^{n-2i} - \frac{x^n}{n}$$

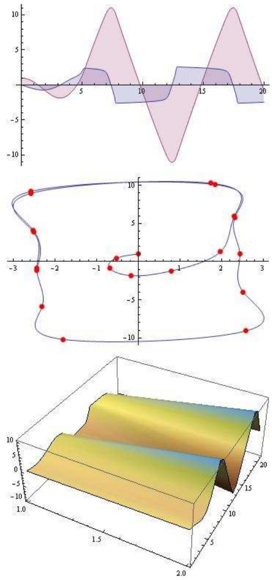

Figure 4: The solutions of the system of differential equations (8) (Example 1; the case a)).

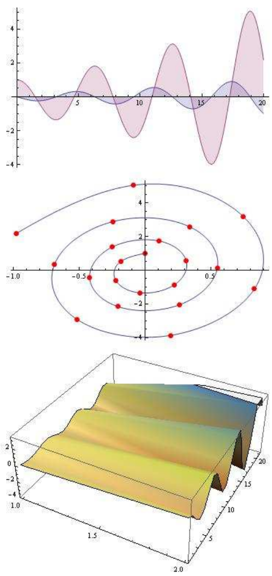

Figure 5: The solutions of the system of differential equations (8) (Example 1; the case b)).

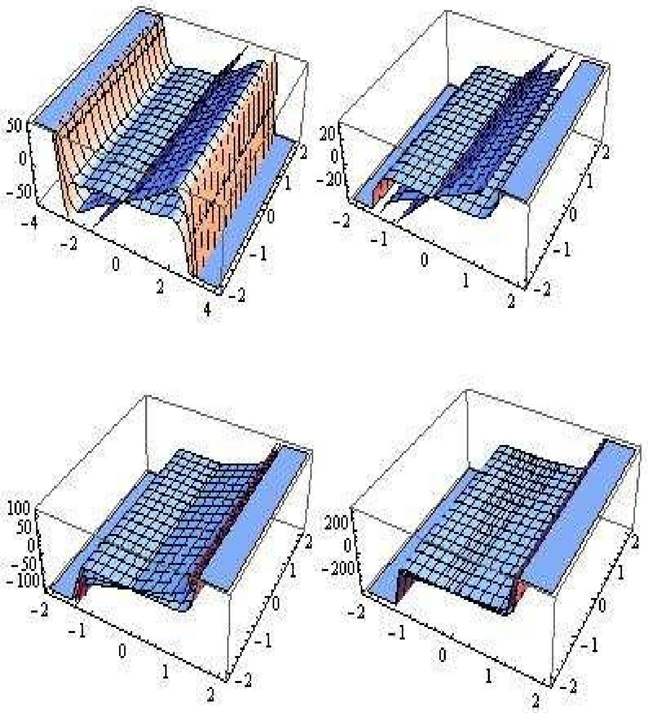

Figure 6: The catastrophe surfaces (x, y, p): a) $n = 7$; b) $n = 11$; c) $n = 15$; d) $n = 19$.

for $n = 3, 7, 11, 15, 19, \ldots,$.

The catastrophe surfaces (x, y, p): a) $n = 7$; b) $n = 11$; c) $n = 15$; d) $n = 19$; for the model (9) for the following values of $p = 1.1, 20, 40$ are shown in Fig. 6.

It is easy to see that the approximate differential system (8) describes a Lienard–type equation of the form:

$$x'' + cQ_{n-1}(x)x' + x = 0. \tag{10}$$

The solutions of the Lienard-type equation (10) for a) $n = 15$; $c = 0.015032$; b) $n = 19$; $c = 0.00871$ are shown in Fig. 7.

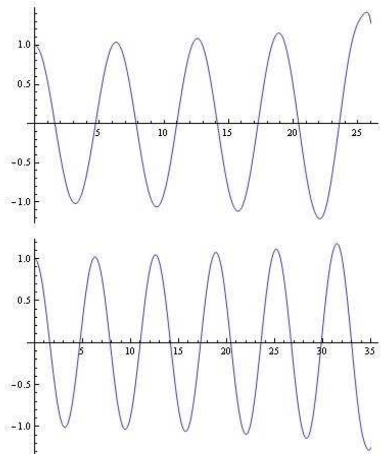

Figure 7: The solutions of the Lienard-type equation (10) for c) $n = 15$; $c = 0.015032$; d) $n = 19$; $c = 0.00871$.

2.3. EXISTENCE OF A LIMIT CYCLE

The case $n = 7$.

Theorem D. Let $n = 7$. The Lienard's equation (10) has a unique limit cycle surrounding the origin.

Proof. Checking the conditions of Lienard's theorem with $g(x) = x$, $f(x) = cQ_6(x)$ we find that

a) $g(x)$ is an odd function;

b) $f(x)$ is an even function;

c)

$$F(x) = c\int_0^x G_6(\xi)d\xi = c\left(\frac{x^7}{7} - x^5 + x^3 - x\right) = cx\left(\frac{x^6}{7} - x^4 + x^2 - 1\right)$$

(see Fig. 8 b). The real roots of the equation $F(x) = 0$ are $x_1 = 0$ and

$$x_{2,3} = \pm\sqrt{\frac{7}{3} + \frac{1}{3}\left(217 - 21\sqrt{57}\right)^{\frac{1}{3}} + \frac{1}{3}\left(7\left(31 + 3\sqrt{57}\right)\right)^{\frac{1}{3}}} \approx \pm 2.45599.$$

d) $F(x) < 0$ for $0 < x < 2.45599$;

e) $F(x) > 0$ and increasing for $x > 2.45599$.

The conditions are satisfied hence exists a unique limit cycle surrounding the origin.

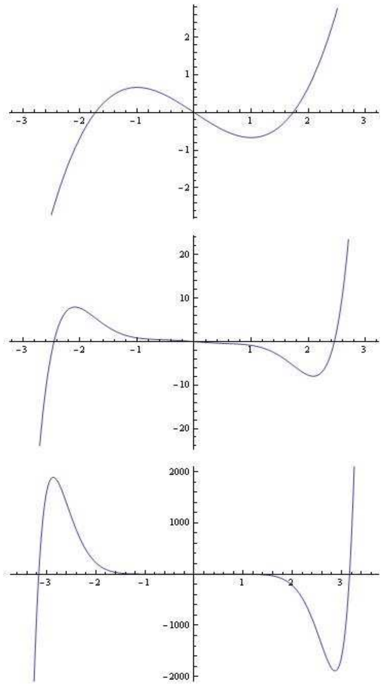

Figure 8: The properties of the function F: a) $n = 3$ (the classical example); b) $n = 7$; c) $n = 11$.

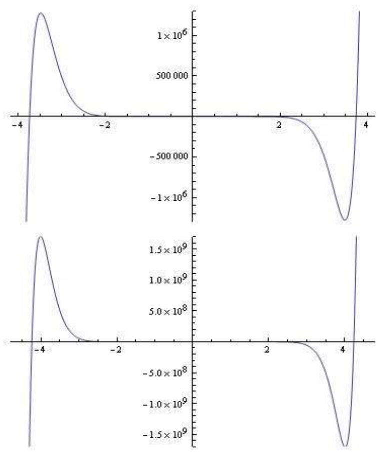

Figure 9: The properties of the function F: a) $n = 15$; b) $n = 19$.

For the *case* $n = 19$ we have

$$F(x) = c\int_0^x G_{18}(\xi)d\xi$$
$$= cx\left(\frac{x^{18}}{19} - x^{16} + x^{14} - x^{12} + x^{10} - x^8 + x^6 - x^4 + x^2 - 1\right)$$

(see Fig. 9 b). The real roots of the equation $F(x) = 0$ are $x_1 = 0$ and

$$x_{2,3} \approx \pm 4.2426406.$$

We see that $F(x) < 0$ for $0 < x < 4.242640$; $F(x) > 0$ and increasing for $x > 4.242640$.

The following conclusion is required from the above analysis:

In the general case denote by

$$F(x) = c\int_0^x G_{n-1}(\xi)d\xi.$$

Suppose that x_n is the only positive root of the equation $F(x) = 0$. Then $F(x) < 0$ for $0 < x < x_n$; $F(x) > 0$ and increasing for $x > x_n$ and differential equation has a unique limit cycle surrounding the origin.

2.4. THE NEW MODEL IN THE LIGHT OF MELNIKOV'S CONSIDERATIONS.

Considered in Section 2.1 new extended relaxation oscillation model can be written as follows

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon H_n(x)) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{11}$$

where $\mu > 0$, $\epsilon > 0$ and

$$H_n(x) = -\mu x + \sum_{i=1}^{[\frac{n}{2}]-1} (-1)^i x^{n-2i} + \frac{x^n}{n}$$

for $n = 3, 7, 11, 15, 19, \ldots,$.

For $\mu = 1$ and $n = 3, 7, 11, 15, 19$ we prove the existence of limit cycle (Theorem D). The proof is based on the verification of the conditions in Lienard's theorem. In this section we will study the model (11) for $n = 7, 11, 15$ in the light of Melnikov's approach.

The case $n = 7$.

The following is valid

Theorem E. The Lienard–type system (11) for $n = 7$, and for all sufficiently small $\epsilon \neq 0$

a) for $0 < \mu < 0.2456027...$ has three hyperbolic limit cycles with radii r_1, r_2 and r_3.

b) for $\mu = 0.245602716$ has a simple limit cycle and limit cycle with multiplicity – two.

Proof. From Theorem C for the 3th degree Melnikov polynomial in r^2 we have:

$$P(r^2, 3) = -\frac{\mu}{2} + \frac{3}{8}r^2 - \frac{5}{16}r^4 + \frac{5}{128}r^6. \tag{12}$$

The polynomial (12) has the roots

$$-\sqrt{\frac{8}{3} + \frac{44(\frac{2}{5})^{\frac{1}{3}}}{3A^{\frac{1}{3}}} + \frac{2}{3}\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}}};$$

$$\sqrt{\frac{8}{3} + \frac{44(\frac{2}{5})^{\frac{1}{3}}}{3A^{\frac{1}{3}}} + \frac{2}{3}\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}}};$$

$$-\sqrt{\frac{8}{3} - \frac{22(\frac{2}{5})^{\frac{1}{3}}}{3A^{\frac{1}{3}}} + \frac{22i\left(\frac{2}{5}\right)^{\frac{1}{3}}}{\sqrt{3}A^{\frac{1}{3}}} - \frac{1}{3}\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}} - \frac{i\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}}}{\sqrt{3}}};$$

$$\sqrt{\frac{8}{3} - \frac{22(\frac{2}{5})^{\frac{1}{3}}}{3A^{\frac{1}{3}}} + \frac{22i\left(\frac{2}{5}\right)^{\frac{1}{3}}}{\sqrt{3}A^{\frac{1}{3}}} - \frac{1}{3}\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}} - \frac{i\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}}}{\sqrt{3}}};$$

$$-\sqrt{\frac{8}{3} - \frac{22(\frac{2}{5})^{\frac{1}{3}}}{3A^{\frac{1}{3}}} - \frac{22i\left(\frac{2}{5}\right)^{\frac{1}{3}}}{\sqrt{3}A^{\frac{1}{3}}} - \frac{1}{3}\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}} + \frac{i\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}}}{\sqrt{3}}};$$

$$\sqrt{\frac{8}{3} - \frac{22(\frac{2}{5})^{\frac{1}{3}}}{3A^{\frac{1}{3}}} - \frac{22i\left(\frac{2}{5}\right)^{\frac{1}{3}}}{\sqrt{3}A^{\frac{1}{3}}} - \frac{1}{3}\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}} + \frac{i\left(\frac{2}{5}\right)^{\frac{2}{3}} A^{\frac{1}{3}}}{\sqrt{3}}}$$

where

$$A = 130 + 135\mu + 3\sqrt{15}\sqrt{135\mu^2 + 260\mu - 72}.$$

For an appropriate choice of the parameter μ (for example $\mu = 0.245602716$) we have a limit cycle and limit cycle with multiplicity two (see Table 1).

This completes the proof of the theorem.

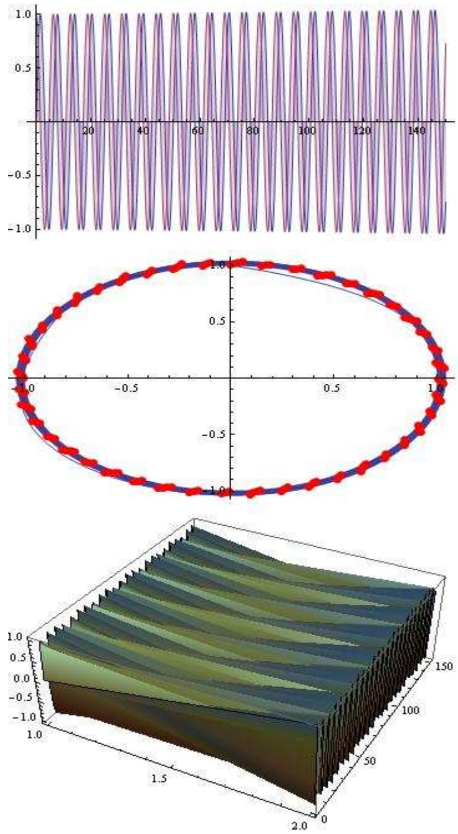

Figure 10: The solutions of the Lienard system and phase plot (Theorem E.; $\epsilon = 0.01$; $\mu_1 = 0.2456027164$; a simple limit cycle and limit cycle with multiplicity two).

The simulations of the Lienard-type system (11) for $\epsilon = 0.01$; $\mu_1 = 0.2456027164$ is shown in Fig. 10. Some values of the radii r_i; $i = 1, 2, 3$ as functions of μ in the interval $0 < \mu < 0.245602716$ are tabulated in Table 1.

In the case $n = 11$ the following is valid

Theorem F. The Lienard–type system (11) for $n = 11$, and for all sufficiently small $\epsilon \neq 0$

a) for $0 < \mu < 0.2959165194$ has three hyperbolic limit cycles with radii r_1, r_2 and r_3;

b) for $\mu \approx 0.29591651949$ has a simple limit cycle and limit cycle with

μ	r_1	r_2	r_3
0.01	0.116123	1.20533	2.55611
0.05	0.266097	1.17479	2.55911
0.1	0.390247	1.13122	2.56282
0.15	0.500756	1.07817	2.56647
0.2	0.618979	1.00577	2.57008
0.2456027	0.829955	0.830182	2.57332
0.245602716	0.83005	0.830087	2.57332

Table 1: The values of μ that result in a three positive roots of the Melnikov polynomial (Theorem E).

multiplicity - two.

Proof. The proof follows the ideas given in this section and will be omitted. We note that for the Melnikov polynomial we have:

$$P(r^2,5) = -\frac{\mu}{2} + \frac{3}{8}r^2 - \frac{5}{16}r^4 + \frac{35}{128}r^6 - \frac{63}{256}r^8 + \frac{21}{1024}r^{10}. \qquad (13)$$

The polynomial (13) has three positive roots for $0 < \mu < 0.2959165194$, and hence three hyperbolic limit cycles.

Some values of the radii r_i; $i = 1, 2, 3$ as functions of μ in the interval $0 < \mu < 0.2959165194$ are tabulated in Table 2.

μ	r_1	r_2	r_3
0.05	0.265664	1.08731	3.30001
0.1	0.387217	1.06798	3.30002
0.15	0.489977	1.04481	3.30004
0.2	0.587282	1.01531	3.30005
0.295	0.836131	0.874608	3.30008
0.29591651949	0.855828	0.855848	3.30009

Table 2: The values of μ that result in a three positive roots of the Melnikov polynomial (Theorem F).

Evidently, for $\mu \approx 0.29591651949$ we obtain one simple limit cycle and a limit cycle with multiplicity–two.

Obviously, research in this direction can be successfully continued. For example

Theorem G. The Lienard–type system (11) for $n = 15$, and for all sufficiently small $\epsilon \neq 0$

a) for $0 < \mu < 0.322625$ has three hyperbolic limit cycles with radii r_1, r_2 and r_3;

b) for $\mu = 0.322625$ has a simple limit cycle and limit cycle with multiplicity - two.

For the Melnikov polynomial we have:

$$P(r^2, 7) = -\frac{\mu}{2} + \frac{3}{8}r^2 - \frac{5}{16}r^4 + \frac{35}{128}r^6 - \frac{63}{256}r^8 + \frac{231}{1024}r^{10} - \frac{429}{2048}r^{12} + \frac{429}{32768}r^{14}. \tag{14}$$

The polynomial (1) has three positive roots for $0 < \mu < 0.322625$, and hence three hyperbolic limit cycles. Evidently, for $\mu = 0.322625$ we obtain one simple limit cycle and a limit cycle with multiplicity–two. Some values of the radii r_i; $i = 1, 2, 3$ as functions of μ in the interval $0 < \mu < 0.322625$ are tabulated in Table 3.

μ	r_1	r_2	r_3
0.1	0.504666	1.052	3.86316
0.15	0.489493	1.03704	3.86316
0.32	0.851798	0.906411	3.86316
0.322625	0.880204	0.880959	3.86316

Table 3: The values of μ that result in a three simple positive roots (or one simple root and root with multiplicity – two) of the Melnikov polynomial (Theorem G).

3. APPENDIX: A PROCEDURE FOR NUMERICAL SOLUTION OF THE POLYNOMIAL $F(X)$ (APPEARING IN THE PLANAR SYSTEM OF LIENARD) WITH HIGH DEGREE

In many cases, the numerical calculation of the zeros of the polynomial $F(x) = a_1x + a_2x^2 + \cdots + a_{2m+1}x^{2m+1}$ (appearing in the planar system of Lienard) and the polynomial of Melnikov $P(r^2,n)$

$$P(r^2,n)=\frac{a_1}{2}+\frac{3}{8}a_3r^2+\cdots+\begin{pmatrix}2n+2\\n+1\end{pmatrix}\frac{a_{2n+1}}{2^{2n+2}}r^{2n}$$

at a sufficiently high degree is very difficult. In this appendix, we provide researchers with bilateral local approximations for all positive roots (simple or multiple) of these algebraic equations. A method, originally due to Sendov [26], for simultaneous approximate calculation of all positive roots of the equation $f(x)=a_0+a_1x+\cdots+a_mx^m=0$ is based on the following theorem given by Poincare:

Theorem. Let f be a polynomial with real coefficients. If k is a large enough natural number, then the number of positive roots of $f(x)$ equal to the number of changes in sign in the sequence of the non–negative coefficients of the polynomial $g(x)=(1+x)^kf(x)$.

Let $0<x_1\le x_2\le\cdots\le x_p$, $p\le m$ be positive roots of $f(x)$ and $(1+x)^kf(x)=\sum_{\nu=0}^{m+k}b_k(\nu)x^\nu$. Let $\nu_k(1)$ denote the smallest integer for which $b_k(\nu_k(1))\ge 0$ and $b_k(\nu_k(1)+1)<0$, $b_k(0)=a_0>0$. In general, $\nu_k(s)$ is the smallest integer for which

$$(-1)^{s-1}b_k(\nu_k(s))\ge 0,$$
$$(-1)^{s-1}b_k(\nu_k(s)+1)<0.$$

Then we obtain the numbers

$$\nu_k(1),\nu_k(2),\ldots,\nu_k(s). \tag{15}$$

The numbers (15) satisfy (see Sendov):

$$\frac{\nu_k(s)}{k-\nu_k(s)+1}\le\xi(k,\nu,s)\le\frac{\nu_k(s)+1}{k-\nu_k(s)}, \tag{16}$$

$$\lim_{k\to\infty}\frac{\nu_k(s)}{k-\nu_k(s)+1}=\lim_{k\to\infty}\xi(k,\nu,s)=x_s, \tag{17}$$
$$f(x_s)=0,\ s=1,2,\ldots,p.$$

A far the asymptotic value (17) is known, we can be obtain more precise local estimations. The estimation obtained (see Kyurkchiev [27])

$$\left|\frac{\nu}{k-\nu+1}-\frac{\frac{\nu}{k-\nu+1}-1}{\left(\frac{\nu}{k-\nu+1}\right)^k}(1+\frac{1}{k})^k-x_s\right|\le\omega_1(k+1)^{-\frac{1}{q}}.$$

```
Print["METHOD - SENDOV:"]
k = Input["Input k:"];
Print[k];
v = CoefficientList[(1 + x)^k (x^8 - 36. x^7 + 546. x^6 - 4536. x^5 +
      22449. x^4 - 67284. x^3 + 118124. x^2 - 109584. x + 40321), x];
step = k;
n = 1;
i = 1;
key = 1;
While[key == 1 && i < k,
 If[v[[i]] < 0 && v[[i+1]] > 0,
  Print["change of sign:"]
   Print[" v", i - 1, "=", v[[i]], " v", i, "=", v[[i+1]]];
  Print["Approximation: v/(k - v + 1) "]
   Print[(i - 1) / (k - i)];
  Print[(i - 1) / (k - i) // N];
 ];
 If[v[[i]] > 0 && v[[i+1]] < 0,
  Print["change of sign:"]
   Print[" v", i - 1, "=", v[[i]], " v", i, "=", v[[i+1]]];
  Print["Approximation: v/(k - v + 1)"]
   Print[(i - 1) / (k - i)];
  Print[(i - 1) / (k - i) // N];
 ];
 i++;
]
Plot[x^8 - 36. x^7 + 546. x^6 - 4536. x^5 + 22449. x^4 - 67284. x^3 +
  118124. x^2 - 109584. x + 40321, {x, 0, 9}]
```

Figure 11: The module in CAS Mathematica (method due to Sendov).

in the case where x_s is a q–multiple root is very important. We propose a software module within the programming environment *CAS Mathematica* for the simultaneous approximation of all positive roots by Sendov's method (see Fig. 11). For example we consider the polynomial

$$f(x) = x^8 - 36x^7 + 546x^6 - 4536x^5 + 22449x^4 - 67284x^3 + 118124x^2 - 109584x + 40321 = 0$$

Numerical results for the all positive roots using our module in CAS Mathematica are visualized on Fig. 12–13.

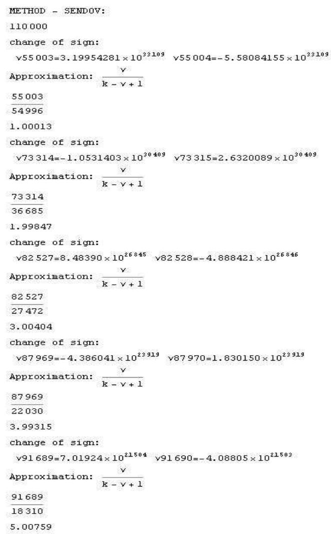

Figure 12: The module in CAS Mathematica (method due to Sendov).

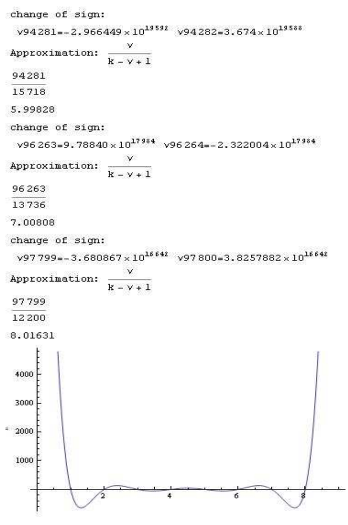

Figure 13: The module in CAS Mathematica (method due to Sendov).

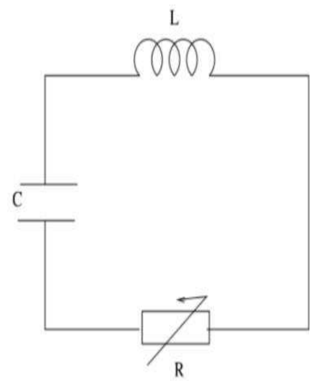

Figure 14: Electric circuit modeling the Van der Pol oscillator.

4. THE HYPOTHETICAL VAN DER POL GENERALIZED OSCILLATOR IN THE AUTONOMOUS REGIME

Consider the polynomial (see Section 2.1)

$$f_n(\frac{i}{i_0}) = \sum_{k=1}^{[\frac{n}{2}]} (-1)^{k+1} \left(\frac{i}{i_0}\right)^{n-2k} - \frac{\left(\frac{i}{i_0}\right)^n}{n}$$

for $n = 3, 7, 11, 15, 19, \dots,$

From

$$U_R = -R_0 i_0 f_n(\frac{i}{i_0}),$$

where U_R is the characteristic intensity–tension of the nonlinear resistance R, and

$$U_L + U_R + U_C = 0,$$

where U_L and U_C are the tension to the limits of the inductor (L) and capacitor (R) respectively:

$$U_L = L\frac{di}{d\tau},$$

$$U_C = \frac{1}{C}\int i d\tau$$

we have

$$L\frac{di}{d\tau} - R_0 i_0 f_n(\frac{i}{i_0}) + \frac{1}{C}\int i d\tau = 0,$$

$$L\frac{d^2i}{d\tau^2} - R_0 f_n'(\frac{i}{i_0})\frac{di}{d\tau} + \frac{i}{C} = 0.$$

Setting $x = \frac{i}{i_0}$; $t = \omega_e \tau$ where $\omega_e = \frac{1}{\sqrt{LC}}$ is an electric pulsation, we obtain generalized Van der Pol oscillator in the autonomous regime.

5. CONCLUDING REMARKS

The results obtained in this article are based on the following algorithms:

i) an algorithm for recurrent generation of an extended oscillator of arbitrary order n fixed by the user;

ii) an algorithm for automatic verification of the conditions in Lienard's theorem for the existence of a boundary cycle;

iii) an algorithm for obtaining reliable estimates for the zeros of the polynomial $F(x)$ (appearing in the Lienard planar system) and the high-order Melnikov polynomials $P(r^2, n)$ for determining their number and type.

Software tools for conducting research and visualization have also been developed, implemented in CAS Mathematica. The proposed modules are a small part of a much more general project for studying the dynamics of strictly nonlinear models.

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