A NOTE ON THE SMOOTH APPROXIMATION TO $|x(1-x)\dots(n-1-x)|$ USING GAUSSIAN ERROR FUNCTION

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ABSTRACT: In this note we present smooth approximation to $|x(1-x)\dots(n-1-x)|$ using $x(1-x)\dots(n-1-x)erf\left(\frac{x(1-x)\dots(n-1-x)}{\mu}\right)$ as $\mu \to 0^+$. The above approximation results are of interest in some applications involving the so-called "polynomial variable transfer", such as modelling and simulation of filters characteristics and antenna charts. Numerical examples are presented using *CAS MATHEMATICA*.

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1. INTRODUCTION

Some techniques for smooth approximation to $|x|$ using sigmoidal and hyperbolic functions can be found in [1]–[4]. For rational approximation to $|x|$, see, [13], [14]. About convolution based smooth approximations to the absolute value function, see [15]. The above approximation results are of interest in some applications involving the so-called "polynomial variable transfer". More precisely, the adaptive functions with "polynomial variable transfer" can be used for modelling and simulation of filters characteristics and antenna charts (in an appropriate interval) [16]. We will note that the tolerance analysis for the amplitude phase error of the main function parameters including the beamwidth and side lobes level is usually determined by computer simulations. Of course, this analysis is closely related to the number and type of polynomial zeros, respectively, of the emitters [17]. For other results, see [18]–[19]. Typical filter characteristics and bounds of the power pattern are depicted on Fig. 1–Fig.2.

The Gaussian error function is defined as follows:

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt.$$

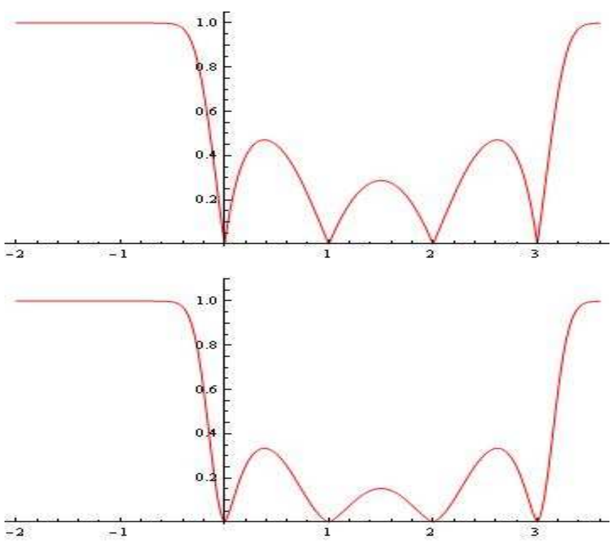

Figure 1: A typical filter characteristics.

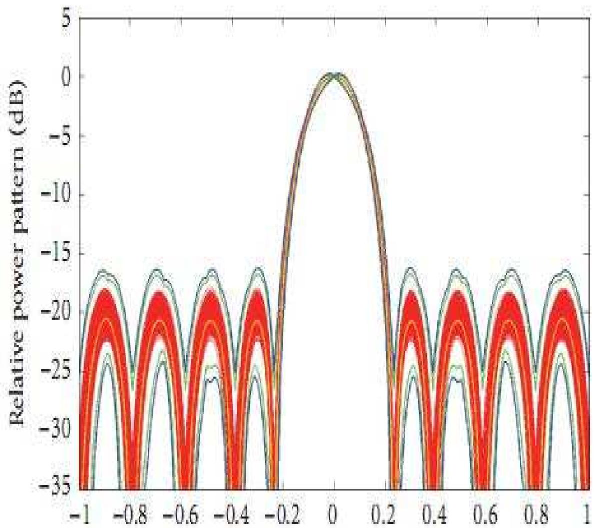

Figure 2: Bounds of the power pattern [17].

Note that, the function $erf(z)$ is a sigmoidal function according to the usual definition given, e.g., [8, 6, 7]. Sigmoidal functions found a wide application field in the theory of artificial neural networks, where they are used as activation functions in neuronal models, see e.g., [5]–[12].

For example, in [4] the authors proved the following result:

- for $\mu > 0$ and $\alpha = \frac{2}{e\sqrt{\pi}} \approx 0.415175$ the approximation $F(x) = x\,erf(\frac{x}{\mu})$ to $|x|$ satisfies:

$$||x| - F(x)| < \alpha\mu.$$

This result can be successfully extended for the analysis of smooth approximation to $|x(1-x)\dots(n-1-x)|$ using Gaussian error function with "polynomial variable transfer", i.e., $x \to f(x) = \sum_{i=0}^{n} a_i x^i$, $a_0 = 0$.

In this article, $F_\mu^n(x) := x(1-x)\dots(n-1-x)erf\left(\frac{x(1-x)\dots(n-1-x)}{\mu}\right)$ is considered for smooth approximation to $F^n(x) := |x(1-x)\dots(n-1-x)|$, as $\mu \to 0^+$, for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $n \geq 1$.

More precisely, we will prove a uniform convergence result, together with a very general upper bound, that are valid for every function $F^n(x)$, $n \in \mathbb{N}$, $n > 0$, and on the whole $\mathbb{R}$.

2. MAIN RESULTS

Here, we establish the main result of the paper.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geq 1$ be fixed. Then the family $(F^n_\mu)_{\mu>0}$ converges uniformly on $\mathbb{R}$ to F^n, as $\mu \to 0^+$.*

Proof. First of all, we define $A^n := [-1, n] \subset \mathbb{R}$, for any fixed $n \in \mathbb{N}$, $n \geq 1$. Let now $\varepsilon > 0$ be fixed. By the uniform continuity of F^n in A^n, there exists $\delta = \delta(\varepsilon) > 0$ (we can choose $\delta < 1$) such that $|F^n(x) - F^n(y)| < \varepsilon$, for ever x, $y \in A^n$. Now, for every fixed

$$x \in [-\delta, \delta] \cup [1-\delta, 1+\delta] \cup ... \cup [n-1-\delta, n-1+\delta] \ =: A^n_\delta,$$

there exists $i = 0, 1, ..., n-1$, such that $|x - i| < \delta$, and hence:

$$\begin{aligned} &\left|F^n_\mu(x) - F^n(x)\right| \\ &= |F^n(x)| \cdot \left| erf\left(\frac{x(1-x)\dots(n-1-x)}{\mu}\right) - sign[x(1-x)\dots(n-1-x)]\right| \\ &= \ |F^n(x)| \cdot \left| erf\left(\frac{F^n(x)}{\mu}\right) - sign[F^n(x)]\right| \leq \ 2\,|F^n(x) - F^n(i)| \ \leq \ 2\,\varepsilon, \end{aligned}$$

for every $\mu > 0$, where $F^n(i) = 0$ and:

$$sign[F^n(x)] \ := \ \begin{cases} +1, & if\ F^n(x) \geq 0, \\ -1, & otherwise. \end{cases}$$

On the other hand, if we set:

$$M \ := \ \min\{F^n(-\delta),\ F^n(\delta),\ F^n(1-\delta),\ F^n(1+\delta),\ ....,\ F^n(n-1+\delta)\} \ > \ 0,$$

and denoting by $\sigma > 0$ a sufficiently large value, such that $erf(z) = 1$ if $z \geq \sigma$, and $erf(z) = -1$ if $z \leq -\sigma$, it turns out that, for every $0 < \mu \leq M/\sigma$ one has:

$$\sigma \ \leq \ \frac{M}{\mu} \ \leq \ \frac{|F^n(x)|}{\mu},$$

for every $x \notin A^n(x)$, thus:

$$\left|F^n_\mu(x) - F^n(x)\right| \ = \ |F^n(x)| \cdot \left| erf\left(\frac{F^n(x)}{\mu}\right) - sign[F^n(x)]\right| \ = \ 0.$$

In conclusion, we proved that in correspondence of $\varepsilon > 0$ there exists the parameter M/σ (depending on ε) such that, for every $0 < \mu \leq M/\sigma$ one has:

$$\left|F^n_\mu(x) - F^n(x)\right| \leq 2\,\varepsilon, \quad x \in \mathbb{R}. \tag{1}$$

This completes the proof. □

The proof of Theorem 1 is constructive and also allows to understand how the parameter $\mu > 0$ must be chosen in order to get any desired order of accuracy in the above uniform approximation. From now on, we always denote by $\sigma > 0$ a parameter, such that $erf(z) = 1$ if $z \geq \sigma$, and $erf(z) = -1$ if $z \leq -\sigma$. We can prove the following.

Theorem 2. *Let $n \in \mathbb{N}$, $n \geq 1$ be fixed. For any $0 < \varepsilon < 1$, and for every $0 < \mu \leq M/\sigma$, where:*

$$M \;:=\; \min\left\{F^n(-\varepsilon),\; F^n(\varepsilon),\; F^n(1-\varepsilon),\; F^n(1+\varepsilon),\; ....,\; F^n(n-1+\varepsilon)\right\} \;>\; 0,$$

we have:

$$\left|F^n_\mu(x) - F^n(x)\right| \leq 2\,\varepsilon\,(1+\varepsilon)...(n-1+\varepsilon), \tag{2}$$

for every

$$x \in [-\varepsilon, \varepsilon] \cup [1-\varepsilon, 1+\varepsilon] \cup ... \cup [n-1-\varepsilon, n-1+\varepsilon] \;=:\; A^n_\varepsilon,$$

and

$$\left|F^n_\mu(x) - F^n(x)\right| = 0, \tag{3}$$

for every $x \notin A^n_\varepsilon$.

The proof of Theorem 2 follows immediately as in Theorem 1. Theorem 2 provide a theoretical upper bound for the error of uniform approximation.

For instance, we can consider the special case $n = 2$; $a_1 = 1$, $a_2 = -1$. Smooth approximation of $F^2(x)$ using F^2_μ can be obtained for various values of $\mu > 0$ (see, e.g., Fig. 3 for a) $\mu = 0.21$; b) $\mu = 0.16$; c) $\mu = 0.1$).

Similarly, smooth approximations of $F^3(x)$ using F^3_μ (i.e., when $n = 3$; $a_1 = 2$, $a_2 = -3$, $a_3 = 1$) for a) $\mu = 0.24$; b) $\mu = 0.17$, have been depicted in Fig. 4.

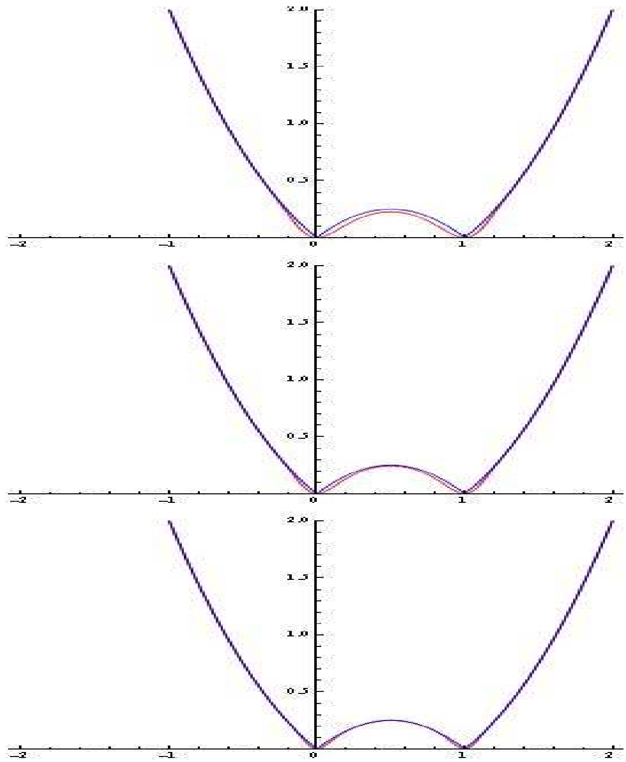

Figure 3: The smooth approximation: a) $\mu = 0.21$; b) $\mu = 0.16$; c) $\mu = 0.1$.

3. CONCLUDING REMARKS

Note that, in the results proved in Theorem 1 and Theorem 2 is included the problem of the smooth approximation to $|x|$ using $F_1(x) = x\, erf(\frac{x}{\mu})$ originally considered in [4].

Formally, we define the Gaussian error function with "polynomial variable transfer" and let us consider $F_2(x) = x\, erf(\frac{f(x)}{\mu})$, where $f(x) = \sum_{i=0}^{n} a_i x^i$, $a_0 = 0$.

For example, let $n = 2$. Fig. 5 shows that for some constraints imposed on the coefficients of the polynomial, the approximation F_2 to $|x|$ gives better results.

The study of issues related to smooth approximation is important in the

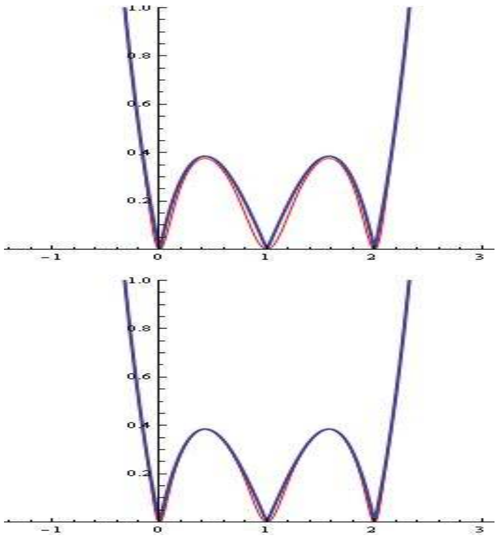

Figure 4: The smooth approximation: a) $\mu = 0.24$; b) $\mu = 0.17$.

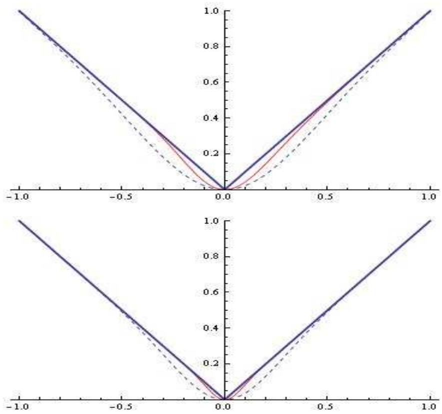

Figure 5: Graph of the functions $F^1_\mu(x)$ - dashed and $F_2(x)$ - red; a) $a_1 = 2.1,\ a_2 = -0.9,\ \mu = 0.5$; b) $a_1 = 3,\ a_2 = -0.01,\ \mu = 0.3$.

field of signal theory, antenna feeder technique, analysis and synthesis of antenna patterns, including noise minimization. For example, consider $F(x) = erf(\frac{f(x)}{\mu})$, $f(x) = -0.9x + 0.3x^2 + 2.7x^3$. Let $x = b\cos\theta + c$, where θ is the azimuthal angle and $b = 2.1$ and $c = -0.7$. A typical radiation pattern using $F(\theta)$ is depicted on Fig. 6.

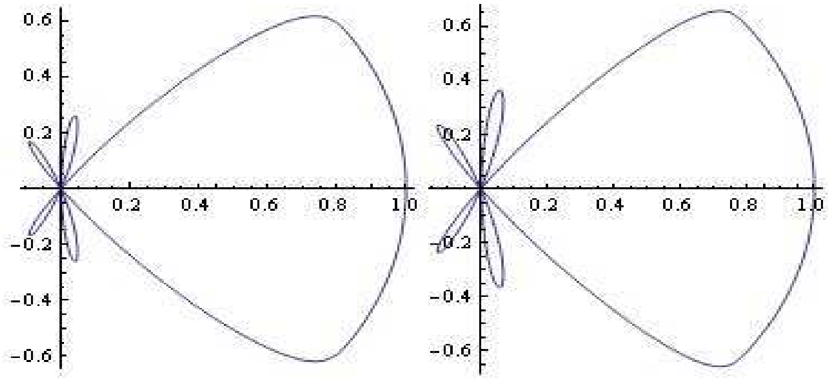

Figure 6: A typical radiation pattern using $F(\theta)$ for a) $\mu = 1$; b) $\mu = 0.7$.

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