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HAUSDORFF APPROXIMATION OF THE HEAVISIDE STEP FUNCTION BY UNIT GOMPERTZ AND COMPLEMENTARY UNIT GOMPERTZ CUMULATIVE FUNCTIONS

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ABSTRACT: In this paper we study the one–sided Hausdorff approximation of the Heaviside step function by a families of Unit–Gompertz (UGo) and Complementary Unit–Gompertz (CGo) cumulative sigmoids [7]. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in ”saturation” study. Numerical examples are presented using *CAS MATHEMATICA*.

AMS Subject Classification: 41A46

Key Words: Unit–Gompertz cumulative model (SIE), Complementary Unit–Gompertz model, ”Saturation” in the Hausdorff sense, upper and lower bounds

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1. INTRODUCTION

In [1] the authors introduced a new probability distribution with support on $(0, 1)$ and named the distribution as Unit–Logistic Distribution (ULD). In [2] the authors introduced an alternative parametrization, where one parameter is the median.

The corresponding cumulative distribution function is written as [3]:

$$M(t) = \left(1 + \left(\frac{\mu(1-t)}{t(1-\mu)}\right)^{\beta}\right)^{-1}$$

where $0 \leq t \leq 1$ and $0 \leq \mu \leq 1$ is the median.

In [4] the authors introduced a new probability distribution with support on $(0, 1)$ and named the distribution as Unit–Weibull Distribution (UWD). The corresponding cumulative distribution function is written as:

$$M1(t) = e^{-\alpha(-\ln t)^{\beta}}$$

where $0 \leq t \leq 1$ and α, $\beta > 0$.

Such cumulative probability distributions can be used with success in approximating parameterized data in the field of "virus-theory", insurance mathematic and population dynamics.

The Topp–Leone (TL) distribution was originally proposed by Topp and Leone (1955) [5] as an alternative to beta distribution and it has been applied for some failure data. The corresponding cumulative distribution function is written as:

$$M_2(t) = t^{\alpha}(2-t)^{\alpha}$$

where $0 \leq t \leq 1$ and $\alpha > 0$.

In [6] we study some intrinsic properties of these families.

Definition 1. In [7] the authors proposed the following new Unit Gompertz distribution (UGo) with cumulative distribution function:

$$F_1(t) = 2^{\frac{t^{-k}-1}{1-a^{-k}}} \tag{1}$$

for $t \in (0, 1)$, $k > 0$ and $a \in (0, 1)$ is the median parameter. i.e. $F_1(a) = \frac{1}{2}$.

Definition 2. In [7] the authors proposed the following new Complementary Unit Gompertz distribution (CUGo) with cumulative distribution function:

$$F_2(t) = 1 - 2^{\frac{(1-t)^{-k}-1}{1-(1-a)^{-k}}} \tag{2}$$

for $t \in (0,1)$, $k > 0$ and $a \in (0,1)$ is the median parameter. i.e. $F_2(a) = \frac{1}{2}$.
Definition 3. The shifted Heaviside step function is defined by

$$h_a(t) = \begin{cases} 0, & \text{if} \quad t < a, \\ [0,1], & \text{if} \quad t = a, \\ 1, & \text{if} \quad t > a \end{cases}$$

Definition 4. The Hausdorff distance [8] (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

In this note we study some properties of the models (1)–(2).

2. MAIN RESULTS

1. For the one–sided Hausdorff distance d between Heaviside function $h_a(t)$ and the function $F_1(t)$ we have the following nonlinear equation

$$F_1(a + d) = 1 - d. \tag{3}$$

From Tylor expansion for $H(d) = F_1(a + d) - 1 + d$ we find:

$$1 - 2^{-\frac{1}{1-a^{-k}} + \frac{a^{-k}}{1-a^{-k}}} + \left(1 - \frac{2^{-\frac{1}{1-a^{-k}} + \frac{a^{-k}}{1-a^{-k}}} a^{-1-k} k \ln 2}{1 - a^{-k}}\right) d + O(d^2) = A + Bd + O(d^2)$$

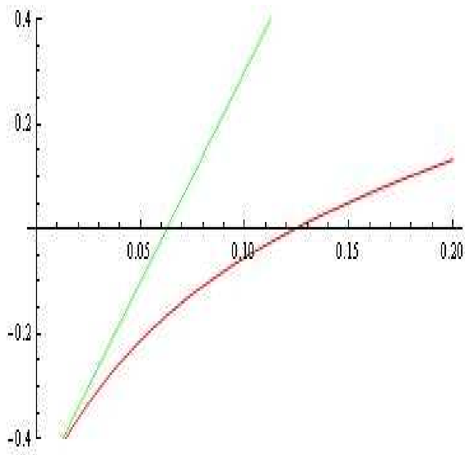


Figure 1: The functions $H(d)$ and $H_1(d)$ for $k = 2$ and $a = 0.1$.

From $F_1(a) = \frac{1}{2}$ we see that

$$A = -\frac{1}{2}, \quad B = 1 - \frac{1}{2}\frac{a^{-1-k}k\ln 2}{1 - a^{-k}}.$$

Consider the function

$$H_1(d) = A + Bd.$$

Let $B > \frac{1}{2.1}e^{1.05}$.

The functions $H(d)$ and $H_1(d)$ are increasing and $H_1(d)$ approximates $H(d)$ with $d \to 0$ as $O(d^2)$ (see, Fig.1).

Further, we have

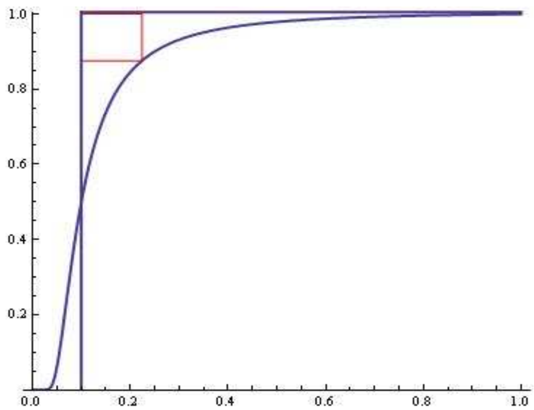
$$H_1\left(\frac{1}{2.1B}\right) < 0,$$

$$H_1\left(\frac{\ln(2.1B)}{2.1B}\right) > 0.$$

Thus, we prove the following theorem gives upper and lower bounds for d.

Theorem 1. Let $B > \frac{1}{2.1}e^{1.05}$. The Hausdorff distance d between Heaviside function $h_a(t)$ and the function $F_1(t)$ satisfies the following inequalities

$$d_l := \frac{1}{2.1B} < d < \frac{\ln(2.1B)}{2.1B} := d_r. \tag{4}$$



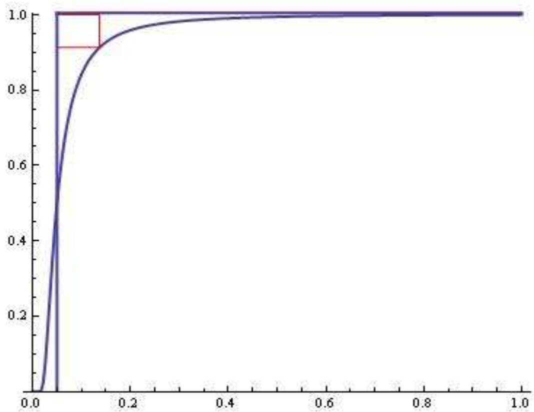


Figure 2: The model (1) for 1) $k = 2$, $a = 0.1$; H-distance $d = 0.124062$, $d_l = 0.0595128$, $d_r = 0.167919$; 2) $k = 2$, $a = 0.05$; H-distance $d = 0.0869186$, $d_l = 0.031964$, $d_r = 0.110057$.

The model (1) for fixed:

1) $k = 2$, $a = 0.1$;

2) $k = 2$, $a = 0.051$

is plotted on Fig. 2.

2. For the one–sided Hausdorff distance d_1 between Heaviside function $h_a(t)$ and the function $F_2(t)$ we have the following nonlinear equation

$$F_2(a + d_1) = 1 - d_1. \tag{5}$$

The reader may formulate the corresponding approximation problem for arbitrary k and a following the ideas given in Theorem 1, and will be omitted.

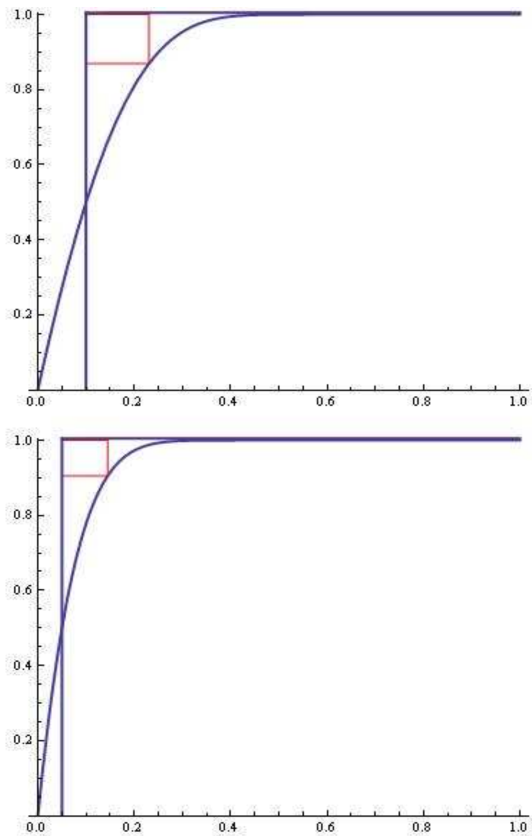


Figure 3: The model (2) for 1) $k = 2$, $a = 0.1$; H-distance $d_1 = 0.130545$ 2) $k = 2$, $a = 0.05$; H-distance $d_1 = 0.094792$.

The model (2) for fixed:

1) $k = 2$, $a = 0.1$;

2) $k = 2$, $a = 0.051$

is plotted on Fig. 3.

A comparison of Figures 2 and 3 shows that the saturation d is better than d_1.

Remark. We formally define the following *Unit Gompertz model with "polynomial variable transfer*:

$$F_1^*(t) = 2^{\frac{(f(t))^{-k}-1}{1-a^{-k}}} \tag{6}$$

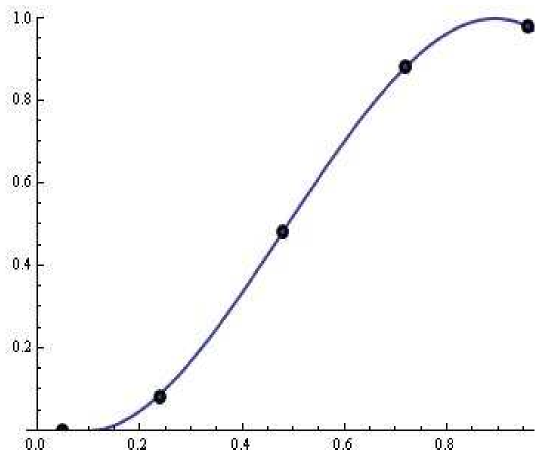


Figure 4: The model (6) for $k = 0.0635215$, $a = 0.52$, $n = 3$, $a_1 = -0.50076$, $a_2 = 4.8642$, $a_3 = -3.41726$.

where $\sum_{i=0}^{n} a_i t^i$; $a_0 = 0$

3. APPLICATIONS

1. We examine the experimental (parameterized) data (Biomass for Xantobacter autotrophycum with electric field).

The appropriate fitting of the data by the model (6) with $k = 0.0635215$, $a = 0.52$, $n = 3$, $a_1 = -0.50076$, $a_2 = 4.8642$, $a_3 = -3.41726$ is visualized on Fig. 4.

2. Let $t = b\cos(\theta) + c$.

Then, for example, typical emitting chart using $F_1(\theta)$ for

$n = 3, a = 0.9, k = 2, a_0 = 0, a_1 = 0.7, a_2 = -2.9, a_3 = 1.6, b = 0.9, c = 0.8;$

is plotted on Fig. 5.

Numerical examples are presented using *CAS MATHEMATICA*.

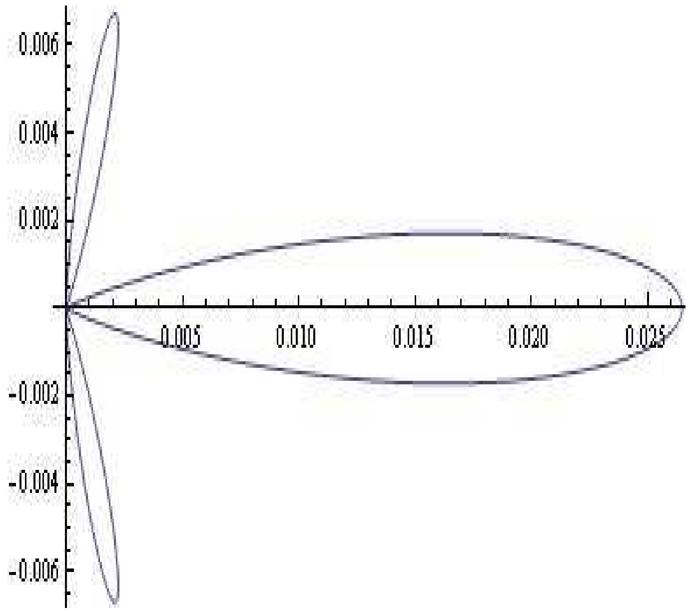


Figure 5: A typical emitting chart using $F_1(\theta)$ for $n = 3,\ a = 0.9,\ k = 2,\ a_0 = 0,\ a_1 = 0.7,\ a_2 = -2.9,\ a_3 = 1.6,\ b = 0.9,\ c = 0.8$.

4. CONCLUDING REMARKS

The use of the new model with many free parameters $a_i;\ i = 1, 2, \ldots,$ makes it attractive for analysis and approximation of specific data from Population Dynamics, Biostatistics, Debugging and Test Theory, Computer viruses propagation and Antenna–feeder Analysis.

We recommend the use of this model at a degree of polynomial $n \leq 10$.

For other results, see [9]–[47].

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