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INVESTIGATIONS ON A NEW GOMPERTZ–EXTENDED–GENERALIZED–EXPONENTIAL (G-EGE) CUMULATIVE FUNCTION

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ABSTRACT: In this paper we study the one–sided Hausdorff approximation of the shifted Heaviside step function by a family of Gompertz–Extended–Generalized Exponential (G-EGE) cumulative function [19]. The model has a certain right of existence insofar as the theory of sigmoidal functions is well developed. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study. Numerical examples are presented using *CAS MATHEMATICA*.

AMS Subject Classification: 41A46

Key Words: Gompertz–Extended–Generalized Exponential (G-EGE) cumulative function, Heaviside step function, Hausdorff distance

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1. INTRODUCTION

The Gompertz model is well known and widely used in many aspects of biology. It has been frequently used to describe the growth of animals and plants, as well as the number or volume of bacteria and cancer cells. The model, referred to at the time as the Gompertz theoretical law of mortality, was first suggested and first applied Benjamin Gompertz in 1825 [1]. The Gompertz model is a special case of the four parameter Richards model and thus belongs to the Richards family of three–parameter sigmoidal growth models. The insurance industry quickly started to use his method of projecting death risk.

The Gompertz and logistic curves are still used in industry, because these curves are well fitted to the cumulative number of faults observed in existing software development processes. Japanese software development companies prefer regression analysis based on deterministic functions such as Gompertz and Gompertz–Makeham–type curves to estimate the number of residual faults.

For some history, and new Gompertz model approach, see [2]–[14].

Some modifications, properties and applications of extended Gompertz and exponentiated extended Gompertz families of distributions can be found in [15]–[16].

In [17], the authors proposed a five–parameters exponentiated generalized extended Gompertz cumulative sigmoid:

$$F(t) = \left(1 - \left(1 - \left(1 - e^{-\frac{\beta}{\gamma}\left(e^{\gamma t}-1\right)}\right)^{\theta}\right)^{a}\right)^{b}, \qquad (1)$$

where $t > 0$, $a > 0$, $b > 0$, $\theta > 0$, $\beta > 0$, $\gamma \geq 0$. Some properties of the family (1) are considered in [18].

Definition 1. In [19] the authors proposed the following Gompertz–Extended–Generalized Exponential (G-EGE) cumulative function

$$F(t) = 1 - e^{\frac{\theta}{\lambda}\left(1-\left(\frac{\beta^{\gamma}-(\beta-1)^{\gamma}}{\beta^{\gamma}-(\beta-e^{-t})^{\gamma}}\right)^{\lambda}\right)}, \qquad (2)$$

where $t > 0$, $\theta > 0$, $\lambda > 0$, $\gamma > 1$, $\beta > 1$.

Definition 2. The *shifted Heaviside step function* is defined by

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases}$$

Definition 3. The Hausdorff distance [21] (the H–distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

In this note we study the Hausdorff approximation of the *shifted Heaviside step function* by the family of type (2).

2. MAIN RESULTS. NUMERICAL EXAMPLES

We consider the following class of this family:

$$\begin{aligned} &M(t) = 1 - e^{\frac{\theta}{\lambda}\left(1-\left(\frac{\beta^{\gamma}-(\beta-1)^{\gamma}}{\beta^{\gamma}-(\beta-e^{-t})^{\gamma}}\right)^{\lambda}\right)}; \\ &M(t_0) = \tfrac{1}{2}. \end{aligned} \tag{3}$$

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid - (3) satisfies the relation

$$M(t_0 + d) = 1 - d. \tag{4}$$

The model (3) for different parameter values θ, λ, γ and β is shown in Figure 1–(a,b,c).

In [20] Bantan, Jamal, Chesneau and Elgarhy introduced a new G–Family of distribution with c.d.f.

$$H(t) = e^{\alpha_1\beta_1\left(1-\frac{1}{G(t)}\right)}\left(2 - e^{\beta_1\left(1-\frac{1}{G(t)}\right)}\right)^{\alpha_1} \tag{5}$$

where $\alpha_1, \beta_1 \in R^+$ and $G(t)$ is a c.d.f. of a baseline continuous distribution.

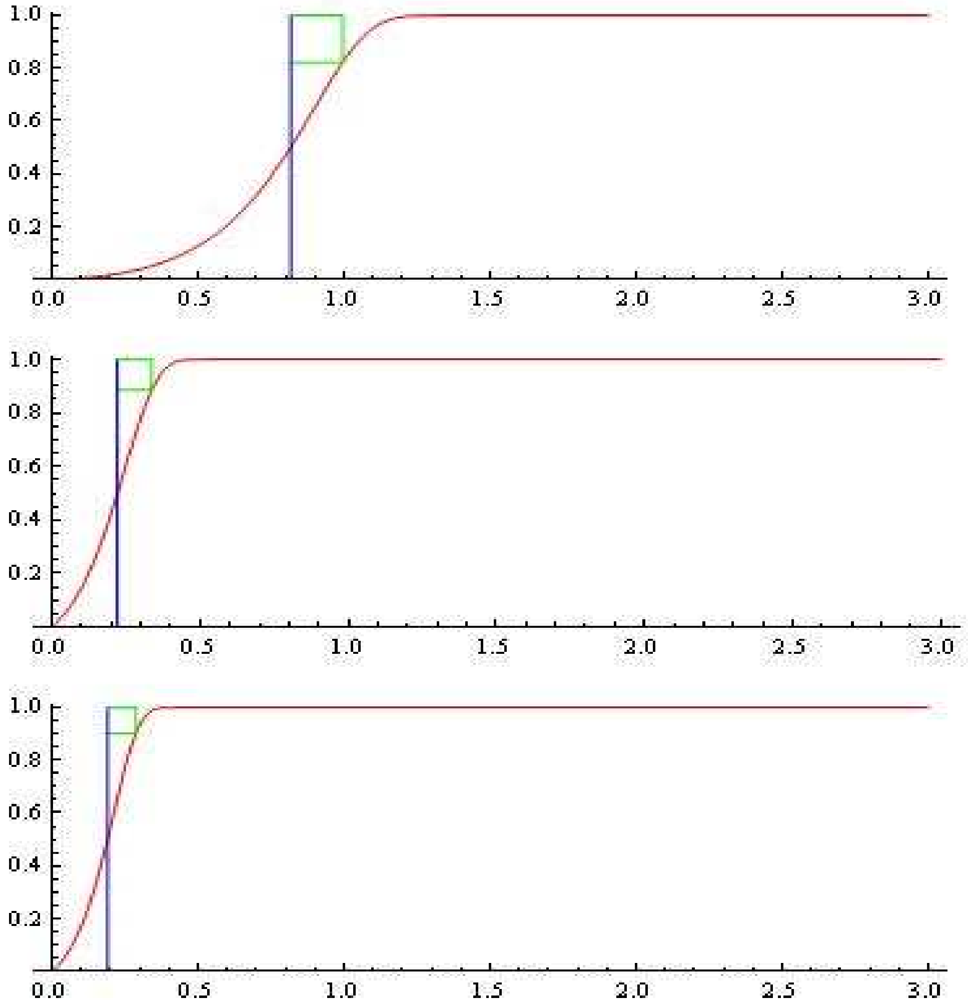

Figure 1: The model (3) for a) $\beta = 1.1$; $\gamma = 3.01$; $\lambda = 7.8$; $\theta = 0.99$, $t_0 = 0.817327$; H–distance $d = 0.177762$; b) $\beta = 9.9$; $\gamma = 1.01$; $\lambda = 9$; $\theta = 0.99$, $t_0 = 0.221002$; H–distance $d = 0.115441$; c) $\beta = 56$; $\gamma = 1.0001$; $\lambda = 11.6$; $\theta = 0.999$, $t_0 = 0.18988$; H–distance $d = 0.0974808$.

The following result shows some inequalities involving $H(t)$ (see, Proposition 1 [20]):

$$e^{\alpha_1\beta_1\left(1-\frac{1}{G(t)}\right)}\left(2-G(t)^{\beta_1}\right)^{\alpha_1} \leq H(t) \leq 2^{\alpha_1} e^{\alpha_1\beta_1\left(1-\frac{1}{G(t)}\right)}. \tag{6}$$

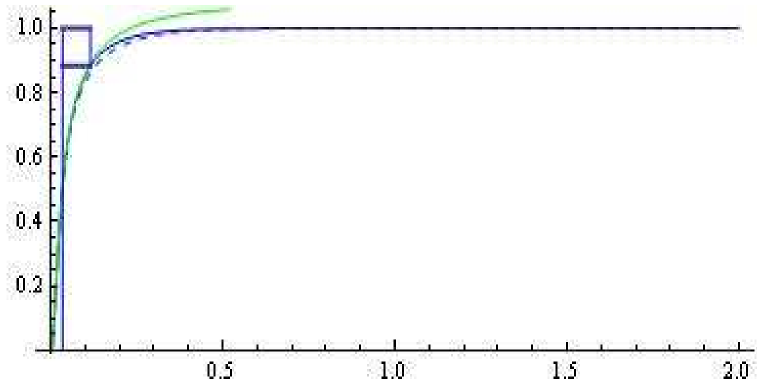

Figure 2: The model (7) and two–sided estimations (6) with $G(t) = M(t)$ for fixed $\alpha_1 = 0.1$, $\beta_1 = 0.3$ and $\beta = 10$; $\gamma = 1.1$; $\lambda = 2.9$; $\theta = 1.1$, $t_0 = 0.0335888$; H–distance $d = 0.115988$.

Definition 4. Formally, we define the following corresponding c.d.f.:

$$M_1(t) = H(t); \quad G(t) = M(t). \tag{7}$$

In this paper we study some properties of the new family.

When studying the intrinsic properties of the $M_1(t)$, it is also appropriate to study the "saturation" to the horizontal asymptote.

Some computational examples are presented on Fig. 2–3.

From Fig. 2–3 it can be seen that these estimations can be used as "confidence bounds", which are extremely useful for the specialists in the choice of model for cumulative data approximating in areas of Biostatistics, Population dynamics, Growth theory, Debugging and Test theory, Computer viruses propagation, Financial and Insurance mathematics.

3. SOME APPLICATIONS

Example 1.

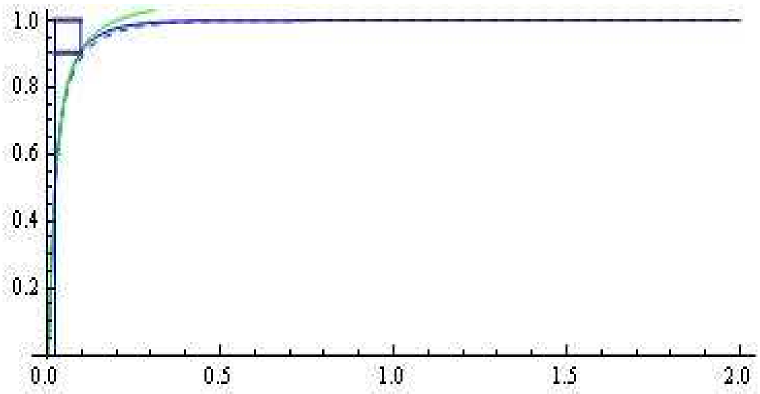

Figure 3: The model (7) and two–sided estimations (6) with $G(t) = M(t)$ for fixed $\alpha_1 = 0.08$, $\beta_1 = 0.29999$ and $\beta = 14$; $\gamma = 1.01$; $\lambda = 2.9$; $\theta = 1.32$, $t_0 = 0.0231156$; H–distance $d = 0.0968153$.

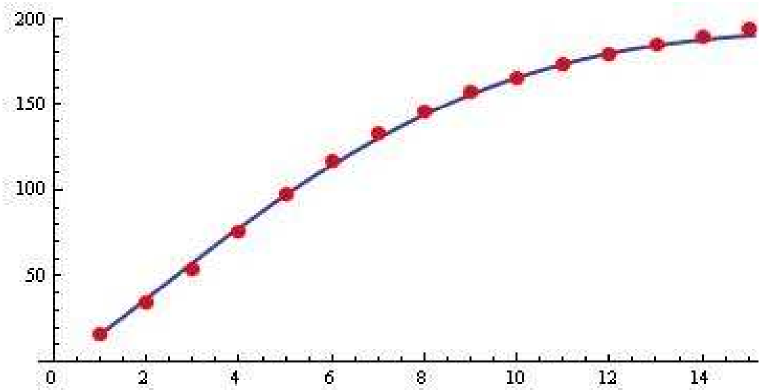

Figure 4: The fitted model $M^*(t) = \omega M(t)$ with $\omega = 194$, $\beta = 1.1$; $\gamma = 1.66778$; $\lambda = 0.1$; $\theta = 0.12$

We examine the data for the growth of red abalone Haliotis Rufescens in Northern California [39].

For this data the fitted model $M^*(t) = \omega M(t)$ for $\omega = 194$, $\beta = 1.1$; $\gamma = 1.66778$; $\lambda = 0.1$; $\theta = 0.12$ is visualized on Fig. 4.

Example 2.

On July 26, 2004 a variant of MyDoom attacks Google, AltaVista and

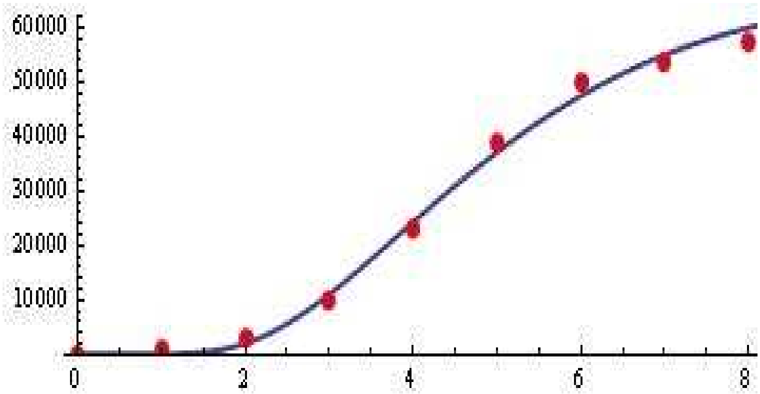

Figure 5: The fitted model $M^*(t) = \omega M(t)$ with $\omega = 66300$, $\beta = 2.01$; $\gamma = 34.4779$; $\lambda = 0.13$; $\theta = 0.32$.

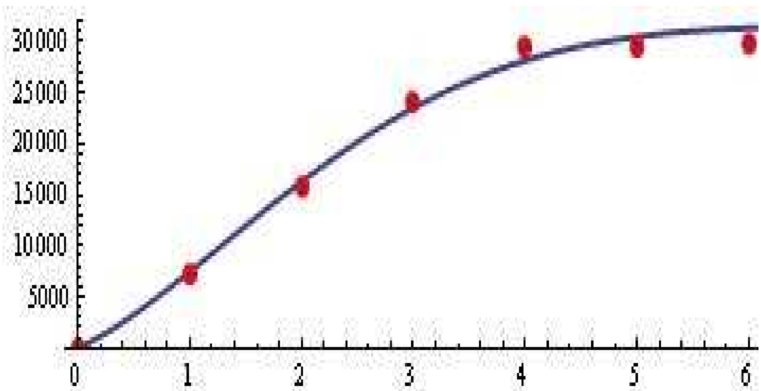

Figure 6: The fitted model $M^*(t) = \omega M(t)$ with $\omega = 31600$, $\beta = 2.01$; $\gamma = 3.12799$; $\lambda = 0.27$; $\theta = 0.39$.

Lycos, completely stopping the function of the popular Google search engine for the larger portion of the workday, and creating noticeable slow-downs in the AltaVista and Lycos engines for hours [40]. We analyze the data given in [40].

The fitted model $M^*(t) = \omega M(t)$ for $\omega = 66300$, $\beta = 2.01; \gamma = 34.4779; \lambda = 0.13$; $\theta = 0.32$ is visualized on Fig. 5.

Example 3.

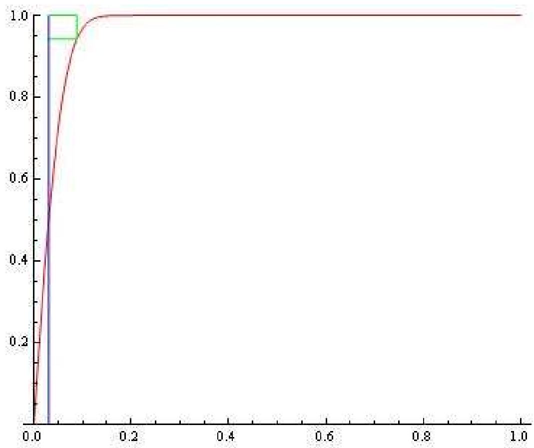

Figure 7: The model (8) for fixed $\alpha = 9$, $a = 1.1$ $\lambda = 0.9$, $\theta = 0.99$, $t_0 = 0.0306208$; H–distance $d = 0.0576797$.

We analyze the data for Welchia worm [41].

For this data the fitted model for estimated parameters: $\omega = 31600$, $\beta = 2.01$; $\gamma = 3.12799$; $\lambda = 0.27$; $\theta = 0.39$ is plotted on Fig. 6.

The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study.

The proposed new model can be successfully used to approximating data from Population Dynamic, Biostatistics and Debugging Theory.

For some approximation, computational and modelling aspects, see [22]–[37], [42]–[47].

The results obtained in this paper can be used when controlling growth in Software Reliability Models, see [8], [9], [38].

Remark.

Definition 5. In [48] Reyad, Selim and Othman proposed a new generator of continuous distributions. The new G–family with baseline cdf $G(t) = 1 - e^{-at}$, $a > 0$ has the following cdf [48]:

$$M_2(t) = 1 - e^{1-\left(1+\lambda(1-e^{-2at})^{\theta}\right)^{\alpha}}, \tag{8}$$

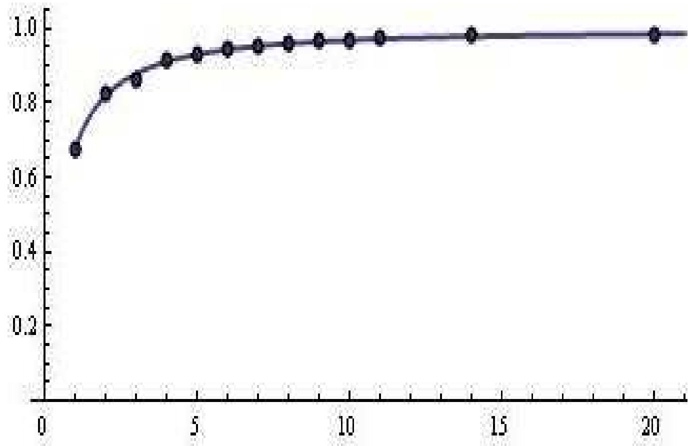

Figure 8: The fitted model (8) for $\alpha = 0.184081$, $a = 0.0328168$ $\lambda = 14913.1$, $\theta = 2$.

where $t > 0$, $\theta > 0$, $\lambda > 0$, and $\alpha > 0$.

Let t_0 is the value for which $M_2(t_0) = \frac{1}{2}$. The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid - (8) satisfies the relation

$$M_2(t_0 + d) = 1 - d. \tag{9}$$

A computational example is presented on Fig. 7.

Studies of "saturation" in the Hausdorff sense at the "median" level show that the new model has its worthy place among the "family of models" designed to analyze and approximate "specific" cumulative data.

Example 4.

We analyze the data for *data_CDF_of_ransoms_received_per_address_in_*C_{CL} [49].

*data_CDF_of_ransoms_received_per_address_in_*C_{CL} := {{1,0.6762},{2,0.8286}, {3,0.8667},{4,0.9143},{5,0.9333},{6,0.9429},{7,0.9524},{8,0.9571},{9,0.9667}, {10,0.9714},{11,0.9733},{14,0.9810},{20,0.9829}}.

For this data the fitted model (8) for estimated parameters: $\alpha = 0.184081$, $a = 0.0328168$ $\lambda = 14913.1$, $\theta = 2$ is presented on Fig. 8.

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