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SOME NEW CLASSES OF GROWTH FUNCTIONS GENERATED BY REACTION NETWORKS AND BASED ON "CORRECTING AMENDMENTS" OF BATEMAN–GOMPERTZ AND BATEMAN–GOMPERTZ–MAKEHAM–TYPE. I.

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ABSTRACT: It is of interest to observe model by Markov [2] based on the following new reaction networks

$$S \overset{k_1(t)}{\longrightarrow} P \overset{k_2(t)}{\longrightarrow} Q$$
$$P + X \overset{k}{\longrightarrow} 2X + P.$$

in the case where, for example, the reaction constants $k_1 = k_1(t)$ and $k_2 = k_2(t)$ are functions of time.

In the present work we will get a modifications of the growth model based on a new reaction scheme.

It is proven, that the new growth models are based on the insertion of "correcting amendments" of Bateman–Gompertz–type and Bateman–Gompertz–Makeham–type.

We examine the data: "cdf of the number of Bitcoin received per address".

Some numerical examples, using *CAS MATHEMATICA* illustrating our results are given.

The generation of limited growth curves with other exponentially variable transfers can be successfully expanded based on the considerations in this article.

AMS Subject Classification: 41A46

Key Words: Bateman–Gompertz type correction, Bateman–Gompertz–Makeham type correction, Reaction networks, Laplace integral transform, Generalized growth model

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1. INTRODUCTION AND PRELIMINARIES

Dynamical models consisting of a systems of ”reaction” differential equations are commonly used in chemistry, there the differential equations are called *reaction equations.*

For some details, see [3]–[9].

In [1]–[2] Markov proposed a class of growth–decay model formulated in terms that include various types of evolution of the resource species:

$$\begin{array}{ccc} S \xrightarrow{k_1} P \xrightarrow{k_2} & Q & \\ P + X & \xrightarrow{k} & 2X + P. \end{array} \tag{1}$$

Reaction network (1) induces the following differential system

$$\begin{cases} \dfrac{ds}{dt} & = & -k_1 s \\ \dfrac{dp}{dt} & = & k_1 s - k_2 p \\ \dfrac{dx}{dt} & = & kxp \end{cases} \tag{2}$$

with $s(0) = s_0$; $p(0) = p_0$; $x(0) = x_0$.

Using Laplace integral transform we have

$$p(t) = \frac{k_1}{k_2 - k_1} s_0 \left(e^{-k_1 t} - e^{-k_2 t} \right) + p_0 e^{-k_2 t}. \tag{3}$$

and from the last equation of differential system (2) $x' = kxp$, Markov generates a new logistic model.

In [10] Kyurkchiev considered the following reaction network (for $n \geq 2$):

$$\begin{aligned}
S &\xrightarrow{k_1} P \xrightarrow{k_2} Q \\
S &\xrightarrow{k_3} P_1 \xrightarrow{k_4} Q_1 \\
&\dots \\
S &\xrightarrow{k_{2n-1}} P_{n-1} \xrightarrow{k_{2n}} Q_{n-1} \\
P_{n-1} + X &\xrightarrow{k} nX + P_{n-1}.
\end{aligned} \tag{4}$$

The proposed reaction networks (4) involve additional species interpreted as environmental resource.

Hence, the general new model can be written for the growth function in the form:

$$\begin{cases} x'(t) = kxP_{n-1}(t) \\ x(0) = x_0 \end{cases} \tag{5}$$

where for $k_i \neq k_j$; $i, j = 1, 2, \dots, 2n$;

$$P_{n-1}(t) = \frac{k_{2n-1} s_0}{k_{2n} - \sum_{i=1}^{n} k_{2i-1}} \left(e^{-\sum_{i=1}^{n} k_{2i-1} t} - e^{-k_{2n} t} \right) + p_{n-1,0} e^{-k_{2n} t}. \tag{6}$$

It is of interest to observe model (1) by Markov in the case where, for example, the reaction constant k_1 is a function of time, i.e. $k_1 = k_1(t)$.

In this article we will get a modifications of the growth model based on a new reaction scheme.

2. MAIN RESULTS

2.1. THE CASE $K_1 = K_1(T)$ AND $K_2 = CONST$

We examine the following generalized reaction network:

$$\begin{aligned}
S &\xrightarrow{k_1(t)} P \xrightarrow{k_2} Q \\
P + X &\xrightarrow{k} 2X + P.
\end{aligned} \tag{7}$$

Consequently the following differential equations can be formulated

$$\begin{cases} \frac{ds}{dt} &= -k_1(t)s \\ \frac{dp}{dt} &= k_1(t)s - k_2 p \\ \frac{dx}{dt} &= kxp \end{cases} \tag{8}$$

with $s(0) = s_0$; $p(0) = p_0$; $x(0) = x_0$.

Suppose that the $k_1(t)$ varies in the following manner

$$k_1(t) = e^{bt}; \ \ (b > 0). \tag{9}$$

For the solution of the first equation in (8)

$$\frac{ds}{dt} + k_1(t)s = 0$$

we obtain

$$s(t) = Ce^{-\int_0^t e^{bt}dt} = Ce^{-\frac{1}{b}\int_0^t e^{bt}d(bt)} = Ce^{-\frac{1}{b}e^{bt}|_0^t} = Ce^{-\frac{1}{b}\left(e^{bt}-1\right)}.$$

From $s(0) = s_0$ we get $C = s_0$ and

$$s(t) = s_0 e^{-\frac{1}{b}\left(e^{bt}-1\right)}. \tag{10}$$

From the second equation of the system (8) we have

$$\frac{dp}{dt} + k_2 p(t) = k_1(t)s(t) = s_0 e^{bt} e^{-\frac{1}{b}\left(e^{bt}-1\right)}.$$

The solution to this equation is

$$\begin{aligned} p(t) &= s_0 e^{-k_2 t}\int_0^t e^{bt} e^{-\frac{1}{b}\left(e^{bt}-1\right)} e^{k_2 t}.dt + Re^{-k_2 t} \\ &= s_0 e^{-k_2 t}\int_0^t e^{(k_2+b)t-\frac{1}{b}e^{bt}+\frac{1}{b}}.dt + Re^{-k_2 t} \end{aligned}$$

and $p(t)$ is preserved as a function of the traditional exponential integral - $Ei(.)$.

The general new model can be written for the growth function in the form:

$$\begin{cases} x'(t) = kxp(t) \\ x(0) = x_0 \end{cases}$$

Remark. We note that the cumulative distribution function for Gompertz distribution

$$F(t|b,\eta) = 1 - e^{-\eta(e^{bt}-1)},$$

where $\eta > 0$ is shape parameter and $b > 0$ is scale parameter has been used of science such as biology, demography, insurance, gerontology, marketing, debugging and test theory.

The corresponding survival function

$$S^*(t|b,\eta) = e^{-\eta(e^{bt}-1)}$$

is of the form (10).

Obviously, the new growth model proposed in this article is based on the insertion of "correcting amendments" of Bateman–Gompertz–type.

Without loss of generality, let $b = 1$ and $k_2 = 1$.

Then

$$\begin{aligned} p(t) &= s_0 e^{-t} \int_0^t e^{2t-e^t+1}.dt + Re^{-k_2 t} \\ &= s_0 e^{-t} \left(-(1+e^t)e^{1-e^t} \right) |_0^t + Re^{-t} \\ &= s_0 e^{-t} \left(-(1+e^t)e^{1-e^t} + 2 \right) + Re^{-t}. \end{aligned}$$

From $p(0) = p_0$ we get $R = p_0$.

So, finally, for the solution $p(t)$ we get

$$p(t) = s_0 e^{-t} \left(-(1+e^t)e^{1-e^t} + 2 \right) + p_0 e^{-t}. \tag{11}$$

More importantly, the solution $x(t)$ of the last equation of the differential system (8)

$$x(t) = x_0 e^{k(p_0+s_0)+k\left(e^{-t}(-p_0-2s_0)+e^{1-e^t-t}s_0\right)} \tag{12}$$

generates a new growth model that we have not encountered described in the literature.

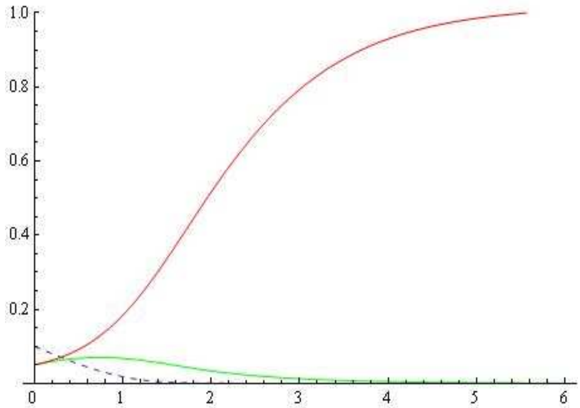

Figure 1: The functions $x(t)$–(red); $p(t)$–(green); $s(t)$–(dashed) for $k = 20.1$; $s_0 = 0.1$; $p_0 = 0.05$; $x_0 = 0.05$.

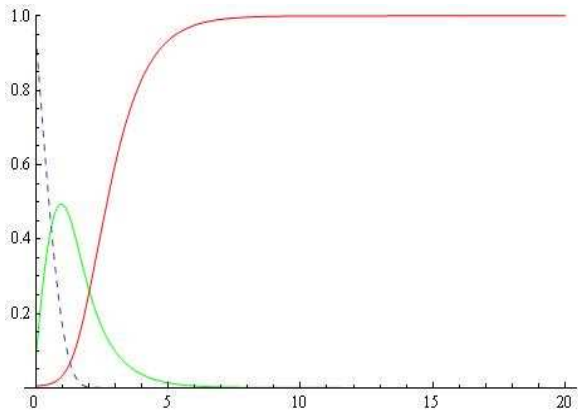

Figure 2: The functions $x(t)$–(red); $p(t)$–(green); $s(t)$–(dashed) for $k = 5.17$; $s_0 = 0.95$; $p_0 = 0.075$; $x_0 = 0.005$.

We illustrate our new model for various parameters k, s_0, p_0 and x_0 (see, Fig. 1–Fig. 2).

An example of the usage of dynamical solution of the system and their graphical representation is illustrated on Fig 3.

Remarks. 1. It is important to study the characteristic - "super saturation" of the model to the horizontal asymptote. These studies are the subject of a future article.

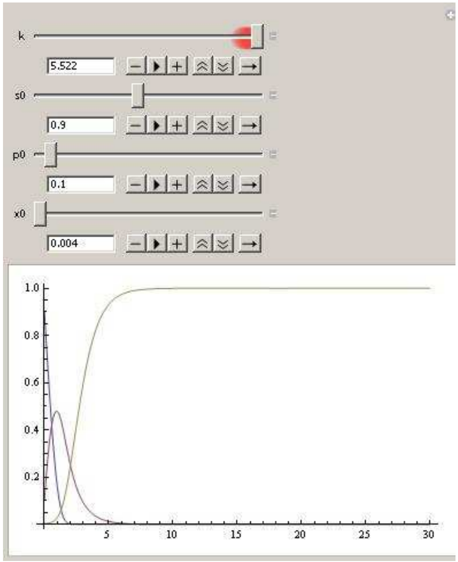

Figure 3: An example of the usage of dynamical solution of the system and their graphical representation. The plots are prepared using *CAS Mathematica*.

We will note only that, for example, for the new model (12) is fulfilled:

$$\lim_{t\to\infty} x(t) = x_0 e^{k(p_0+s_0)}.$$

2. The new growth model can be used with success (of course, after extensive research) in the field of analysis of Computer Viruses Propagation and Debugging and Test Theory.

2.1.1. SOME APPLICATIONS

It is well known that in many cases the existing modifications to the classical logistics model do not give very reliable results in approximating "specific data".

We examine the following "specific datasets":

1. Approximating cdf of the number of Bitcoin received per address [32]

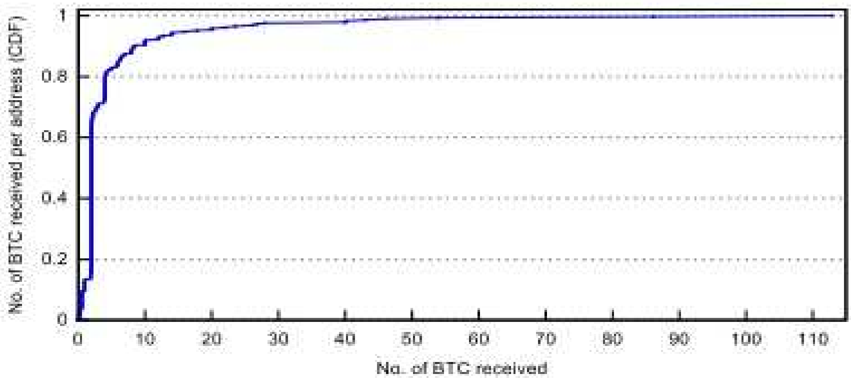

Figure 4: CDF of Bitcoin received (in ransoms) per address in C_{CL} [32].

We consider the following data (see, [32]:

$data\_CDF\_of\_Bitcoin\_received\_(inransoms)\_per\_address\_in\_C_{CL}$

$:= \{\{1, 0.0857\}, \{2, 0.1238\}, \{3, 0.6571\}, \{4, 0.6854\}, \{5, 0.8381\},$

$\{6, 0.8476\}, \{7, 0.8810\}, \{8, 0.9095\}, \{9, 0.9143\}, \{10, 0.9333\},$

$\{12, 0.9429\}, \{14, 0.9571\}, \{18, 0.9667\}, \{20, 0.9762\}, \{23, 0.9810\},$

$\{27, 0.9857\}, \{40, 0.9905\}, \{46, 0.9952\}, \{59, 0.9981\}\}.$

Fig. 4 show cdf of the number of Bitcoin received per address respectively [32].

The model (12) for $p_0 = 0.01$, $s_0 = 0.1$, $x_0 = 0.00160594$, $k = 58.2891$ is visualized on Fig. 5.

2. We consider the following data (see, [32]:

$data\_CDF\_1 := \{\{1, 0.6762\}, \{2, 0.8286\}, \{3, 0.8667\}, \{4, 0.9143\},$

$\{5, 0.9333\}, \{6, 0.9429\}, \{7, 0.9524\}, \{8, 0.9571\}, \{9, 0.9667\},$

$\{10, 0.9714\}, \{11, 0.9733\}, \{14, 0.9810\}, \{20, 0.9829\}, \{23, 0.9857\},$

$\{25, 0.9885\}, \{55, 0.9905\}, \{70, 0.9952\}, \{83, 1\}\}.$

The model (12) for $p_0 = 0.011$, $s_0 = 0.11$, $x_0 = 0.524987$, $k = 5.1585$ is visualized on Fig. 6.

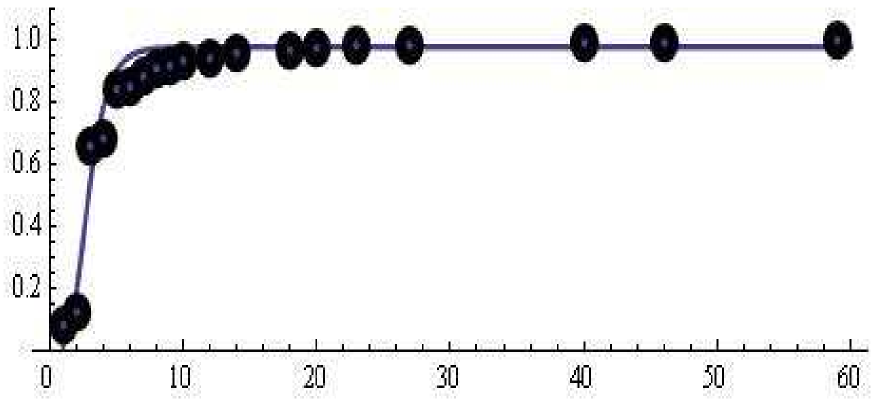

Figure 5: The fitted model (12) for approximation of the data: "cdf of the number of Bitcoin received per address" [32].

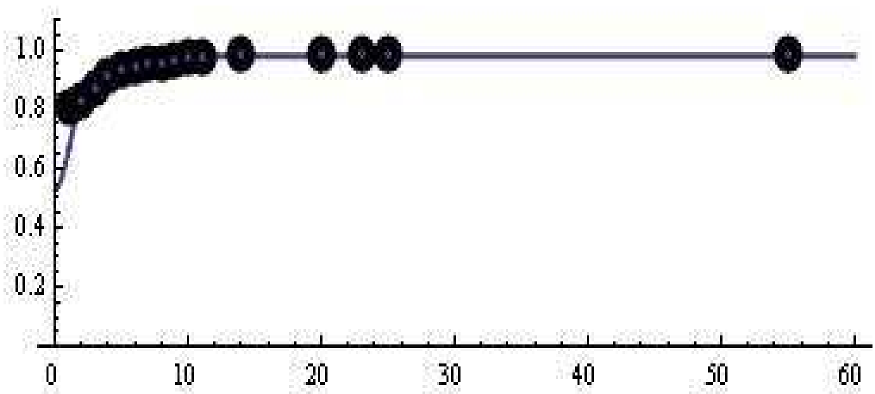

Figure 6: The fitted model (12) for approximation of the data: "data_CDF_1" [32].

3. Storm worm was one of the most biggest cyber threats of 2008 [33]. We consider the following data:

$data\_Storm\_IDs$

$:= \{\{1, 0.843\}, \{4, 0.926\}, \{5, 0.954\}, \{6, 0.967\}, \{7, 0.976\},$

$\{8, 0.981\}, \{9, 0.985\}, \{10, 0.991\}, \{22, 0.995\}, \{38, 0.997\}, \{51, 0.998\},$

$\{64, 0.9985\}, \{74, 0.999\}, \{83, 1\}, \{100, 1\}\}$

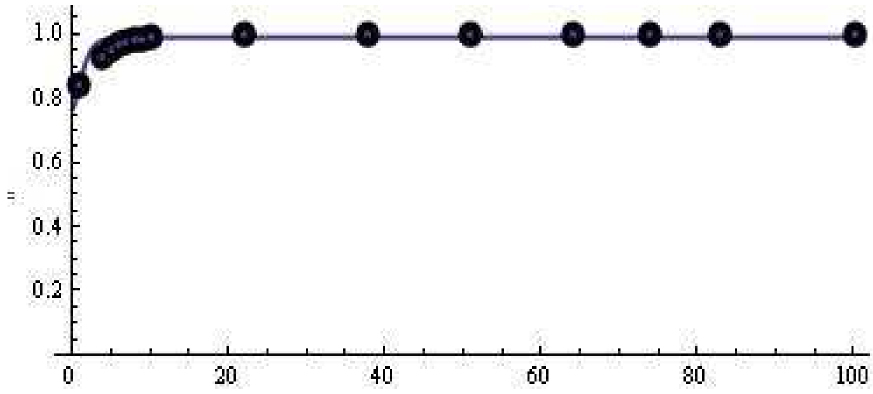

Figure 7: The fitted model (12) for approximation of the data: "data-Storm".

The model (12) for $p_0 = 0.011$, $s_0 = 0.11$, $x_0 \approx 0.764$, $k = 2.13738$ is visualized on Fig. 7.

The experiments show that in some cases the use of the growth model proposed in this article is satisfactory.

Specialists working in this scientific field have a say.

2.1.2. THE CASE $K_1(T) = \alpha + \beta E^{\gamma T}$

It is known that the survival function corresponding to Gompertz-Makeham distribution is of the form

$$S^*(t) = e^{-\alpha t - \frac{\beta}{\gamma}\left(e^{\gamma t} - 1\right)}. \tag{13}$$

Let $k_1(t)$ in our model is defined by $k_1(t) = \alpha + \beta e^{\gamma t}$.

Then for the solution of the first equation in (8) we have

$$s(t) = Ce^{-\alpha t - \frac{\beta}{\gamma} e^{\gamma t}}.$$

From $s(0) = s_0$ we get $C = s_0 e^{\frac{\beta}{\gamma}}$ and

$$s(t) = s_0 e^{-\alpha t - \frac{\beta}{\gamma}\left(e^{\gamma t} - 1\right)},$$

i.e. the solution $s(t)$ is of the form (13).

For the solution $p(t)$ we get

$$p(t) = s_0 e^{-k_2 t} \int_0^t (\alpha + \beta e^{\gamma t}) e^{-\alpha t - \frac{\beta}{\gamma}(e^{\gamma t}-1)} e^{k_2 t}.dt + R_1 e^{-k_2 t}.$$

Obviously, the solution $p(t)$ is preserved as a function of the traditional exponential integral - $Ei(.)$.

For example, if $\alpha = \beta = \gamma = k_2 = 1$, then for $p(t)$ we have

$$p(t) = s_0 e^{-t} e(-e^{-e^t} + Ei(-e^t))|_0^t + R_1 e^{-t}$$

and the solution of the latter equation of the differential system (8) $x' = kxp$ generates a *new growth model based on the insertion of "correcting amendments" of Bateman–Gompertz–Makeham–type.*

Of course, the detailed examination of the two models proposed must take into account the critical remarks in the literature regarding applicability of the Gompertz–Makeham law in Human Populations (see, for instance [31]).

For other results, see [10]–[30].

2.1.3. THE CASE $K_1(T) = \frac{1}{\alpha}\left(\frac{T}{\alpha}\right)^{\beta-1} E^{\left(\frac{T}{\alpha}\right)^\beta}$

It is known that the survival function corresponding to Modified Makeham Distribution is of the form [29]

$$S^*(t) = e^{1-e^{\left(\frac{t}{\alpha}\right)^\beta}}.$$

Let $k_1(t)$ in our model is defined by $k_1(t) = \frac{1}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{\left(\frac{t}{\alpha}\right)^\beta}$.

Then for the solution of the first equation in (8) we have

$$s(t) = Ce^{-e^{\left(\frac{t}{\alpha}\right)^\beta}}.$$

From $s(0) = s_0$ we get $C = s_0 e$ and

$$s(t) = s_0 e^{1-e^{\left(\frac{t}{\alpha}\right)^\beta}}.$$

For the solution $p(t)$ we get

$$p(t) = s_0 \frac{1}{\alpha} e^{-k_2 t} \int_0^t \left(\frac{t}{\alpha}\right)^{\beta-1} e^{1+\left(\frac{t}{\alpha}\right)^\beta - e^{\left(\frac{t}{\alpha}\right)^\beta}} e^{k_2 t}.dt + R_2 e^{-k_2 t}$$

and the solution of the latter equation of the differential system (8) $x' = kxp$ generates a *new growth model based on the insertion of "correcting amendments" of Bateman–Modified–Makeham–type.*

2.2. THE CASE $K_1 = K_1(T)$ AND $K_2 = K_2(T)$

We examine the following generalized reaction network:

$$\begin{aligned} S \quad &\xrightarrow{k_1(t)} P \xrightarrow{k_2(t)} \quad Q \\ P + X \quad &\xrightarrow{k} \quad 2X + P. \end{aligned} \tag{14}$$

Suppose that

$$k_1(t) = e^{bt};\ \ k_2(t) = e^{at};\ \ (b, a > 0). \tag{15}$$

From the corresponding differential system we have

$$s(t) = s_0 e^{-\frac{1}{b}\left(e^{bt}-1\right)}. \tag{16}$$

$$p(t) = s_0 e^{-\frac{1}{a}e^{at}} \int_0^t e^{bt-\frac{1}{b}(e^{bt}-1)+\frac{1}{a}e^{at}}.dt + R_2 e^{-\frac{1}{a}e^{at}}.$$

Without loss of generality, let $b = a = 1$.

Then, for the solution $p(t)$ we get

$$p(t) = s_0 e^{-e^t} e(e^t - 1) + ep_0 e^{-e^t}. \tag{17}$$

More importantly, the solution $x(t)$ of the last equation of the differential system

$$x(t) = x_0 e^{-ek\left(-\frac{s_0}{e}+(p_0-s_0)Ei[-1]\right)+ek\left(-e^{-e^t}s_0+(p_0-s_0)Ei[-e^t]\right)} \tag{18}$$

generates a new growth model.

We illustrate the new model for various parameters k, s_0, p_0 and x_0 (see, Fig. 8–Fig. 9).

Remark. The studies in this article can be extended to the reaction mechanism, proposed by Kyurkchiev [10] (see, reaction network (4)) in the case where $k_3 = k_3(t),\ k_5 = k_5(t), \ldots, k_{2n-1} = k_{2n-1}(t)$.

This requires of course, a lot of serious research on the proposed new model and we will miss it here.

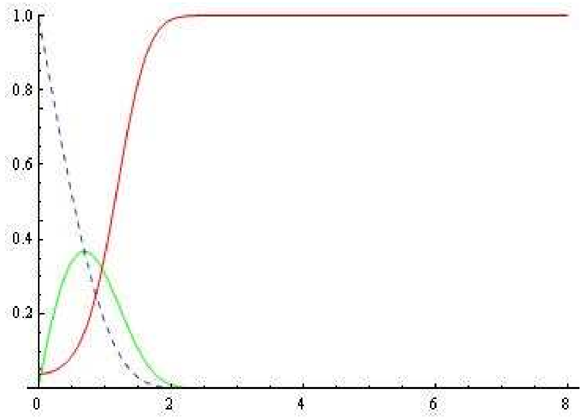

Figure 8: The functions $x(t)$–(red); $p(t)$–(green); $s(t)$–(dashed) for $p_0 = 0.01$, $s_0 = 0.99$, $x_0 = 0.039$, $k = 8$.

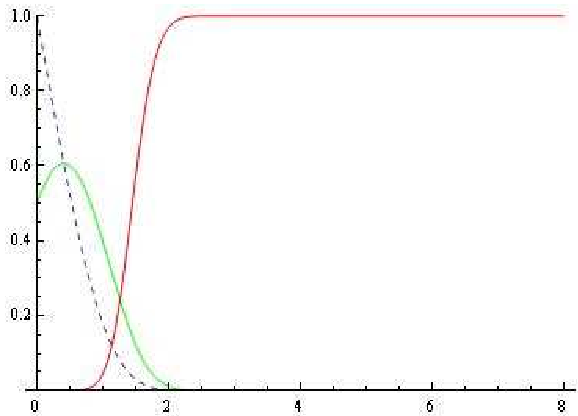

Figure 9: The functions $x(t)$–(red); $p(t)$–(green); $s(t)$–(dashed) for $p_0 = 0.5$, $s_0 = 0.999$, $x_0 = 0.000000808$, $k = 20$.

3. CONCLUSIONS

The generation of limited growth curves with other exponentially variable transfers can be successfully expanded based on the considerations in this article.

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