Communications in Applied Analysis, **23**, No. 3 (2019), 483-495 ISSN: 1083-2564

SOLVING BVPs OF SINGULARLY PERTURBED DISCRETE SYSTEMS

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ABSTRACT: In this article, we study boundary value problems of a large class of non-linear discrete systems at two-time-scales. Algorithms are given to implement asymptotic solutions for any order of approximation.

AMS Subject Classification: 93C05, 93C55, 93C70, 93C73
Key Words: time-scale; iterative methods; boundary value problem; discrete system; asymptotic expansions

Received: June 11, 2019 ; **Accepted:** December 16, 2019 ;
Published: January 4, 2020. **doi:** 10.12732/caa.v23i3.8
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION

With the development of the digital computer, the theory of discrete time systems has now a significant impact in control theory. Since about forty years, there has been an intensive research in the area of discrete systems at two-time-scales, for instance, see [2, 3, 5, 6, 7, 20]. Recently, in [14, 18], we considered some classes of non-linear discrete systems at two-time-scales, we gave iterative methods to solve two-point boundary value problems (TPBVP).

The TPBVPs constitute an important class of problems that occur frequently in optimal control, and many discrete TPBVPs have not yet been fully studied. This article is devoted to the resolution of TPBVPs of a broad class of systems at two-time-scales. Consider the systems

$$\begin{cases} x(t+1) = f\left(\varepsilon x(t), \varepsilon^p y(t), \varepsilon, t\right), \\ y(t+1) = g\left(\varepsilon^{1-p} x(t), y(t), \varepsilon, t\right), \end{cases} \qquad t \in I_{N-1},\ p = 0, 1, \tag{1}$$

$$x(t=0) = \alpha(\varepsilon), \qquad y(t=N) = \beta(\varepsilon), \tag{2}$$

where $|\varepsilon| \ll 1$, $I_N = \{0, 1, \cdots, N\}$; the mappings $f : \mathcal{U} \to X$ and $g : \mathcal{U} \to Y$ are supposed to be n-differentiable in their arguments where $\mathcal{U} := F(I_N, X) \times G(I_N, Y) \times (-1, 1) \times I_N$, $F(I_N, X)$ and $G(I_N, Y)$ respectively denote the space of all mappings of I_N into the Banach spaces $(X, \|.\|)$ and $(Y, \|.\|)$. We suppose that $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ have the asymptotic representations

$$\alpha(\varepsilon) = \sum_{j=0}^{n} \varepsilon^j \alpha^{(j)} + \mathcal{O}(\varepsilon), \quad \beta(\varepsilon) = \sum_{j=0}^{n} \varepsilon^j \beta^{(j)} + \mathcal{O}(\varepsilon). \tag{3}$$

The linear case of system (1) includes two models known in control theory, the *C-model* ($p = 0$) and the *R-model* ($p = 1$) [2, 12]. Our results consist in giving sufficient conditions for the existence of solutions to the discrete boundary value problems (1)$-$(2), and to describe iterative algorithms to find the coefficients of the series

$$x(t,\varepsilon) = \sum_{j=0}^{n} \varepsilon^j x^{(j)}(t) + \mathcal{O}(\varepsilon^{n+1}),\ y(t,\varepsilon) = \sum_{j=0}^{n} \varepsilon^j y^{(j)}(t) + \mathcal{O}(\varepsilon^{n+1}), \tag{4}$$

fixing the asymptotic solutions. Several models of discrete boundary value problems been studied in previous works using a similar technique [8, 9, 10, 11, 12, 13, 15, 16, 17, 19]. The plan of this paper is as follows. In Section 2, we study the case $p = 0$ which represents a non-linear extension of the C-model and Section 3 is devoted to the non-linear extension of the R-model obtained by setting $p = 1$ into (1). We conclude the paper with a brief conclusion. Throughout this paper, for an abbreviated writing, $D_1^{k_1} D_2^{k_2} \cdots D_p^{k_p} f$ denotes the partial derivative $\frac{\partial^{k_1+k_2+\cdots+k_p} f(x_1,x_2,\cdots,x_p)}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_p^{k_p}}$.

2. C-MODEL

Consider the non-linear extension of the *C-Model* [20],

$$\begin{cases} x(t+1) = f\left(\varepsilon x(t), y(t), \varepsilon, t\right), \\ y(t+1) = g\left(\varepsilon x(t), y(t), \varepsilon, t\right), \end{cases} \quad t \in I_{N-1}. \tag{5}$$

subject to the boundary values (2). The reduced problem is obtained by setting the small parameter to zero:

$$\begin{cases} x(t+1) = f\left(0, y(t), 0, t\right), \\ y(t+1) = g\left(0, y(t), 0, t\right), \end{cases} \quad t \in I_{N-1}, \tag{6}$$

$$x(0) = \alpha(0), \qquad y(N) = \beta(0). \tag{7}$$

We notice that in (6), the boundary layer occurs at the initial value; we solve (6)–(7) using only the final value thus we need the following hypothesis.
H1 Suppose $D_2 g\left(0, y(t), 0, t\right) \neq 0,\ 0 \leq t \leq N-1,\ \forall (x,y) \in X \times Y$.

Proposition 1. *If H1 holds, then problem* (6)−(7) *has a unique solution.*

BVP (5) − (2) is transformable into a system of equations depending on a parameter: $\mathcal{G}\left(\varepsilon, \chi\right) = 0$ if we use the notations,

$$\chi = \left(x(0), y(0), \ldots, x(N), y(N)\right), \ \mathcal{G} : (-1,1) \times X^{2N+2} \longrightarrow X^{2N+2},$$

$$\mathcal{G}\left(\varepsilon, \chi\right) = \left(f_0\left(\varepsilon, \chi\right), g_0\left(\varepsilon, \chi\right), \ldots, f_N\left(\varepsilon, \chi\right), g_N\left(\varepsilon, \chi\right)\right),$$

$$\begin{cases} f_0\left(\varepsilon, \chi\right) = x(0) - \alpha(\varepsilon),\ g_0\left(\varepsilon, \chi\right) = y(N) - \beta(\varepsilon) \\ f_{t+1}\left(\varepsilon, \chi\right) = x(t+1) - f\left(\varepsilon x(t), y(t), \varepsilon, t\right), \\ g_{t+1}\left(\varepsilon, \chi\right) = y(t+1) - g\left(\varepsilon x(t), y(t), \varepsilon, t\right), \quad 0 \leq t \leq N-1. \end{cases}$$

If hypothesis H1 is satisfied, we can apply the Implicit Function Theorem [1]. Thus, for a sufficiently small ε, we can find a function

$$\Phi(\varepsilon) = \left(\phi_0(\varepsilon), \psi_0(\varepsilon), \phi_1(\varepsilon), \psi_1(\varepsilon), \ldots, \phi_N(\varepsilon), \psi_N(\varepsilon)\right),$$

of class C^n, such that $\mathcal{G}\left(\varepsilon, \Phi(\varepsilon)\right) = 0$. Therefore, we have

$$\begin{cases} \phi_{t+1}(\varepsilon) = f\left(\varepsilon \phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t\right), \\ \psi_{t+1}(\varepsilon) = g\left(\varepsilon \phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t\right), \end{cases} \quad t \in I_{N-1}, \tag{8}$$

$$\phi_0(\varepsilon) = \alpha(\varepsilon), \quad \psi_0(\varepsilon) = \beta(\varepsilon). \tag{9}$$

Applying Faa di Bruno's formula [4] to (8), we find the following lemma; for a brief writing, we omit the arguments for f and g.

Lemma 2. *Suppose that ϕ and ψ satisfy (8), and that all necessary derivatives are defined. Then we have for $n \geq 2$,*

$$\begin{gathered}\frac{d^n\phi_{t+1}(\varepsilon)}{d\varepsilon^n} - nD_1 f\frac{d^{n-1}\phi_t(\varepsilon)}{d\varepsilon^{n-1}} - D_2 f\frac{d^n\psi_t(\varepsilon)}{d\varepsilon^n} = \\ \underline{\sum_0\cdots\sum_n}\frac{n!D_1^{p_1}D_2^{p_2}D_3^{p_3}f\times\prod_{i=1}^n(i\frac{d^{i-1}\phi_t(\varepsilon)}{d\varepsilon^{i-1}}+\varepsilon\frac{d^i\phi_t(\varepsilon)}{d\varepsilon^i})^{q_{i1}}(\frac{d^i\psi_t(\varepsilon)}{d\varepsilon^i})^{q_{i2}}(\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^n(i!)^{k_i}\prod_{i=1}^n\prod_{j=1}^3 q_{ij}!},\end{gathered} \tag{10}$$

$$\begin{gathered}\frac{d^n\psi_{t+1}(\varepsilon)}{d\varepsilon^n} - nD_1 g\frac{d^{n-1}\phi_t(\varepsilon)}{d\varepsilon^{n-1}} - D_2 g\frac{d^n\psi_t(\varepsilon)}{d\varepsilon^n} = \\ \underline{\sum_0\cdots\sum_n}\frac{n!D_1^{p_1}D_2^{p_2}D_3^{p_3}g\times\prod_{i=1}^n(i\frac{d^{i-1}\phi_t(\varepsilon)}{d\varepsilon^{i-1}}+\varepsilon\frac{d^i\phi_t(\varepsilon)}{d\varepsilon^i})^{q_{i1}}(\frac{d^i\psi_t(\varepsilon)}{d\varepsilon^i})^{q_{i2}}(\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^n(i!)^{k_i}\prod_{i=1}^n\prod_{j=1}^3 q_{ij}!},\end{gathered} \tag{11}$$

where the coefficients k_i, q_{ij} and p_j, $i = 0, \cdots, n$, $j = 1, 2, 3$, are all nonnegative integer solutions of the Diophantine equations

$$\begin{gathered}\sum_0 \to k_1 + 2k_2 + \cdots + nk_n = n, \\ \sum_i \to q_{i1} + q_{i2} + q_{i3} = k_i, \quad i = 1, 2, \cdots, n, \\ p_j = q_{1j} + q_{2j} + \cdots + q_{nj}, \quad j = 1, 2, 3, \\ k = p_1 + p_2 + p_3 = k_1 + k_2 + \cdots + k_n,\end{gathered} \tag{12}$$

in $\underline{\sum_0\cdots\sum_n}$ we fix $k_n = 0$; the case $k_n = 1$ is removed to the left side,

$$\left(\varepsilon^{(i)}\right)^{q_{i3}} := \begin{cases} 1, & i = 1 \quad \vee \quad q_{i3} = 0, \\ 0, & i \geq 2 \quad \wedge \quad q_{i3} \neq 0. \end{cases} \tag{13}$$

Proof. Notice that for $i \geq 1$,

$$\frac{d^i\left(\varepsilon\phi_t(\varepsilon)\right)}{d\varepsilon^i} = i\frac{d^{i-1}\phi_t(\varepsilon)}{d\varepsilon^{i-1}} + \varepsilon\frac{d^i\phi_t(\varepsilon)}{d\varepsilon^i}, \tag{14}$$

$$\frac{d^i\left(\varepsilon\psi_t(\varepsilon)\right)}{d\varepsilon^i} = i\frac{d^{i-1}\psi_t(\varepsilon)}{d\varepsilon^{i-1}} + \varepsilon\frac{d^i\psi_t(\varepsilon)}{d\varepsilon^i}, \tag{15}$$

we expand Faa Di Bruno's Formula into (8) and we shift the terms that correspond to $k_n = 1$ on the left hand side. □

2.1. ASYMPTOTIC SOLUTIONS

In accordance with the Maclaurin polynomial expansions, we have

$$\phi_t(\varepsilon) = \sum_{j=0}^{n} \frac{\varepsilon^j}{j!}\frac{d^j\phi_t}{d\varepsilon^j}(0) + \mathcal{O}(\varepsilon^{n+1}),\ \psi_t(\varepsilon) = \sum_{j=0}^{n} \frac{\varepsilon^j}{j!}\frac{d^j\psi_t}{d\varepsilon^j}(0) + \mathcal{O}(\varepsilon^{n+1}). \quad (16)$$

An iterative process giving the coefficients of (4) is found when we substitute for $0 \le t \le N$, $l = 0, 1$, into (10) and (11) by

$$x^{(i)}(t+l) = \frac{1}{i!}\frac{d^i\phi_{t+l}}{d\varepsilon^i}(0), \quad y^{(i)}(t+l) = \frac{1}{i!}\frac{d^i\psi_{t+l}}{d\varepsilon^i}(0). \quad (17)$$

The *zero* order approximation coefficients are the solution sequence of the *reduced problem* (6)−(7); using the final value

$$y^{(0)}(N) = \beta(0) = \beta^{(0)}, \quad (18)$$

hypothesis **H1** allows to find from the recurrence

$$y^{(0)}(t+1) = g\left(0), y^{(0)}(t), 0, t\right), \quad (19)$$

the values $y^{(0)}(t)$, $t = 0, \cdots, N-1$, then in a next step we can set

$$x^{(0)}(t+1) = f\left(0, y^{(0)}(t), 0, t\right), \quad t = 0, \cdots, N-1, \quad (20)$$

whereas

$$x^{(0)}(0) = \alpha(0) = \alpha^{(0)}. \quad (21)$$

For 1^{st} order approximation, using the final value

$$y^{(1)}(N) = \beta^{(1)}, \quad (22)$$

hypothesis H1 allows computing $y^{(1)}(t)$, $t \in I_{N-1}$, from the formula

$$y^{(1)}(t) = [D_2 g_t]^{-1}[y^{(1)}(t+1) - D_1 g_t x^{(0)}(t) - D_3 g_t]; \quad (23)$$

then $x^{(1)}(t)$, $t \in I_{N-1}$, can be deduced from

$$x^{(1)}(t+1) = D_1 f_t x^{(0)}(t) + D_2 f_t y^{(1)}(t) + D_3 f_t,\ t \in I_{N-1}, \quad (24)$$

$$x^{(1)}(0) = \alpha^{(1)}, \quad (25)$$

where

$$f_t := f(0, y^{(0)}(t), 0, t), \quad g_t := g\left(0, y^{(0)}(t), 0, t\right). \tag{26}$$

The same calculation technique is used for higher order coefficients. For 2^{nd} order development, with the final value

$$y^{(2)}(N) = \beta^{(2)}, \tag{27}$$

we can calculate

$$\begin{array}{c} y^{(2)}(t) = [D_2 g_t]^{-1}[y^{(2)}(t+1) - D_1 g_t x^{(1)}(t) - D_2 D_3 g_t y^{(1)}(t) \\ -D_1 D_2 g_t x^{(0)}(t) y^{(1)}(t) - D_1 D_3 g_t x^{(0)}(t) - \frac{1}{2!} D_1^2 g_t \left(y^{(1)}(t)\right)^2 \\ -\frac{1}{2!} D_3^2 g_t - \frac{1}{2!} D_1^2 g_t \left(x^{(0)}(t)\right)^2], \end{array} \tag{28}$$

after that, it is possible to compute for $t \in I_{N-1}$,

$$\begin{array}{c} x^{(2)}(t+1) = D_2 g_t y^{(2)}(t) + D_1 g_t x^{(1)}(t) + \frac{1}{2!} D_1^2 f_t \left(x^{(0)}(t)\right)^2 \\ +\frac{1}{2!} D_3^2 f_t + \frac{1}{2!} D_1^2 f_t \left(y^{(1)}(t)\right)^2 + D_1 D_3 f_t x^{(0)}(t) \\ +D_1 D_2 f_t x^{(0)}(t) y^{(1)}(t) + D_2 D_3 f_t y^{(1)}(t), \end{array} \tag{29}$$

whereas

$$x^{(2)}(0) = \alpha^{(2)}. \tag{30}$$

In general, to find the n order coefficients: set

$$y^{(n)}(N) = \beta^{(n)}, \tag{31}$$

then compute

$$\begin{array}{c} y^{(n)}(t) = [D_2 g_t]^{-1}[y^{(n)}(t+1) - D_1 g_t x^{(n-1)}(t) \\ -\underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p1} D_2^{p2} D_3^{p3} g_t \times \prod_{i=1}^n (x^{(i-1)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\delta_i)^{q_{i3}}}{\prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}]. \end{array} \tag{32}$$

Next, find for $t \in I_{N-1}$,

$$\begin{array}{c} x^{(n)}(t+1) = D_1 f_t x^{(n-1)}(t) + D_2 f_t y^{(n)}(t) \\ +\underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p1} D_2^{p2} D_3^{p3} f_t \times \prod_{i=1}^n (x^{(i-1)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}, \end{array} \tag{33}$$

while

$$x^{(n)}(0) = \alpha^{(n)}. \tag{34}$$

The effectiveness of the suggested algorithm is illustrated in the following theorem.

Theorem 3. *If H1 holds, there exists* $\epsilon > 0$*, such that for all* $|\varepsilon| < \epsilon$*, BVP* (5)−(2) *has a unique solution satisfying* (4)*; the coefficients* $x^{(0)}(t)$*,* $y^{(0)}(t)$*,* $x^{(1)}(t)$*,* $y^{(1)}(t)$*,* $x^{(2)}(t)$*,* $y^{(2)}(t)$*,* $x^{(n)}(t)$*,* $y^{(n)}(t)$*, are found according to the orderly process* (18)−(19)*,* (20)−(21)*,* (22)−(23)*,* (24)−(25)*,* (27)−(28)*,* (29)−(30)*,* (31)−(32)*,* (33)−(34)*.*

Proof. Let $\tilde{\chi} = (\varepsilon, \chi)$, $|\varepsilon| \leq \delta < 1$, and $\mathrm{G}(\tilde{\chi}) = (\varepsilon, \mathcal{G}(\tilde{\chi}))$ where $\mathcal{D}\mathrm{G}$ denotes its jacobian matrix. Hypothesis H1 guarantees that $\mathcal{D}\mathrm{G}$ is invertible at

$$\tilde{\chi}^{(0)} = \left(0, x^{(0)}(0), y^{(0)}(0), x^{(0)}(1), y^{(0)}(1), \ldots, x^{(0)}(N), y^{(0)}(N)\right),$$

since

$$\det \mathcal{D}\mathrm{G}\left(\tilde{\chi}^{(0)}\right) = \prod_{t=0}^{N-1} \mathcal{D}_2 g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right) \neq 0,$$

which testifies that the *reduced problem* (6)−(7) has a unique solution. We can choose $\xi > 0$ such that, if $\|\tilde{\chi} - \tilde{\chi}^{(0)}\| < \xi$, then

$$\|\mathcal{D}\mathrm{G}(\tilde{\chi}) - \mathcal{D}\mathrm{G}\left(\tilde{\chi}^{(0)}\right)\| < \frac{1}{2}\|\left(\mathcal{D}\mathrm{G}\left(\tilde{\chi}^{(0)}\right)\right)^{-1}\|^{-1}, \tag{35}$$

since $\mathcal{D}\mathrm{G}$ is continuous. Let $\epsilon = \frac{\xi}{2}\|\left(\mathcal{D}\mathrm{G}\left(\tilde{\chi}^{(0)}\right)\right)^{-1}\|^{-1}$, the mapping

$$\Theta_\tau(\tilde{\chi}) = \tilde{\chi} - \left(\mathcal{D}\mathrm{G}\left(\tilde{\chi}^{(0)}\right)\right)^{-1}(\mathrm{G}(\tilde{\chi}) - \tau)$$

is a contraction that maps $B\left(\tilde{\chi}^{(0)}, \xi\right)$ to itself when $|\varepsilon| < \epsilon$ and $\|\tau\| < \epsilon$. Therefore Θ_τ has a unique fixed point $\tilde{\chi}$. As a consequence, for τ fixed, $\|\tau\| < \epsilon$, there exists a unique $\tilde{\chi}$ such that $\tau = \mathrm{G}(\tilde{\chi})$ and $\|\tilde{\chi} - \tilde{\chi}^{(0)}\| < \xi$, i.e., G is 1-to-1 from $\mathrm{G}^{-1}(B(0,\epsilon))$ into $B(0,\epsilon)$. Definitely we have $(\varepsilon, 0, \cdots, 0) \in B(0,\epsilon)$ if $|\varepsilon| < \epsilon$; there exists a unique $(\varepsilon, \Phi(\varepsilon))$ in $B\left(\tilde{\chi}^{(0)}, \xi\right) / (\varepsilon, 0, \cdots, 0) = \mathrm{G}(\varepsilon, \Phi(\varepsilon))$, where $\Phi(\varepsilon) = (\phi_0(\varepsilon), \psi_0(\varepsilon), \cdots, \phi_N(\varepsilon), \psi_N(\varepsilon))$. We confirmed that for $|\varepsilon| < \epsilon$, there exists a unique $\phi(\varepsilon)$ which verifies $\mathcal{G}(\varepsilon, \Phi(\varepsilon)) = 0$, thus BVP (1)−(2) has a unique solution. Moreover, the function Φ is n-continuously differentiable over $(-\epsilon, \epsilon)$, just as G and G^{-1}; its derivatives are fixed in Lemma 2. The proof is completed. □

If the functionals f and g are *smooth* and the asymptotic expansions for the boundary conditions are convergent, the problems described above are defined for any order n.

H2 Suppose that $\|\alpha_k^{(i)}\| \leq \frac{A}{\delta^i}$, $\|\beta^{(i)}\| \leq \frac{B}{\delta^i}$, A and B are constants.

Theorem 4. *If assumptions H1 and H2 hold, f and g are smooth functions, then there exists $\epsilon > 0$, for all $|\varepsilon| < \epsilon$, BVP (1)–(2) has a unique solution satisfying*

$$x(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n x^{(n)}(t), \quad y(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n y^{(n)}(t),$$

where $x^{(0)}(t)$, $y^{(0)}(t)$, $x^{(1)}(t)$, $y^{(1)}(t)$, $x^{(2)}(t)$, $y^{(2)}(t)$, $x^{(n)}(t)$, $y^{(n)}(t)$, are found according to the orderly process (18)–(19), (20)–(21), (22)–(23), (24)–(25), (27)–(28), (29)–(30), (31)–(32), (33)–(34).

3. R-MODEL

This section is devoted to solving the problem

$$\begin{cases} x(t+1) = f\left(\varepsilon x(t), \varepsilon y(t), \varepsilon, t\right), \\ y(t+1) = g\left(x(t), y(t), \varepsilon, t\right), \end{cases} \quad t \in I_{N-1}, \tag{36}$$

to which boundary conditions (2) are attributed. We employ parameter elimination to obtain the reduced system

$$\begin{cases} x^{(0)}(t+1) = f\left(0, 0, 0, t\right), \\ y^{(0)}(t+1) = g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right), \end{cases} \quad t \in I_{N-1}, \tag{37}$$

$$x^{(0)}(0) = \alpha(0), \tag{38}$$

$$y^{(0)}(N) = \beta(0). \tag{39}$$

We notice that equation

$$x^{(0)}(t+1) = f\left(0, 0, 0, t\right), \quad t \in I_{N-1} \tag{40}$$

simply defines the values $x^{(0)}(1)$, …, $x^{(0)}(N)$, and the boundary layer is located at the initial value (38). In next step, it may be possible to solve backward the difference equation

$$y^{(0)}(t+1) = g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right), \quad t \in I_{N-1} \tag{41}$$

with the final value (39), if we assume the following hypothesis.
H3 $D_2 g\left(x(t), y(t), 0, t\right) \neq 0$, $0 \leq t \leq N-1$, $\forall (x,y) \in X \times Y$.

Proposition 5. *If H3 holds, then problem* (37)−(38)−(39) *has a unique solution.*

Using the same notations as in Section 2, we can find for a sufficiently small ε, functions $\phi_0(\varepsilon), \psi_0(\varepsilon), \ldots, \phi_N(\varepsilon), \psi_N(\varepsilon)$, of class C^n, such that

$$\begin{cases} \phi_{t+1}(\varepsilon) = f\left(\varepsilon\phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t\right), \\ \psi_{t+1}(\varepsilon) = g\left(\varepsilon\phi_t(\varepsilon), \psi_t(\varepsilon), \varepsilon, t\right), \end{cases} \quad t \in I_{N-1}, \tag{42}$$

$$\phi_0(\varepsilon) = \alpha(\varepsilon), \quad \psi_0(\varepsilon) = \beta(\varepsilon). \tag{43}$$

Lemma 6. *Suppose that ϕ and ψ satisfy* (42)*, and that all necessary derivatives are defined. Then we have for $n \geq 2$,*

$$\begin{gathered} \frac{d^n\phi_{t+1}(\varepsilon)}{d\varepsilon^n} - nD_1 f \frac{d^{n-1}\phi_t(\varepsilon)}{d\varepsilon^{n-1}} - nD_2 f \frac{d^{n-1}\psi_t(\varepsilon)}{d\varepsilon^{n-1}} = \\ \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{n! D_1^{p1} D_2^{p2} D_3^{p3} f \times \prod_{i=1}^n \left(i \frac{d^{i-1}\phi_t(\varepsilon)}{d\varepsilon^{i-1}} + \varepsilon \frac{d^i\phi_t(\varepsilon)}{d\varepsilon^i}\right)^{q_{i1}}}{\prod_{i=1}^n (i!)^{k_i} \prod_{i=1}^n \prod_{j=1}^3 q_{ij}!} \\ \times \left(i \frac{d^{i-1}\psi_t(\varepsilon)}{d\varepsilon^{i-1}} + \varepsilon \frac{d^i\psi_t(\varepsilon)}{d\varepsilon^i}\right)^{q_{i2}} (\varepsilon^{(i)})^{q_{i3}}, \end{gathered} \tag{44}$$

$$\begin{gathered} \frac{d^n\psi_{t+1}(\varepsilon)}{d\varepsilon^n} - D_1 g \frac{d^n\phi_t(\varepsilon)}{d\varepsilon^n} - D_2 g \frac{d^n\psi_t(\varepsilon)}{d\varepsilon^n} = \\ \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{n! D_1^{p1} D_2^{p2} D_3^{p3} g \times \prod_{i=1}^n \left(\frac{d^i\psi_t(\varepsilon)}{d\varepsilon^i}\right)^{q_{i1}} \left(\frac{d^i\phi_t(\varepsilon)}{d\varepsilon^i}\right)^{q_{i2}} (\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^n (i!)^{k_i} \prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}, \end{gathered} \tag{45}$$

where the coefficients k_i, q_{ij} and p_j, $i = 0, \cdots, n$, $j = 1, 2, 3$, are all nonnegative integer solutions of the Diophantine equations (12)*, and* (13)*.*

3.1. ASYMPTOTIC SOLUTIONS

The *zero* order approximation coefficients being the solution sequence of the *reduced problem* (37)−(38)−(39), we can determine the coefficients of higher order by using Lemma 6 and substitution (17). Then, we can immediately deduce the process for solving BVP (36)−(2). For 1^{st} order coefficients, while

$$x^{(1)}(0) = \alpha^{(1)} \tag{46}$$

is fixed, first we determine

$$x^{(1)}(t+1) = D_1 f_t x^{(0)}(t) + D_2 f_t y^{(0)}(t) + D_3 f_t, \ t \in I_{N-1}, \tag{47}$$

then after we can iterate backward

$$y^{(1)}(t) = [D_2 g_t]^{-1} [y^{(1)}(t+1) - D_1 g_t x^{(1)}(t) - D_3 g_t], \ t \in I_{N-1}, \tag{48}$$

starting from

$$y^{(1)}(N) = \beta^{(1)}, \tag{49}$$

where

$$f_t := f(0,0,0,t), \quad g_t := g\left(x^{(0)}(t), y^{(0)}(t), 0, t\right). \tag{50}$$

For 2^{nd} order coefficients, we follow the following process. 1st fix

$$x^{(2)}(0) = \alpha^{(2)} \tag{51}$$

then compute $x^{(2)}(1), \cdots, x^{(2)}(N)$, from the formula

$$\begin{gathered} x^{(2)}(t+1) = D_1 f_t x^{(1)}(t) + D_2 f_t y^{(1)}(t) + \\ \tfrac{1}{2!} D_1^2 f_t \left(x^{(0)}(t)\right)^2 + \tfrac{1}{2!} D_3^2 f_t + \tfrac{1}{2!} D_2^2 f_t \left(y^{(0)}(t)\right)^2 \\ + D_1 D_2 f_t x^{(0)}(t) y^{(0)}(t) + D_1 D_3 f_t x^{(0)}(t) + D_2 D_3 f_t y^{(0)}(t), \end{gathered} \tag{52}$$

then after we can iterate backward

$$\begin{gathered} y^{(2)}(t+1) = D_1 g_t x^{(2)}(t) + D_2 g_t y^{(2)}(t) \\ + \tfrac{1}{2!} D_1^2 g_t \left(x^{(1)}(t)\right)^2 + \tfrac{1}{2!} D_2^2 g_t \left(y^{(1)}(t)\right)^2 + \tfrac{1}{2!} D_3^2 g_t \\ + D_1 D_2 f_t x^{(1)}(t) y^{(1)}(t) + D_1 D_3 g_t x^{(1)}(t) + D_2 D_3 g_t y^{(1)}(t), \end{gathered} \tag{53}$$

departing from

$$y^{(2)}(N) = \beta^{(2)}. \tag{54}$$

For any n order approximation, $n \geq 1$, we have the ordred process:

$$x^{(n)}(0) = \alpha^{(n)}, \tag{55}$$

$$\begin{gathered} x^{(n)}(t+1) = D_1 f_t x^{(n-1)}(t) + D_2 f_t y^{(n-1)}(t) \\ + \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p1} D_2^{p2} D_3^{p3} f_t \times \prod_{i=1}^n (x^{(i-1)}(t))^{q_{i1}} (y^{(i-1)}(t))^{q_{i2}} (\varepsilon^{(i)})^{q_{i3}}}{\prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}, \end{gathered} \tag{56}$$

which incorporate terms of order less than n, then we can compute

$$\begin{gathered} y^{(n)}(t) = [D_2 g_t]^{-1} [y^{(n)}(t+1) - D_1 g_t x^{(n)}(t) \\ - \underline{\sum_0 \sum_1 \cdots \sum_n} \frac{D_1^{p1} D_2^{p2} D_3^{p3} g_t \times \prod_{i=1}^n (x^{(i)}(t))^{q_{i1}} (y^{(i)}(t))^{q_{i2}} (\delta_i)^{q_{i3}}}{\prod_{i=1}^n \prod_{j=1}^3 q_{ij}!}], \end{gathered} \tag{57}$$

and iterate backward with the final value

$$y^{(n)}(N) = \beta^{(n)}. \tag{58}$$

Theorem 7. *If H3 holds,* $\exists\epsilon > 0$*, such that* $\forall|\varepsilon| < \epsilon$*, BVP* (36)−(2) *has a unique solution satisfying* (4)*; the coefficients* $x^{(0)}(t)$, $y^{(0)}(t)$, $x^{(1)}(t)$, $y^{(1)}(t)$, $x^{(2)}(t)$, $y^{(2)}(t)$, $x^{(n)}(t)$, $y^{(n)}(t)$, *are found according to the orderly process* (38)−(40), (41)−(39), (46)−(47), (48)−(49), (51)−(52), (53)−(54), (55)−(56), (57)−(58).

Theorem 8. *If assumptions H2 and H3 hold, f and g are smooth functions, then* $\exists\epsilon > 0$*, such that* $\forall|\varepsilon| < \epsilon$*, BVP* (36)−(2) *has a unique solution which satisfy*

$$x(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n x^{(n)}(t), \quad y(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n y^{(n)}(t),$$

where the coefficients $x^{(0)}(t)$, $y^{(0)}(t)$, $x^{(n)}(t)$, $y^{(n)}(t)$, *are found according to the orderly process* (38)−(40), (41)−(39), (46)−(47), (48)−(49), (51)−(52), (53)−(54), (55)−(56), (57)−(58).

4. CONCLUSION

BVPs for non-linear extensions of *C-model* and *R-model* of two-time-scales discrete-time systems are studied by a perturbation technique. Iterative algorithms are described to find asymptotic solutions at any order. This paper provides a generalization for the results of [12].

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