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A NOTE ON THE THREE–PARAMETER "GENERALIZED ODD LOG–LOGISTIC EXPONENTIAL (GOLLEX) C.D.F." SOME EXTENSIONS

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ABSTRACT: In [2] the authors propose a new three–parameter "Generalized Odd Log–Logistic Exponential (GOLLEx) distribution based on a previous excellent result by Cordeiro et al. [1].

The authors' assertion that probability distribution produces very good results encouraged us to conduct further studies on "saturation" in Hausdorff sense of the corresponding commutative function to the horizontal asymptote hoping to partially contribute to uncovering some of the "intrinsic properties" of this apparently good model.

We define a new four parameter cdf of transmuted generalized odd log–logistic exponential (TGOLLEx) family.

We construct a family of recurrence generated cdf of the transmuted–transmuted generalized odd log–logistic exponential (TTGOLLEx) family.

We also analyze some experimental data: "actual data to estimate the number of software residual faults"; "TROPICO R Failure Data"; '*data_Jurkat_T_cell_human_leukemia*"; "*data_Storm_IDs*".

Numerical examples, illustrating our results are presented using programming environment *CAS Mathematica*.

AMS Subject Classification: 41A46
Key Words: c.d.f. of the generalized odd log–logistic exponential (GOLLEx) distribution, new four parameter cdf of transmuted generalized odd log–logistic

exponential (TGOLLEx) family, family of recurrence generated cdf of the transmuted–transmuted generalized odd log–logistic exponential (TTGOLLEx), Heaviside step–function $h_{t_0}(t)$, Hausdorff distance

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1. INTRODUCTION AND PRELIMINARIES

Definition 1. *Cordeiro et al. [1] proposed a new class of distributions called the generalized odd log–logistic –G (GOLL–G) family with two shape parameters. The cdf is defined by:*

$$M(t) = \frac{G(t)^{\alpha\theta}}{G(t)^{\alpha\theta} + \left(1 - G(t)^{\theta}\right)^{\alpha}} \tag{1}$$

where α, $\theta \in R^{+}$ *and* $G(t)$ *is a baseline cdf.*

Definition 2. *In [2] the authors introduced a new three parameter distribution called generalized odd log–logistic exponential (GOLLEx) distribution with cdf:*

$$M_1(t) = \frac{\left(1 - e^{-\lambda t}\right)^{\alpha\theta}}{\left(1 - e^{-\lambda t}\right)^{\alpha\theta} + \left(1 - (1 - e^{-\lambda t})^{\theta}\right)^{\alpha}} \tag{2}$$

where α, θ, $\lambda \in R^{+}$, $t > 0$.

Definition 3. *The shifted Heaviside step function is defined by*

$$h_{t_0}(t) = \begin{cases} 0, & if \quad t < t_0, \\ [0,1], & if \quad t = t_0, \\ 1, & if \quad t > t_0 \end{cases} \tag{3}$$

Definition 4. *[3] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B|| = max(|t_A-t_B|, |x_A-x_B|)$.

For other results, see [9]–[10].

In this paper we study the Hausdorff approximation of the Heaviside function $h_{t_0}(t)$ by the cdf of the (GOLLEx) family (2).

We also analyze a new four parameter cdf of transmuted generalized odd log–logistic exponential (TGOLLEx) family.

The model have been tested with real-world data.

Definition 5. *We define the following new four parameter cdf of transmuted generalized odd log–logistic exponential (TGOLLEx) family:*

$$M_2(t) = M_1(t)\,(\mu + 1 - \mu M_1(t)) \tag{4}$$

where $\mu \leq 1$.

In this note some comparisons between $M_1(t)$ and $M_2(t)$ are made.

In Section 2.4 we define a new family of recurrence generated cdf's.

2. MAIN RESULTS

2.1. A NOTE ON THE CDF OF (GOLLEX)

The investigation of the characteristic "supersaturation" of the cdf (2) to the horizontal asymptote is important.

The quantile function is defined by [2]:

$$Q(u) = -\frac{1}{\lambda}\ln\left(1 - \left(1 + \left(\frac{1-u}{u}\right)^{\frac{1}{\alpha}}\right)^{-\frac{1}{\theta}}\right) \tag{5}$$

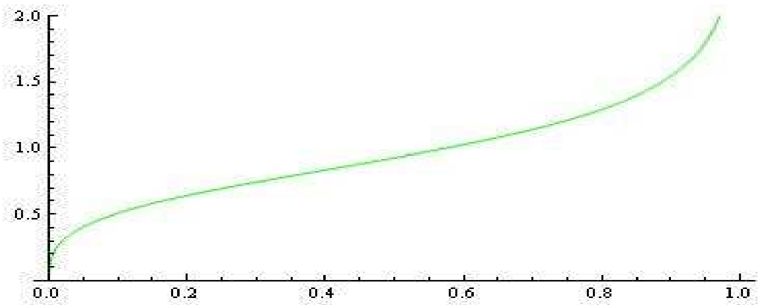

Figure 1: The function $Q(u)$ for $\alpha = 3.6$, $\theta = 0.7$, $\lambda = 0.5$.

(see, Fig. 1 for $\alpha = 3.6$, $\theta = 0.7$, $\lambda = 0.5$).

The median is obtained by substituting $u = 0.5$ in (5).

Let t_0 is the value for which $M_1(t_0) = \frac{1}{2}$, i.e. $t_0 = u(0.5)$.

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the cdf (2) satisfies the relation

$$M_1(t_0 + d) = 1 - d. \tag{6}$$

For given θ, α, λ, the nonlinear equation $M_1(t_0 + d) - 1 + d = 0$ has unique positive root – d.

The cdf (2) for $\lambda = 2.2$, $\theta = 0.8$, $\alpha = 1.99$ and $t_0 = 0.247955$ is visualized on Fig. 2.

From the nonlinear equation (6) we have: $d = 0.185681$.

The cdf (2) for $\lambda = 2.9$, $\theta = 0.3$, $\alpha = 2.3$ and $t_0 = 0.0360296$ is visualized on Fig. 3.

From the nonlinear equation (6) we have: $d = 0.103484$.

From the above examples, it can be seen that the "supersaturation" by the (cdf) $M_1(t)$ is faster.

Obviously, this "advantage" can actually be used to approximate some specific "cumulative data" from different branches of science.

In the next Section, we will support what is said by analyzing real datasets.

2.2. APPLICATIONS

Example 1. We analyze the following "actual data to estimate the number

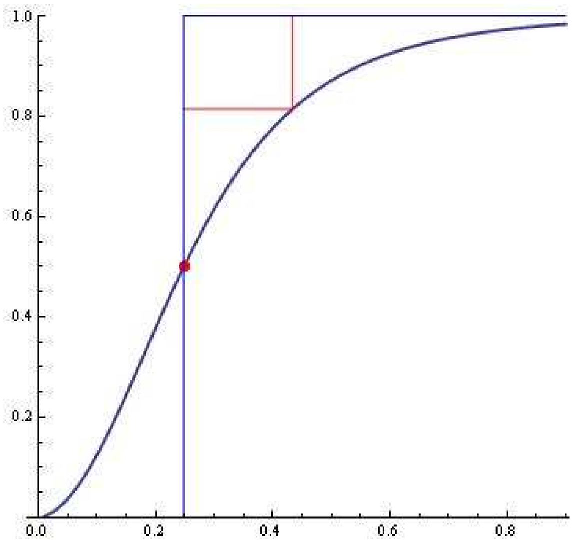

Figure 2: The cdf (2) for $\lambda = 2.2$, $\theta = 0.8$, $\alpha = 1.99$ and $t_0 = 0.247955$; H–distance $d = 0.185681$.

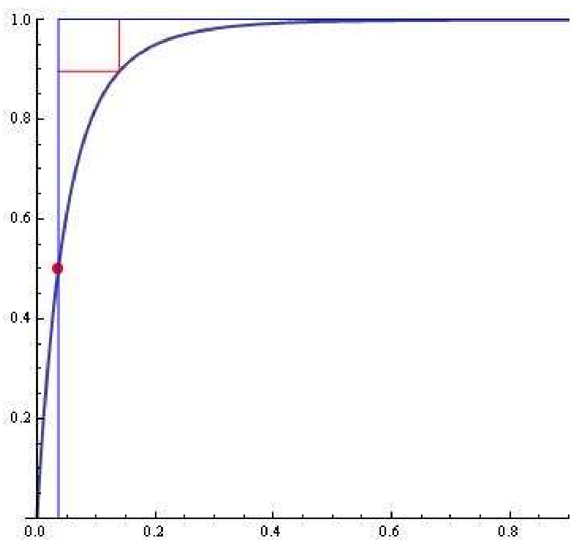

Figure 3: The cdf (2) for $\lambda = 2.9$, $\theta = 0.3$, $\alpha = 2.3$ and $t_0 = 0.0360296$; H–distance $d = 0.103484$.

Week	Cumulative number of software faults	Week	Cumulative number of software faults
1	248	31	4351
2	262	32	4401
3	372	33	4439
4	526	34	4488
5	742	35	4548
6	958	36	4596
7	1215	37	4629
8	1471	38	4680
9	1738	39	4713
10	1936	40	4749
11	1971	41	4783
12	2147	42	4817
13	2258	43	4849
14	2418	44	4877
15	2567	45	4901
16	2688	46	4928
17	2809	47	4950
18	2925	48	4970
19	3026	49	4998
20	3205	50	5024
21	3348	51	5060
22	3476	52	5085
23	3573	53	5088
24	3719	54	5090
25	3750	55	5110
26	3952	56	5129
27	4048	57	5139
28	4137	58	5167
29	4251	59	5186
30	4301		

Figure 4: the "actual data to estimate the number of software residual faults" [4]–[5].

of software residual faults" [4]–[5] (see, Fig. 4).

After that using the model

$$M_1^*(t) = \omega M_1(t)$$

for $\alpha = 0.829671$, $\theta = 2.2$, $\lambda = 0.09$ and $\omega = 5200$ we obtain the fitted model (see, Fig. 5).

Example 2. TROPICO R Failure Data

TROPICO-R software is divided into several implementation modules.

We will analyze the following real data "Cumulative number of failures (CNF) per periods of 10 days (time unit)" [6] (see Fig. 6).

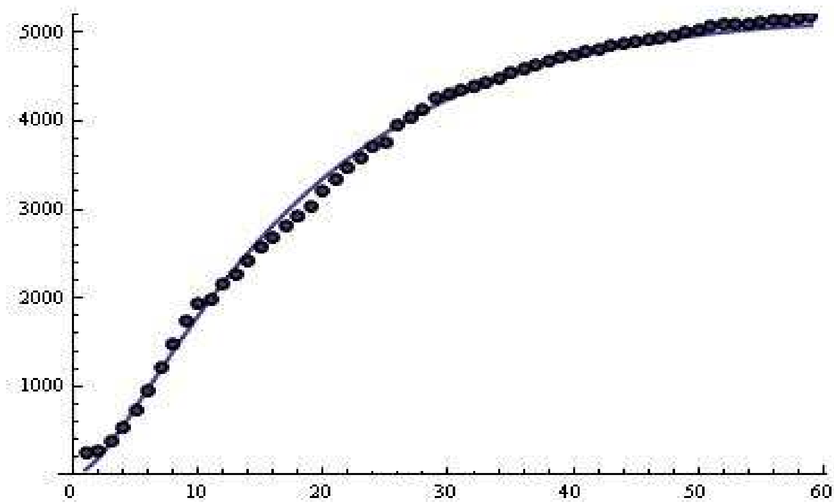

Figure 5: The fitted model $M_1^*(t)$.

The model for values of the parameters $\lambda = 0.089$, $\alpha = 0.534955$, $\theta = 4.6$, $\omega = 470$ is shown in Fig. 7.

Example 3. *data_Jurkat_T_cell_human_leukemia*

Here we will use the data of Jurkat T cell human leukemia [26], [27].

data_Jurkat_T_cell_human_leukemia

$:= \{\{1, 151786\}, \{1.5, 187500\}, \{2, 250000\}, \{2.5, 419643\}, \{3, 553571\},$

$\{3.5, 866071\}, \{4, 1357143\}, \{4.5, 1223214\}, \{5, 1767857\}, \{5.5, 1589286\},$

$\{6, 1705357\}, \{6.5, 2026786\}, \{7, 2125000\}, \{7.5, 2375000\}, \{8, 2571429\},$

$\{8.5, 2410714\}, \{9, 2446429\}, \{9.5, 2678571\}\};$

The model for values of the parameters $\lambda = 0.55$, $\alpha = 0.7$, $\theta = 8.91634$, $\omega = 2900000$ is shown in Fig. 8.

Validation		Field trial		Operation	
Time unit	CNF	Time unit	CNF	Time unit	CNF
1	7	31	301	43	356
2	8	32	302	44	367
3	36	33	310	45	373
4	45	34	317	46	373
5	60	35	319	47	378
6	74	36	323	48	381
7	82	37	324	49	383
8	98	38	338	50	384
9	106	39	342	51	384
10	115	40	345	52	387
11	120	41	350	53	387
12	134	42	352	54	387
13	139			55	388
14	142			56	393
15	145			57	398
16	153			58	400
17	157			59	407
18	174			60	413
19	183			61	414
20	196			62	417
21	200			63	419
22	214			64	420
23	223			65	429
24	246			66	440
25	257			67	443
26	277			68	448
27	283			69	454
28	286			70	456
29	292			71	456
30	297			72	456
				73	457
				74	458
				75	459
				76	459
				77	459
				78	460
				79	460
				80	460
				81	461

Figure 6: Cumulative number of failures (CNF) per periods of 10 days (time unit) [6]

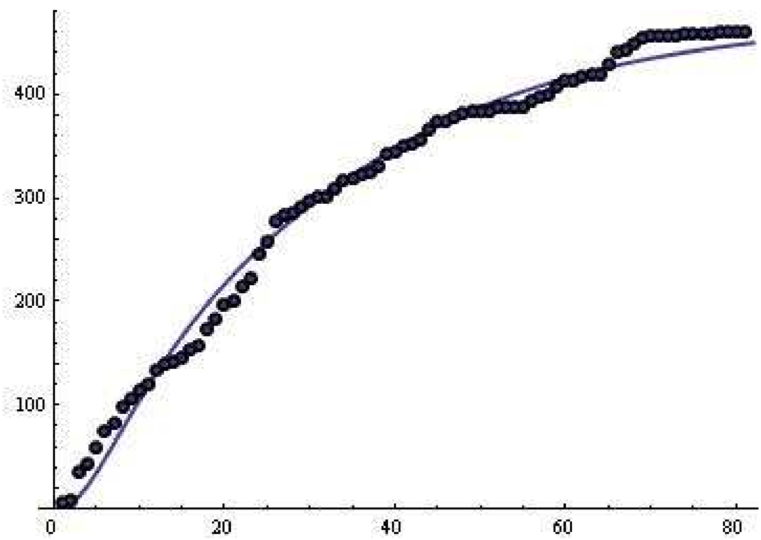

Figure 7: The fitted model $M_1^*(t)$

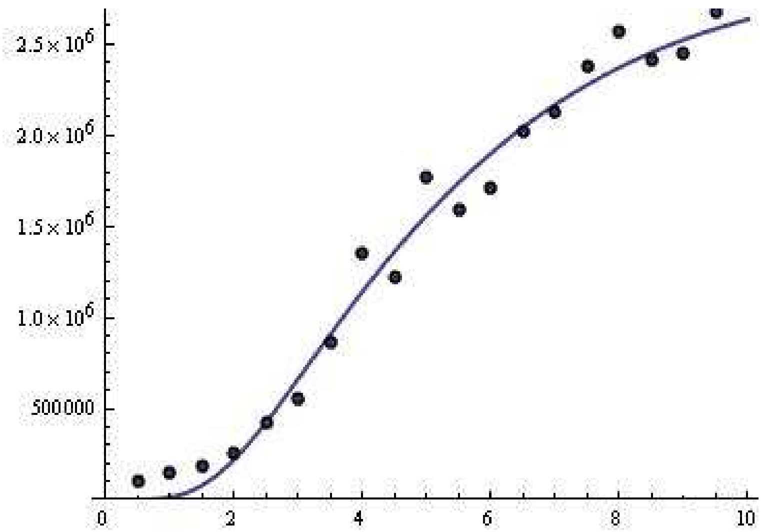

Figure 8: The fitted model $M_1^*(t)$

2.3. A NOTE ON THE NEW FOUR PARAMETER CDF OF TRANSMUTED GENERALIZED ODD LOG–LOGISTIC EXPONENTIAL (TGOLLEX) FAMILY

We consider the following new cdf of the transmuted generalized odd log–logistic exponential (TGOLLEx) family:

$$M_2(t) = M_1(t)\left(\mu + 1 - \mu M_1(t)\right)$$

where $\mu \leq 1$.

Let t_1 is the value for which $M_2(t_1) = \frac{1}{2}$.

The one–sided Hausdorff distance d_1 between the function $h_{t_1}(t)$ and the $M_2(t)$ satisfies the relation

$$M_2(t_1 + d_1) = 1 - d_1. \tag{7}$$

For example, let $\alpha = 2.3$, $\theta = 0.3$, $\lambda = 2.9$ and $\mu = 0.9$, then $t_1 = 0.0185778$ and from nonlinear equation (7) we find $d_1 = 0.0662901$ (see, Fig. 9).

A comparison between $M_1(t)$ and $M_2(t)$ for fixed $\alpha = 2.3$, $\theta = 0.3$, $\lambda = 2.9$ and $\mu = 0.9$ is shown on Fig. 10.

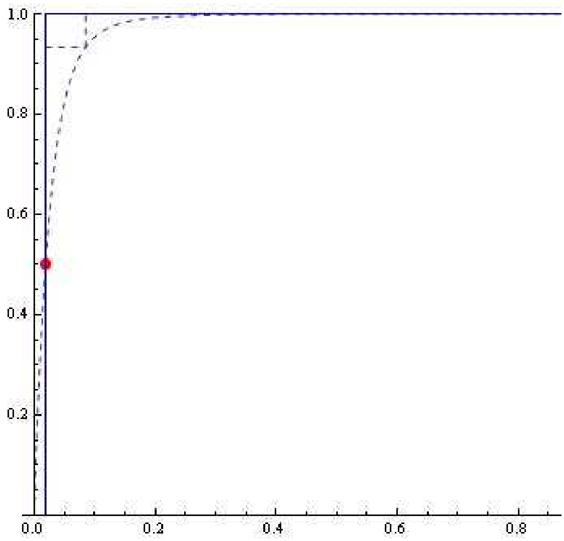

Figure 9: The cdf – $M_2(t)$ for $\alpha = 2.3$, $\theta = 0.3$, $\lambda = 2.9$, $\mu = 0.9$ and $t_1 = 0.0185778$; H–distance $d_1 = 0.0662901$.

From the Fig. 10, it can be seen that the "supersaturation" by the (cdf) $M_2(t)$ is faster.

Example 4. Storm worm one of the most biggest cyber threats of 2008.

We analyze the following data [7]

$$data\_Storm\_IDs := \{\{1, 0.843\}, \{4, 0.926\}, \{5, 0.954\}, \{6, 0.967\},$$
$$\{7, 0.976\}, \{8, 0.981\}, \{9, 0.985\}, \{10, 0.991\}, \{22, 0.995\},$$
$$\{38, 0.997\}, \{51, 0.998\}, \{64, 0.9985\}, \{74, 0.999\}, \{83, 1\}, \{100, 1\},$$
$$\{367, 1\}\}$$

The cdf $M_2(t)$ for $\alpha = 2.3$, $\theta = 0.190889$, $\lambda = 0.067$ and $\mu = 0.1$ is visualized on Fig. 11.

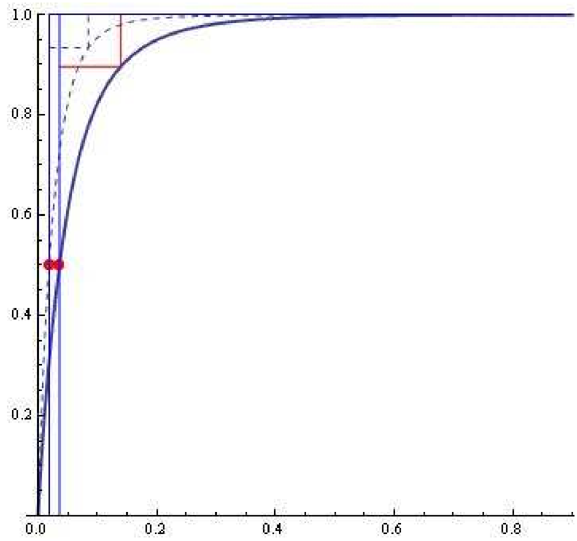

Figure 10: Comparison between $M_1(t)$ and $M_2(t)$ for fixed $\alpha = 2.3$, $\theta = 0.3$, $\lambda = 2.9$ and $\mu = 0.9$.

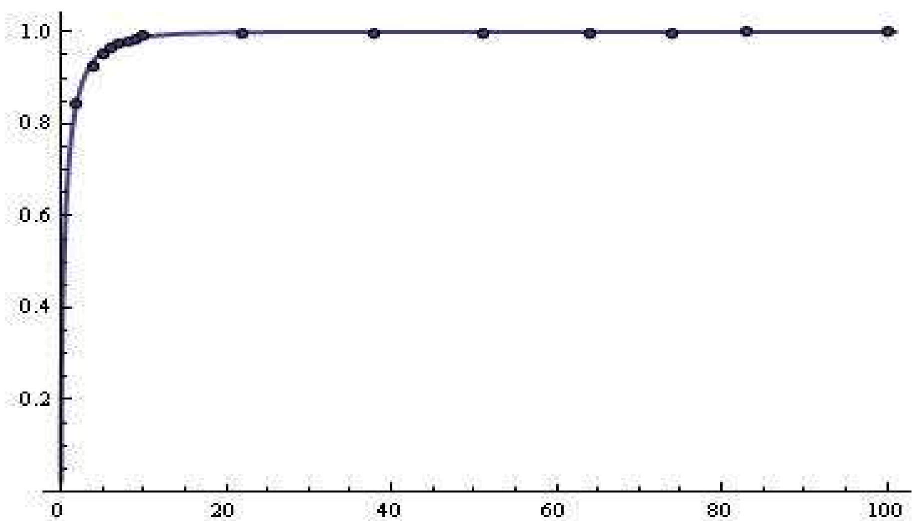

Figure 11: The fitted model $M_2(t)$.

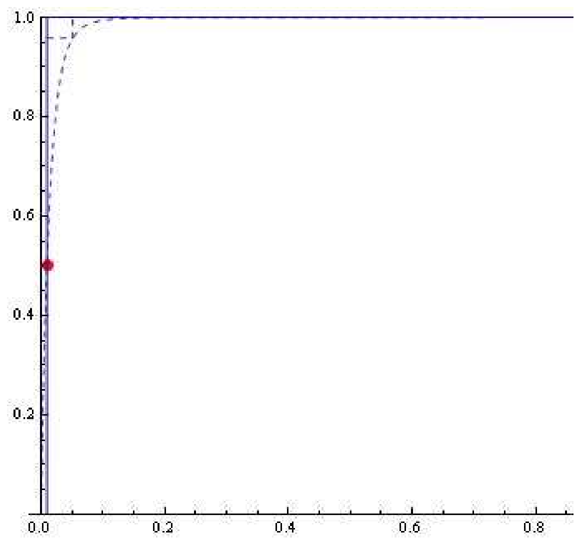

Figure 12: The cdf – $M_3(t)$ for $\alpha = 2.3$, $\theta = 0.3$, $\lambda = 2.9$, $\mu = 0.9$, $\mu_1 = 0.91$ and $t_2 = 0.00942567$; H–distance $d_2 = 0.041676$.

2.4. A NOTE ON THE NEW CDF OF TRANSMUTED–TRANSMUTED GENERALIZED ODD LOG–LOGISTIC EXPONENTIAL (TTGOLLEX) FAMILY

Following the ideas given in [24] we define the following new cdf of the transmuted–transmuted generalized odd log–logistic exponential (TTGOLLEx) family:

$$M_3(t) = M_2(t)\left(\mu_1 + 1 - \mu_1 M_2(t)\right). \tag{8}$$

where $\mu_1 \leq 1$.

For the one–sided Hausdorff distance d_2 between the function $h_{t_2}(t)$ ($M_2(t_2) = 0.5$; $\alpha = 2.3$, $\theta = 0.190889$, $\lambda = 0.067$, $\mu = 0.1$; $\mu_1 = 0.91$) and $M_3(t)$ we find $d_2 = 0.041676$ (see, Fig. 12).

Analogously, for

$$M_4(t) = M_3(t)\left(\mu_2 + 1 - \mu_2 M_3(t)\right). \tag{9}$$

where $\mu_2 \leq 1$ we have for $\alpha = 2.3$, $\theta = 0.3$, $\lambda = 2.9$, $\mu = 0.9$, $\mu_1 = 0.91$, $\mu_2 = 0.92$ and $t_3 = 0.00463245$; H–distance $d_3 = 0.0225022$.

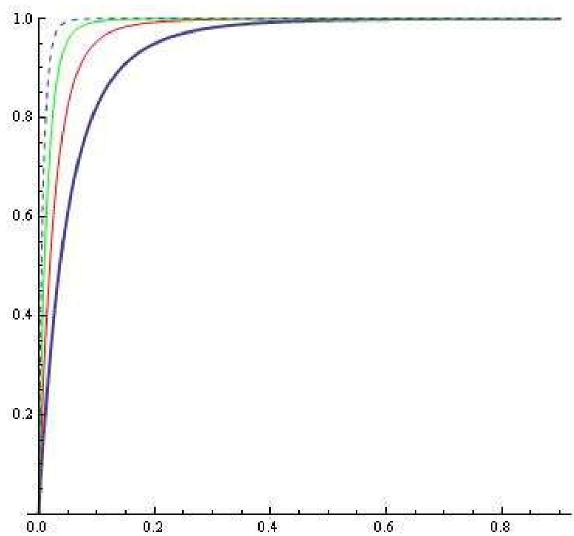

Figure 13: Comparison between $M_1(t)$–thick, $M_2(t)$–red, $M_3(t)$–green and $M_4(t)$–dashed for fixed $\alpha = 2.3$, $\theta = 0.3$, $\lambda = 2.9$ and $\mu = 0.9$, $\mu_1 = 0.91$, $\mu_2 = 0.92$.

The cumulative distribution functions: $M_1(t)$, $M_2(t)$, $M_3(t)$ and $M_4(t)$ are visualized on Fig. 13.

We define the following family of recurrence generated cdf's:

$$\begin{aligned} &M_{i+1}(t) = M_i(t)\left(\mu_{i-1} + 1 - \mu_{i-1} M_i(t)\right) \\ &i = 1, 2, \ldots,; \;\; \mu_0 = \mu \end{aligned} \tag{10}$$

based on the function $M_1(t)$.

Evidently, $\{d_i\}_1^\infty \to 0$.

Based on the methodology proposed in [25], the reader may formulate the corresponding approximation problems.

3. CONCLUDING REMARKS

Finally, we note that the studied models $M_{i+1}(t)$; $i = 1, 2, \ldots$, produces extremely good results, generally when approximating specific "cumulative data" from Computer Viruses Propagation, Debugging Theory and Population Dynamics.

A survey and new generalized classes of distributions can be found in [8].

For other approximation and modelling results, see [11]–[23].

We hope that the results will be useful for specialists in this scientific area.

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