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ON THE "SATURATION" OF THE MARSHALL–OLKIN INVERSE LOMAX CUMULATIVE FUNCTION

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ABSTRACT: In this paper we study the characteristic - "supersaturation" of the cumulative function of the Marshall–Olkin Inverse Lomax distribution to the horizontal asymptote with respect to the Hausdorff distance.

The model lead to greater flexibility in modeling of various specific data types.

Numerical examples using *CAS Mathematica* are given.

Key Words: "supersaturation" of the Marshall–Olkin inverse lomax (MOIL) model, Heaviside function, Hausdorff distance

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1. INTRODUCTION

The cumulative distribution function of the Marshall–Olkin family is given by [1]:

$$F(t) = \frac{G(t)}{\theta + (1-\theta)G(t)}; \ \ t > 0; \ \theta > 0 \tag{1}$$

where $G(t)$ is the (cdf) of a given random variable.

For instance, let $G(t)$ is the exponential distribution of the form: $G(t) = 1 - e^{-\lambda t}$; $t > 0$, $\lambda > 0$.

Then the family of distributions is given by:

$$F(t) = \frac{1 - e^{-\lambda t}}{\theta + (1-\theta)(1 - e^{-\lambda t})}. \tag{2}$$

In [5] is considered a growth function based on "amendments" of "Marshal–Olkin - type".

Some results for Marshall–Olkin, Marshall–Olkin extended generalized Lindley distribution, Marshall–Olkin logistic exponential and Marshall–Olkin alpha power family of distributions are given in [2]–[4].

Definition 1. The cumulative function of the Marshall–Olkin Inverse Lomax distribution is defined by (see, for instance [6]):

$$M(t) = \frac{1}{\theta\left(1 + \frac{\beta}{t}\right)^{\alpha} + 1 - \theta}, \tag{3}$$

where $\alpha > 0$, $\beta > 0$, $\theta > 0$.

It is important to study the characteristic - "supersaturation" of the model (3) to the horizontal asymptote with respect to the Hausdorff distance.

Definition 2. The shifted Heaviside step function is defined by

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0. \end{cases} \tag{4}$$

Definition 3. [7], [8] The Hausdorff distance (the H–distance) $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \tag{5}$$

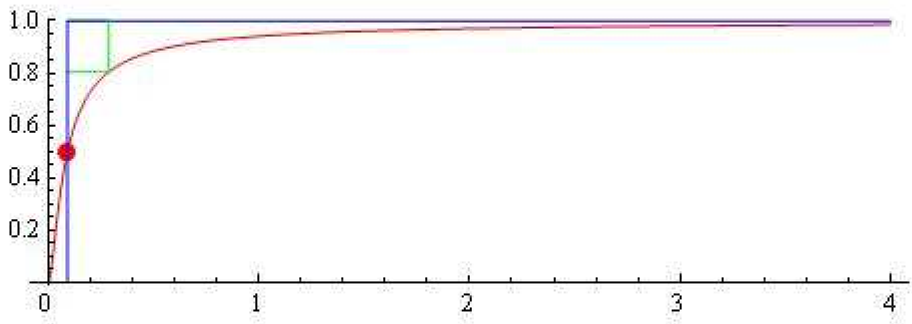


Figure 1: The $M(t)$ for $\alpha = 2$; $\beta = 0.1$; $\theta = 0.3$; $t_0 = 0.09245$; The H–distance $d = 0.196113$.

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t, x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

2. MAIN RESULTS

Let

$$t_0 = \frac{\beta}{\left(\frac{1+\theta}{\theta}\right)^{\frac{1}{\alpha}} - 1}.$$

Evidently, $M(t_0) = \frac{1}{2}$.

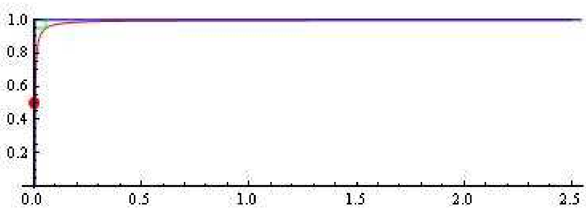
For the Hausdorff distance d between the Heaviside function h_{t_0} and $M(t)$ we have

$$M(t_0 + d) = 1 - d. \tag{6}$$

The saturation is characterized by the value d, which is calculated from the nonlinear equation (6).

Approximations of the $h_{t_0}(t)$ by model (3) for various α, β, θ are visualized on Fig. 1–Fig. 3.

```
α = 1.2
β = 0.005
θ = 0.5
F[t_] :=  1 / (θ * (1 + β / t)^{α} + 1 - θ)
g3 = Plot[F[t], {t, 0, 2.5}, PlotRange → {-0.01, 1}, AspectRatio → 0.3,
   PlotStyle → {Red}];
t0 = β / (((1 + θ) / θ)^{1 / α} - 1)
t0 = 0.00333767
F[t0] // N
d = 0.05192361
data = {{t0, 0.5}}
Show[g3, Plot[1, {t, t0, t0 + d},
   PlotRange → Full, PlotStyle → {Green}], Plot[1 - d, {t, t0, t0 + d},
   PlotRange → Full, PlotStyle → {Green}],
 ListPlot[data, Joined → True, Mesh → Full,
  MeshStyle → Directive[PointSize[Large], Red]],
 ListLinePlot[{{t0, 0}, {t0, 0.999}}, PlotStyle → {Blue}],
 ListLinePlot[{{t0, 0.999}, {4, 0.999}}, PlotStyle → {Blue}],
 ListLinePlot[{{t0 + d, 1 - d}, {t0 + d, 0.999}}, PlotStyle → {Green}],
 AspectRatio → 0.3]
Clear[d]
FindRoot[F[t0 + d] - 1 + d, {d, 0.4}]

1.2

0.005

0.5

{0.00333767}

0.00333767

{0.5}

0.0519236

{{0.00333767, 0.5}}
```




```
{d → 0.0519236}
```

Figure 2: The $M(t)$ (The module in CAS Mathematica).

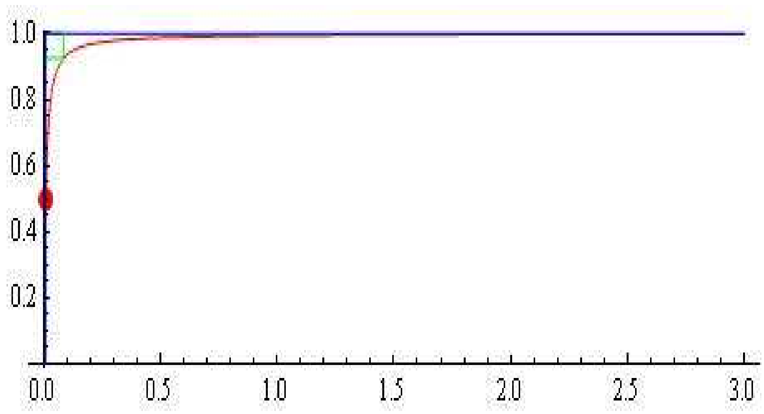


Figure 3: The $M(t)$ for $\alpha = 1.7$; $\beta = 0.01$; $\theta = 0.4$; $t_0 = 0.00917862$; The H–distance $d = 0.0763781$.

3. SOME APPLICATIONS

The model lead to greater flexibility in modeling of various specific data types.

By further analyzing the addresses of CryptoLocker cluster (C_{CL}) in [9] it is discovered that approximately 83.16% Bitcoin addresses received maximum two payments as well as 13.33% Bitcoin addresses received no more than one Bitcoin.

It is known [9] that an address

16i7w5G2aoq8zqLDR3VJnawZ8VmYFZjVsd

collected 112.94 BTC while other address

1HFLn7JP7FZrufvNKkQPEfAWGjKUdFZEmy

collected 83 ransom payments.

Fig. 4 shows cdf of the number of ransoms per address [9].

For example we photographed the data in Fig. 4:

$$data\_CDF\_of\_ransoms\_received\_per\_address\_in\_C_{CL} :=$$

$$\{\{1, 0.6762\}, \{2, 0.8286\}, \{3, 0.8667\}, \{4, 0.9143\}, \{5, 0.9333\},$$

$$\{6, 0.9429\}, \{7, 0.9524\}, \{8, 0.9571\}, \{9, 0.9667\}, \{10, 0.9714\},$$

$$\{11, 0.9733\}, \{14, 0.9810\}, \{20, 0.9829\}, \{23, 0.9857\}, \{25, 0.9885\},$$

$$\{55, 0.9905\}, \{70, 0.9952\}, \{83, 1\}\}.$$

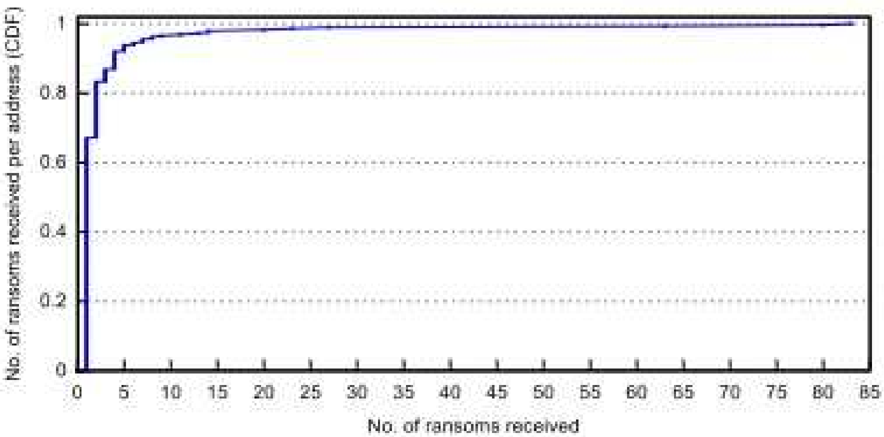


Figure 4: CDF of ransoms received per address in C_{CL} [9].

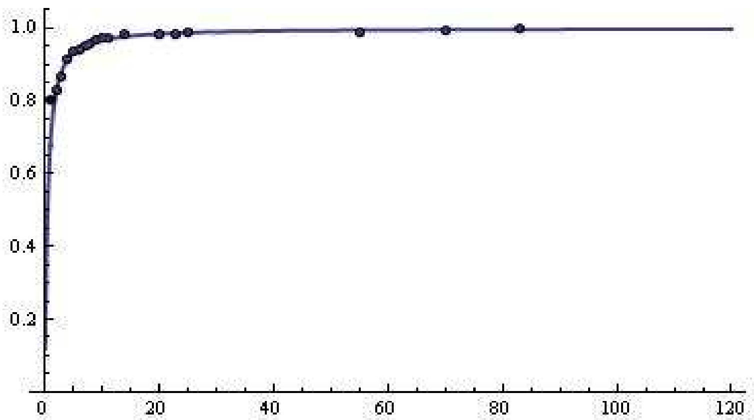


Figure 5: The fitted model $M(t)$.

The fitted model for $\alpha = 1.19523$; $\beta = 31.6463$; $\theta = 0.00756503$ is visualized on Fig. 5.

4. CONCLUSION

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered family of functions.

The module offers the following possibilities:

1. calculation of the H-distance between the h_{t_0} and the model $M(t)$;
2. generation of the functions under user defined values of the parameters α, β and θ;
3. software tools for animation and visualization.

For other results, see [10]–[26].

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