*Communications in Applied Analysis*, **23**, No. 2 (2019), 343-358 ISSN: 1083-2564

# STABILITY IN DYNAMIC EQUATIONS ON TIME SCALES TITLE OF PAPER LINE TWO

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**ABSTRACT:** We are interested in the stability, uniform stability, and exponential stability of the zero solution of different types of dynamic systems on time scales. The approach is based on suitable Lyapunov functionals and certain inequalities.

**AMS Subject Classification:** 39A10
**Key Words:**

**Received:** June 13, 2017 ; **Accepted:** November 13, 2018 ;
**Published:** March 2, 2019. **doi:** 10.12732/caa.v23i2.6
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

## 1. INTRODUCTION

In this paper, we use suitable Lyapunov functionals and give conditions for the stability, uniform stability, and exponential stability of the zero solution of different types of dynamic systems on time scales. Our considered systems will be variations of the general dynamic system

$$x^{\Delta}(t) = G(t, x(t), x(\delta(t))),\ t \in [t_0, \infty)_{\mathbb{T}} \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$ which is unbounded above and $0 \in \mathbb{T}$. Here the function $G$ is rd-continuous, where $x \in \mathbb{R}^n$ and $G : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ with $G(t, 0, 0) = 0$. the delay function $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ is strictly increasing, invertible and delta differentiable such that $\delta(t) < t$, $\left|\delta^{\Delta}(t)\right| < \infty$ for $t \in \mathbb{T}$, and $\delta(t_0) \in \mathbb{T}$.

For each $t_0 \in \mathbb{T}$ and for a given $rd-$continuous initial function $\psi := [\delta(t_0), t_0]_{\mathbb{T}} \to \mathbb{R}^n$, we say that $x(t) := x(t; t_0, \psi)$ is the solution of (1.1) if $x(t) = \psi(t)$ on $[\delta(t_0), t_0]_{\mathbb{T}}$ and satisfies (1.1) for all $t \geq t_0$.

We say $V : [\delta(t_0), \infty)_{\mathbb{T}} \times \mathbb{R}^n \mapsto [0, \infty)$ is a *type I Lyapunov functional* on $[\delta(t_0), \infty)_{\mathbb{T}} \times \mathbb{R}^n$ when

$$V(t, x) = \sum_{i=1}^{n} \Big(V_i(x_i) + U_i(t)\Big),$$

where each $V_i : \mathbb{R} \mapsto \mathbb{R}$ and $U_i : [\delta(t_0), \infty)_{\mathbb{T}} \mapsto \mathbb{R}$ are continuously differentiable. Next, we extend the definition of the derivative of a type I Lyapunov function to type I Lyapunov functionals. If $V$ is a type I Lyapunov functional and $x$ is a solution of equation (1.1), then we have

$$\begin{aligned} [V(t, x)]^{\Delta} &= \sum_{i=1}^{n} \Big(V_i(x_i(t)) + U_i(t)\Big)^{\Delta} \\ &= \int_0^1 \nabla V\left[x(t) + h\mu(t)G(t, x(.))\right].G(t, x(.))dh + \sum_{i=1}^{n} U_i^{\Delta}(t) \end{aligned}$$

where $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$ is the gradient operator. This motivates us to define $\dot{V} : [\delta(t_0), \infty)_{\mathbb{T}} \times \mathbb{R}^n \mapsto \mathbb{R}$ by

$$\dot{V}(t, x) = [V(t, x)]^{\Delta}.$$

Continuing in the spirit of [12], we have

$$\dot{V}(t, x) = \begin{cases} \sum\limits_{i=1}^{n} \frac{V_i\left(x_i + \mu(t)G_i(t, x(\cdot))\right) - V_i(x_i)}{\mu(t)} + \sum\limits_{i=1}^{n} U_i^{\Delta}(t), & \text{when } \ \mu(t) \neq 0, \\ \nabla V(x).G(t, x(\cdot)) + \sum\limits_{i=1}^{n} U_i^{\Delta}(t), & \text{when } \ \mu(t) = 0. \end{cases}$$

We also use a continuous strictly increasing function $W_i : [0, \infty) \mapsto [0, \infty)$ with $W_i(0) = 0$, $W_i(s) > 0$ if $s > 0$ for each $i \in \mathbb{Z}^+$.

## 2. CALCULUS ON TIME SCALES WITH PRELIMINARY RESULTS

An introduction with applications and advances in dynamic equations are given in [7, 8]. In this section, we only mention necessary basic results on time scales. We have two jump operators, namely the *forward jump operator* and the *backward jump operator*

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\}, \quad \rho(t) := \sup\{s < t : s \in \mathbb{T}\}$$

for all $t \in \mathbb{T}$, respectively. Therefore, there might be four types of points in a time scale, i.e., $\sigma(t) > t$ (*right-scattered point* $t$ ), $\rho(t) < t$ (*left-scattered* point $t$), $\sigma(t) = t$ (*right-dense* point $t$), and $\rho(t) = t$ (*left-dense* point $t$). Also $\mu : \mathbb{T} \mapsto [0, \infty)$ defined by $\mu(t) := \sigma(t) - t$ gives the distance between two points in a time scale.

Assume $x : \mathbb{T} \mapsto \mathbb{R}^n$. Then we define $x^\Delta(t)$ to be the vector (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U$ of $t$ such that

$$\left|[x_i(\sigma(t)) - x_i(s)] - x_i^\Delta(t)[\sigma(t) - s]\right| \le \epsilon\, |\sigma(t) - s|$$

for all $s \in U$ and for each $i = 1, 2, \cdots n$. We call $x^\Delta(t)$ the *delta derivative* of $x(t)$ at $t$, and it turns out that $x^\Delta(t) = x'(t)$ if $\mathbb{T} = \mathbb{R}$ and $x^\Delta(t) = x(t+1) - x(t)$ if $\mathbb{T} = \mathbb{Z}$. If $G^\Delta(t) = g(t)$, then the Cauchy integral is defined by

$$\int_a^t g(s)\Delta s = G(t) - G(a).$$

It can be shown that for each continuous function $f : \mathbb{T} \mapsto \mathbb{R}^n$ at $t \in \mathbb{T}$

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \quad \text{for right-scattered point} t$$

and if the limit exists

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \quad \text{for right-dense point} t.$$

The product and quotient rules are given by

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) \tag{2.1}$$

and
$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)} \quad \text{if} \quad g(t)g^{\sigma}(t)\neq 0.$$
for differentiable functions $f,g:\mathbb{T}\mapsto\mathbb{R}^n$ at $t\in\mathbb{T}$. We also have the following simple useful formula

$$f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t), \quad \text{where} \quad f^{\sigma}=f\circ\sigma. \tag{2.2}$$

We say $f:\mathbb{T}\mapsto\mathbb{R}$ is *rd-continuous* provided $f$ is continuous at each right-dense point $t\in\mathbb{T}$ and whenever $t\in\mathbb{T}$ is left-dense $\lim_{s\to t^-}f(s)$ exists as a finite number.

The following chain rule is due to Poetzsche.

**Theorem 2.1.** *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be continuously differentiable and suppose $g:\mathbb{T}\mapsto\mathbb{R}$ is delta differentiable. Then $f\circ g:\mathbb{T}\mapsto\mathbb{R}$ is delta differentiable and the formula*

$$(f\circ g)^{\Delta}(t)=\left\{\int_0^1 f'\left(g(t)+h\mu(t)g^{\Delta}(t)\right)dh\right\}g^{\Delta}(t) \tag{2.3}$$

*holds.*

Next, we define an exponential function on a time scale in order to obtain exponential stability of the zero solution of (1.1). We say that $p:\mathbb{T}\mapsto\mathbb{R}$ is *regressive* provided $1+\mu(t)p(t)\neq 0$ for all $t\in\mathbb{T}^{\kappa}$. We define the set $\mathcal{R}$ of all regressive and rd-continuous functions. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by $\mathcal{R}^+=\{p\in\mathcal{R}:1+\mu(t)p(t)>0 \text{ for all } t\in\mathbb{T}^{\kappa}\}$. If $p\in\mathcal{R}$, then the *exponential function* $e_p(t,t_0)$ is for each fixed $t_0\in\mathbb{T}$ the unique solution of the initial value problem

$$x^{\Delta}=p(t)x, \;\; x(t_0)=1$$

on $\mathbb{T}$. Under the addition on $\mathcal{R}$ defined by

$$(p\oplus q)(t)=p(t)+q(t)+\mu(t)p(t)q(t), \; t\in\mathbb{T}^{\kappa}$$

is an Abelian group (see [7]), where the additive inverse of $p$, denoted by $\ominus p$, is defined by
$$(\ominus p)(t)=\frac{-p(t)}{1+\mu(t)p(t)}, \; t\in\mathbb{T}^{\kappa}.$$

We also define the "circle minus" subtraction $\ominus$ on $\mathcal{R}$ by

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t),\ t \in \mathbb{T}^{\kappa}. \tag{2.4}$$

Therefore,

$$(p \ominus q)(t) = \frac{p(t) - q(t)}{1 + \mu(t)q(t)},\ t \in \mathbb{T}^{\kappa}.$$

We use the following properties of the exponential function $e_p(t,s)$ which are proved in Bohner and Peterson [7].

**Theorem 2.2.** *If $p, q \in \mathcal{R}$, then for $t, s, r, t_0 \in \mathbb{T}$*

*(i)* $e_p(t,t) \equiv 1$ *and* $e_0(t,s) \equiv 1$;

*(ii)* $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;

*(iii)* $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s) = e_p(s,t)$;

*(iv)* $\frac{e_p(t,s)}{e_q(t,s)} = e_{p \ominus q}(t,s)$;

*(v)* $e_p(t,s)e_q(t,s) = e_{p \oplus q}(t,s)$;

*(vi)* $e_p(t,r)e_p(r,s) = e_p(t,s)$.

The following properties of the exponential function on $\mathbb{T}$ can be found in [5].

**Theorem 2.3.** *Let $t_0 \in \mathbb{T}$.*

*(i) If $p \in \mathcal{R}^+$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.*

*(ii) If $p \geq 0$, then $e_p(t,t_0) \geq 1$ for all $t \geq t_0$. Therefore, $e_{\ominus p}(t,t_0) \leq 1$ for all $t \geq t_0$.*

Also, it follows from Bernoulli's inequality ([7], Theorem 6.2]) that for any time scale, if the constant $\lambda \in \mathcal{R}^+$, then

$$0 < e_{\ominus\lambda}(t,t_0) \leq \frac{1}{1 + \lambda(t - t_0)}, \quad t \geq t_0.$$

It follows that

$$\lim_{t\to\infty} e_{\ominus\lambda}(t,t_0) = 0.$$

In particular, if $\mathbb{T} = \mathbb{R}$, then $e_{\ominus\lambda}(t, t_0) = e^{-\lambda(t-t_0)}$ and if $\mathbb{T} = \mathbb{Z}^+$, then $e_{\ominus\lambda}(t, t_0) = (1+\lambda)^{-(t-t_0)}$. For the growth of generalized exponential functions on time scales, see Bodine and Lutz [4].

## 3. STABILITY

We begin this section by stating stability definitions. Then, we use Lyapunov functions/functionals and obtain result concerning the stability of the zero solution.

We define

$$E_{t_0} = [\delta(t_0), t_0]_{\mathbb{T}}$$

which we call the *the initial interval.* For $x \in R^n$, $|x|$ denotes the Euclidean norm of $x$. For any $n \times n$ matrix $A$, $|A|$ will denote any compatible norm so that $|Ax| \leq |A||x|$. Let $C(t)$ denote the set of rd-continuous functions $\phi : [\delta(t_0), t]_{\mathbb{T}} \to R^n$ and $\|\phi\| = \sup\{|\phi(s)| : \delta(t_0) \leq s \leq t\}$.

**Definition 3.1.** The zero solution of (1.1) is stable if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\phi \in E_{t_0} \to \mathbb{R}^n, \phi \in C(t) : \|\phi\| < \delta]$ implies $|x(t, t_0, \phi)| < \varepsilon$ for all $t_0 \geq 0$.

**Definition 3.2.** The zero solution of (1.1) is uniformly stable (US) if it is stable and $\delta$ is independent of $t_0$.

**Definition 3.3.** The zero solution of (1.1) is asymptotically stable (AS) if it is stable and if for each $t_0 \geq 0$ there is an $\eta > 0$ such that $[\phi \in E_{t_0} \to \mathbb{R}^n, \phi \in C(t) : \|\phi\| < \eta]$ implies that any solution of (1.1) $|x(t, t_0, \phi)| \to 0$ as $t \to \infty$. If this is true for any $t_0$ and any $\eta$, then $x = 0$ is globally asymptotically stable (GAS).

**Definition 3.4.** The zero solution of (1.1) is uniformly asymptotically stable (UAS) if it is uniformly stable and if there exists an $\eta > 0$ and for each $\nu > 0$ there exists a $T > t_0$ such that $[t_0 \geq 0, \phi \in E_{t_0} \to \mathbb{R}^n, \phi \in C(t) : \|\phi\| < \eta, t \geq T]$ implies that any solution of (1.1) satisfies $|x(t, t_0, \phi)| < \nu$.

**Definition 3.5.** We say that the zero solution of (1.1) is *exponentially asymptotically stable* on $[\delta(t_0), \infty)_{\mathbb{T}}$ if there exist a positive constant $d$, a constant

$C \in \mathbb{R}^+$, and an $M > 0$ such that for any solution $x(t, t_0, \phi)$ of (1.1),

$$\|x(t, t_0, x_0)\| \leq C\Big(|\phi|, t_0\Big)(e_{-M}(t, t_0))^d, \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}$$

where $C\Big(|\phi|, t_0\Big)$ is a constant depending on $|\phi|$ and $t_0$, $\phi$ is a given continuous and bounded initial function. It is uniformly asymptotically stable if $C$ is independent of $t_0$.

Let $C_{rd}(\mathbb{T}, \mathbb{R})$ be the space of all right-dense continuous functions from $\mathbb{T}$ into $\mathbb{R}$

Let $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ with $f(t, 0) = 0$ and consider the nonlinear dynamical system

$$x^{\Delta} = A(t)x(t) + f(t, x),\ x(t_0) = x_0,\ t \in [t_0, \infty)_{\mathbb{T}} \tag{3.1}$$

where $A(t)$ is $k \times k$ matrix such that all of its entries belong to $C_{rd}(\mathbb{T}, \mathbb{R}$ and and $x$ is $k \times 1$ vector. If $D$ is a matrix, then $|D|$ means the sum of the absolute values of the elements. For what to follow we write $x$ for $x(t)$.
In the case $f(t, x) = 0$, then stability results can be found in [10].
For what to follow we write $A$ to represent $A(t)$. For the next theorems, we take the initial function to be $x_0$ at $t_0$ with $E_{t_0} = \{t_0\}$, and hence the stability definitions are understood to be modified accordingly. This is due to the fact that (3.1) has no delay.
The next theorem will show that the stability of (3.1) depends only on the stability of linear part when the function $f$ is uniformly bounded in $t$ and $x$.

**Theorem 3.6.** *Assume there are positive constants $\alpha^2, \beta^2$ and $k \times k$ positive definite constant symmetric matrix $B$ such that*

$$\alpha^2 x^T x \leq x^T B x \leq \beta^2 x^T x. \tag{3.2}$$

*Also, assume that there are constants $\gamma_1, \gamma_2$ so that*

$$2|B||f(t, x)| \leq \gamma_1 |x| \tag{3.3}$$

*and*

$$2\mu(t)|f(t, x)||BA| \leq \gamma_2 |x| \tag{3.4}$$

*uniformly for* $t \in [t_0, \infty)_{\mathbb{T}}$.
*If there exists positive constant* $\xi$, *where* $-\frac{\xi}{\beta^2} \in \mathcal{R}^+$ *and*

$$A^T B + BA + \mu(t) A^T BA \le -\xi I, \tag{3.5}$$

*where* $-\xi + \gamma_1 + \gamma_2 \le 0$, *then the zero solution of* (3.1) *is uniformly stable.*

**Proof.** Let the matrix $B$ be defined by (3.10) and define

$$V(t,x) = x^T Bx. \tag{3.6}$$

Here $x^T x = x^2 = (x_1^2 + x_2^2 + \cdots + x_k^2)$. Using the product rule given in (2.1) we have along the solutions of (3.1) that

$$\begin{aligned}
\dot{V}(t,x) &= (x^\Delta)^T Bx + (x^\sigma)^T Bx^\Delta \\
&= (x^\Delta)^T Bx + (x + \mu(t)x^\Delta)^T Bx^\Delta \\
&= (x^\Delta)^T Bx + x^T Bx^\Delta + \mu(t)(x^\Delta)^T Bx^\Delta \\
&= x\big[A^T B + BA + \mu(t)A^T BA\big]x^T \\
&+ 2x^T Bf(t,x) + \mu(t) f^T(t,x)BAx + \mu(t)x^T A^T Bf(t,x) \\
&+ \mu(t) f^T(t,x)Bf(t,x) \\
&= x\big[A^T B + BA + \mu(t)A^T BA\big]x^T \\
&+ 2x^T Bf(t,x) + 2\mu(t) f^T(t,x)BAx + \mu(t) f^T(t,x)Bf(t,x) \\
&\le -\xi|x|^2 + 2|x||B||f(t,x)| + 2\mu(t)|f(t,x)||BA||x| + \mu(t)|B|f^2(t,x) \\
&\le (-\xi + \gamma_1 + \gamma_2)|x|^2 \le 0.
\end{aligned}$$

Since $V$ is decreasing, we have from (3.6) that for $t \ge t_0$

$$x^T(t)Bx(t) \le x^T(t_0)Bx(t_0) = x_0^T Bx_0.$$

Thus, by (3.7) we have

$$\alpha^2 x^T x \le x^T(t)Bx(t) \le x_0^T Bx_0.$$

Let $\epsilon > 0$ be given and chose $\delta = \frac{\epsilon}{\sqrt{|B|}}$. For $|x_0| < \delta$, we obtain from the above inequality that

$$|x(t)| \le |x_0|\sqrt{|B|} < \delta\sqrt{|B|} < \epsilon.$$

□

The next theorem will show that the stability of (3.1) depends only on the stability of linear part when the function $f$ is small enough near zero.

**Theorem 3.7.** *Assume there are positive constants* $\alpha^2, \beta^2$ *and* $k \times k$ *positive definite constant symmetric matrix* $B$ *such that*

$$\alpha^2 x^T x \leq x^T B x \leq \beta^2 x^T x. \tag{3.7}$$

*Also, assume that*

$$\lim_{|x| \to 0} \frac{|f(t,x)|}{|x|} = 0, \tag{3.8}$$

*and*

$$\lim_{|x| \to 0} \frac{\mu(t)|f(t,x)||BA|}{|x|} = 0, \tag{3.9}$$

*uniformly for* $t \in [t_0, \infty)_{\mathbb{T}}$.
*If there exists positive constant* $\xi$, *where* $-\frac{\xi}{\beta^2} \in \mathcal{R}^+$ *and*

$$A^T B + BA + \mu(t) A^T BA \leq -\xi I, \tag{3.10}$$

*then the zero solution of* (3.1) *is uniformly exponentially asymptotically stable.*

**Proof.** Let the matrix $B$ be defined by (3.10) and define

$$V(t,x) = x^T B x.$$

Here $x^T x = x^2 = (x_1^2 + x_2^2 + \cdots + x_k^2)$. Using the product rule given in (2.1) we have along the solutions of (3.1) that

$$\begin{aligned}
\dot{V}(t,x) &= (x^\Delta)^T Bx + (x^\sigma)^T Bx^\Delta \\
&= (x^\Delta)^T Bx + (x + \mu(t)x^\Delta)^T Bx^\Delta \\
&= (x^\Delta)^T Bx + x^T Bx^\Delta + \mu(t)(x^\Delta)^T Bx^\Delta \\
&= x\big[A^T B + BA + \mu(t)A^T BA\big]x^T \\
&+ 2x^T Bf(t,x) + \mu(t)f^T(t,x)BAx + \mu(t)x^T A^T Bf(t,x) \qquad (3.11)\\
&+ \mu(t)f^T(t,x)Bf(t,x) \\
&= x\big[A^T B + BA + \mu(t)A^T BA\big]x^T \\
&+ 2x^T Bf(t,x) + 2\mu(t)f^T(t,x)BAx + \mu(t)f^T(t,x)Bf(t,x) \\
&\leq -\xi|x|^2 + 2|x||B||f(t,x)| + 2\mu(t)|f(t,x)||BA||x| + \mu(t)|B|f^2 \qquad (3.12)
\end{aligned}$$

by (3.10). Conditions (3.8) and (3.9) allow us to find $\gamma > 0$ small enough so that

$$|x| \leq \gamma, \text{ for } t \in [t_0, \infty)_{\mathbb{T}},$$

which implies that

$$2|B||f(t,x)| \leq \frac{\xi}{4}|x|, \; 2\mu(t)|f(t,x)||BA| \leq \frac{\xi}{4}|x|, \; \mu(t)|B|f^2(t,x) \leq \frac{\xi}{4}|x|^2. \tag{3.13}$$

Substituting (3.13) into (3.11) and by making use of (3.7), we arrive at

$$\begin{aligned} \dot{V}(t,x) &\leq -\frac{\xi}{4}x^T x \\ &\leq -\frac{\xi}{4\beta^2}x^T Bx \\ &= -\frac{\xi}{4\beta^2}V(t,x). \end{aligned} \tag{3.14}$$

It is easy to see that (3.14) gives

$$V(t,x) \leq V(t_0, x(t_0))e_{-\frac{\xi}{\beta^2}}(t,t_0) \tag{3.15}$$

Thus, by (3.7) and (3.14) we have that

$$\begin{aligned} \alpha^2 x^T x &\leq V(t,x) \leq V(t_0, x(t_0))e_{-\frac{\xi}{\beta^2}}(t,t_0) \\ &\leq \beta^2 x^T(t_0)x(t_0)e_{-\frac{\xi}{\beta^2}}(t,t_0) \end{aligned}$$

Since, $x^T x = |x|^2$, we obtain from the last inequality that

$$|x(t)| \leq \frac{\beta}{\alpha}|x_0|\big(e_{-\frac{\xi}{\beta^2}}(t,t_0)\big)^{\frac{1}{2}}.$$

$\square$

**Example 3.8.** Let $\mathbb{T}$ be a time scale such that $0 \leq \mu(t) < \frac{4}{5}, \mu(t)|g(t)| \leq M_1$ and $|g(t)| \leq M_2$, for all $t \in [t_0, \infty)_{\mathbb{T}}$, for positive constants $M_1$ and $M_2$. We consider the nonlinear system

$$x^{\Delta} = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} x + \frac{g(t)}{\sqrt{2}} \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 + x_2^2 \end{pmatrix} \tag{3.16}$$

Choose $B = I$. Then

$$A^T B + BA + \mu(t) A^T BA = (5\mu(t) - 4)I \leq -\xi I, \text{ for some positive } \xi.$$

Since $B = I$, condition (3.7) is satisfied with $\alpha^2 = \beta^2 = 1$. Thus, $-\frac{\xi}{\beta^2} = -\xi$. In addition, $1 - \mu(t)\xi > 0$, which implies that $-\frac{\xi}{\beta^2} \in \mathcal{R}^+$. Also, it is easy to check that conditions (3.8) and (3.9) are satisfied. Thus, we have shown that the zero solution of the system (3.16) is uniformly exponentially stable.

Next we display couple time scales that satisfy the requirement of Example 3.1.

**1)** Let $H_0 = 0$, and $H_n = \sum_{k=1}^{n} \frac{1}{2k}$, for $n \in \mathbb{N}$. Consider the time scale

$$\mathbb{T}_1 = \{H_n : n \in \mathbb{N}_0\},$$

then $\mu(H_n) = H_{n+1} - H_n = \frac{1}{2n+2}$.

**2)** For $a, b > 0$, with $b < \frac{4}{5}$, we consider the time scale

$$\mathbb{T}_2 = \cup_{k=0}^{\infty} [k(a+b),\ k(a+b)+a].$$

Then

$$\mu(t) = \begin{cases} 0, & t \in \cup_{k=0}^{\infty} [k(a+b),\ k(a+b)+a] \\ b, & t \in \cup_{k=0}^{\infty} \{k(a+b)+a\}. \end{cases}$$

**Example 3.9.** Let $\mathbb{T}$ be a time scale such that $\mu(t)|g(t)||BA| \leq M_1$ and $|g(t)| \leq M_2$, for all $t \in [t_0, \infty)_{\mathbb{T}}$, for positive constants $M_1$ and $M_2$. Consider the nonlinear system

$$x^{\Delta} = \begin{pmatrix} -1 & 0 \\ 0 & -a(t) \end{pmatrix} x + \frac{g(t)}{\sqrt{2}} \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 + x_2^2 \end{pmatrix} \tag{3.17}$$

Choose $B = I$. Then

$$A^T B + BA + \mu(t) A^T BA = \begin{pmatrix} \mu(t) - 2 & 0 \\ 0 & (\mu(t)a(t) - 2)a(t) \end{pmatrix} := A^*$$

Now for any $2 \times 2$ matrix

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

to be negative definite, we must have $-d_{11} > 0$ and $det(D) > 0$.
Hence, $2 - \mu(t) > 0$, which implies that $0 < \mu(t) < 2$. Moreover, $det(A^*) = (\mu(t) - 2)((\mu(t)a(t) - 2)a(t)) > 0$, for $a(t) > 1$ and $\mu(t) < \dfrac{2}{a(t)}$.

As before, since $B = I$, condition (3.7) is satisfied with $\alpha^2 = \beta^2 = 1$. Thus, $-\frac{\xi}{\beta^2} = -\xi$. In addition, $1 - \mu(t)\xi > 0$, which implies that $-\frac{\xi}{\beta^2} \in \mathcal{R}^+$. Thus for $a(t) > 1$ and , $\mu(t) < \dfrac{2}{a(t)}$, we have the zero solution of (3.17) is uniformly exponentially stable.

Next we turn our attention to the nonlinear delay dynamical system

$$x^\Delta = Ax(t) + f(t, x) + g(t, x(\delta(t))\delta^\Delta(t), t \in [t_0, \infty)_{\mathbb{T}} \tag{3.18}$$

where $f(t, 0)$, $g(t, 0) = 0$. Due to the presence of the delay function $g$ a different Lyapunov functional will have to be used.
First we state the following Lemma which we need in the differentiation of the next Lyapunov functional.

**Lemma 3.10.** *([1, Lemma 3.], [2]) Suppose that $\mathbb{T}$ is a time scale having a strictly increasing, and invertible delay function $\delta : [t_0, \infty]_{\mathbb{T}} \to [\delta(t_0), \infty]_{\mathbb{T}}$, $t_0 \in \mathbb{T}$ such that $\delta(t) < t$ and $\left|\delta^\Delta(t)\right| < \infty$. Then for a given $rd-$continuous function $f : \mathbb{T} \to \mathbb{R}$, we have*

$$\left(\int_{\delta(t)}^{t} f(s)\Delta s\right)^\Delta = f(t) - f(\delta(t))\delta^\Delta(t). \tag{3.19}$$

**Theorem 3.11.** *Assume the conditions of Theorem 3.2 hold. If*

$$\lim_{|x|\to 0} \frac{|g(\delta^{-1}(t), x)|}{|x|} = 0, \tag{3.20}$$

$$\lim_{|x|\to 0} \frac{\mu(t)|g(t, x(\delta(t))|}{|x|} = 0 \tag{3.21}$$

$$\lim_{|x|\to 0} \frac{\delta^\Delta(t)|g(t, x(\delta(t))|}{|x|} = 0, \tag{3.22}$$

*and*

$$\lim_{|x|\to 0} \frac{\mu(t)\delta^\Delta(t)|g(t, x(\delta(t))|}{|x|} = 0 \tag{3.23}$$

*uniformly for* $t \in [t_0, \infty)_{\mathbb{T}}$, *then*
*a) the zero solution of* (3.18) *is (AS).*

*b) In addition, if there exists a positive constant* $M$ *such*

$$t - \delta(t) \le M, \ \textit{for all}\, t \in [t_0, \infty)_{\mathbb{T}} \tag{3.24}$$

*then the zero solution of* (3.18) *is (US).*

**Proof.** Consider the Lyapunov functional

$$V(t,x) = x^T Bx + \int_{\delta(t)}^{t} g^2(\delta^{-1}(s), x(s))\Delta s. \tag{3.25}$$

Then by a similar argument as in the proof of Theorem 2.2 and by Lemma 3.1, we have along the solutions of (3.18) that

$$\begin{aligned}
\dot{V}(t,x) &= (x^{\Delta})^T Bx + (x^{\sigma})^T Bx^{\Delta} \\
&+ g^2(\delta^{-1}(t), x(t)) - g^2(t, x(\delta(t))\delta^{\Delta}(t) \\
&\le -\xi|x|^2 + 2|x||B||f(t,x)| + 2\mu(t)|f(t,x)||BA||x| + \mu(t)|B|f^2(t,x) \\
&+ g^2(\delta^{-1}(t), x(t)) - g^2(t, x(\delta(t))\delta^{\Delta}(t) \\
&+ \Big(2|x||B||g(t, x(\delta(t))| + 2\mu(t)|x||BA||g(t, x(\delta(t))| \\
&+ 2\mu(t)|f(t, x(t))||B||g(t, x(\delta(t))| \\
&+ \mu(t)|B|g^2(t, x(\delta(t))\delta^{\Delta}(t)\Big)\delta^{\Delta}(t)
\end{aligned} \tag{3.26}$$

Next we simplify (3.26). As before, for $\gamma > 0$ with

$$|x| \le \gamma, \ \text{for}\ t \in [t_0, \infty)_{\mathbb{T}},$$

we have that

$$g^2(\delta^{-1}(t), x(t)) \le \frac{\xi}{8}|x|^2, \ \text{by (3.20)}$$

$$2|B||g(t, x(\delta(t))|\delta^{\Delta}(t) \le \frac{\xi}{8}|x|, \ \text{by (3.21)}$$

$$2\mu(t)|BA||g(t, x(\delta(t))|\delta^{\Delta}(t) \le \frac{\xi}{8}|x|, \ \text{by (3.23)}$$

$$2\mu(t)|f(t, x(t))||B||g(t, x(\delta(t))|\delta^{\Delta}(t) \le \frac{\xi}{8}|x|^2, \ \text{by (3.23) and (3.8)}$$

and

$$\mu(t)|B|g^2(t, x(\delta(t))(\delta^\Delta(t))^2 \le \frac{\xi}{8}|x|^2 \text{ by (3.22) and (3.23).}$$

Putting everything together, we arrive at

$$\dot{V}(t,x) \quad \le \quad -\frac{7}{8}\xi|x|^2 \tag{3.27}$$

Let $\varepsilon \in (0, \gamma)$ be given. We will find a $\delta > 0$ so that for any bounded initial function $\phi : [\delta(t_0), t_0]_{\mathbb{T}} \to R^n$ with $||\phi|| < \delta$, we have $x(t, t_0, \phi)| < \varepsilon$. Due to (3.27), $V$ is decreasing and hence for $t \ge t_0$ we have that

$$\begin{aligned}
\alpha^2||x||^2 \le V(t,x) \quad &\le \quad V(t_0, \phi) \\
&\le \quad \beta^2||\phi||^2 + \int_{\delta(t_0)}^{t_0} \frac{\xi}{8}|\phi|^2 \Delta s \\
&\le \quad \beta^2||\phi||^2 + (t_0 - \delta(t_0))\frac{\xi}{8}||\phi||^2 \qquad (3.28)\\
&\le \quad (\beta^2 + t_0\frac{\xi}{8})||\phi||^2.
\end{aligned}$$

Or,

$$|x(t, t_0)\phi)| \le \varepsilon, \text{ for } \delta = \Big\{\frac{1}{1/\alpha^2(\beta^2 + \frac{t_0\xi}{8})}\Big\}^{1/2}\varepsilon.$$

This shows the zero solution is stable.
Next we show that $x(t, t_0, \phi)| \to 0$ as $t \to \infty$. We will prove this by contradiction. Suppose $|x(t, t_0, \phi)| \nrightarrow 0$, then there exists a large $T \in \mathbb{T}$ and positive and small $\rho$ such that $|x(t, t_0, \phi)| > \rho$ for all $t \ge T$. Next we integrate (3.27) from $T$ to $t$ and get

$$\begin{aligned}
0 \le V(t,x) \quad &\le \quad V(T,x) - \frac{7}{8}\xi \int_T^t |x(s)|^2 \Delta s \\
&\le \quad V(T,x) - \frac{7}{8}\xi(t-T)\rho \to -\infty, \text{ as } t \to \infty,
\end{aligned}$$

which is a contradiction. This completes the proves of (AS).
As for part b), by (3.24) we have from (3.28) that

$$|x(t, t_0\phi)| \le \Big\{1/\alpha^2(\beta^2 + \frac{M\xi}{8})\Big\}^{1/2}\delta \le \varepsilon, \tag{3.29}$$

for $\delta = \Big\{\frac{1}{1/\alpha^2(\beta^2+\frac{M\xi}{8})}\Big\}^{1/2}\varepsilon$, which prove the zero solution is (US). □

**Remark 3.12.** The existence of $t_0$ on the right-hand side of (3.28) prevented us from proving (US) of the zero solution. In the case $\mathbb{T} = \mathbb{R}, \delta(t) = t - r$ for positive constant $r$, we have that $t - \delta(t) = t - (t - r) = r$. As a consequence one can easily conclude (US) from (3.28) by letting $\delta = \left\{ \frac{\varepsilon}{1/\alpha^2(\beta^2 + \frac{\xi}{8})} \right\}^{1/2}$. This is the exact reason why we have to ask for (3.24). Also, note that in (3.24) we do not require that $t - \delta(t) \in \mathbb{T}$.

We end the paper with the following open problem.

## OPEN PROBLEM

In term of Definition 3.4, what can be said about the uniform asymptotic stability (UAS) of the zero solution of (3.18).

## REFERENCES


[1] M. Adivar and Y. Raffoul, Stability and periodicity in dynamic delay equations, *Computers and Mathematics with Application* **58** (2009) 264–272.

[2] M. Adivar and Y. Raffoul, A note on "Stability and periodicity in dynamic delay equations", *Computers and Mathematics with Application* **59** (2010) 3351–3354.

[3] M. Adivar and Y. Raffoul, Existence results for periodic solutions of integro-dynamic equations on time scales. *Annali di Mathematica*, DOI 10.1007/s1023-008-0088-z.

[4] S. Bodine and D. A. Lutz, Exponential functions on time scales: Their asymptotic behavior and calculation,*Dynam. Systems Appl.*, **12**, (2003), 23–43.

[5] E. Akın–Bohner, M. Bohner and F. Akın. Pachpatte inequalities on time scales, *Journal of Inequalities in Pure and Applied Mathematics*, **6(1)**: 1-23 (2005).

[6] E. Akın–Bohner, and Y. Raffoul. Boundeness in functional dynamic equations on time scales. *Adv. Difference Equ.*, 2006(1–18), 2006.

[7] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications.* Birkhäuser, Boston, 2001.

[8] M. Bohner and A. Peterson. *Advances in Dynamic Equations on Time Scales.* Birkhäuser, Boston, 2003.

[9] M. Bohner and Y. Raffoul. Volterra Dynamic Equations on Time Scales, Preprint.

[10] J. J. DaCunha. Lyapunov Stability and Floquet Theory for Nonautonomous Linear Dynamic Systems on Time Scales, Doctoral Dissertation, 2004, timescales.org.

[11] A. Peterson and Y. Raffoul. Exponential stability of dynamic equations on time scales. Adv. Difference Equ., No. 2, (2005), pp. 133-144.

[12] A. Peterson and C.C. Tisdell. Boundedness and uniqueness of solutions to dynamic equations on time scales, J. Diff. Equations Appl., Vol. 10, No. 13-15 (2004), pp. 1295-1306.

[13] Y. Raffoul. Boundedness in nonlinear functional differential equations with applications to volterra integrodifferential, *J. Integral Equations Appl.*, Vol. 16, No. 4, Winter (2004).

[14] Y. Raffoul. Boundedness in Nonlinear Differential Equations, *Nonlinear Studies*, **10** (2003), pp. 343-350.