Communications in Applied Analysis, **23**, No. 2 (2019), 331-342 ISSN: 1083-2564

WEIGHTED COMPOSITION OF QUASI $*$-PARANORMAL OPERATORS

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ABSTRACT: An operator $T \in B(H)$ is said to be quasi $*$-paranormal operator if $\|T^*Tx\|^2 \leq \|T^3x\| \|Tx\|$ for all $x \in H$. In this paper, quasi $*$-paranormal composition operators on L^2 space and Hardy space is characterized.

AMS Subject Classification: 47B33, 47B37

Key Words: quasi $*$-paranormal operators, composition operators, conditional expectation, Hardy space

Received: August 29, 2018; **Accepted:** January 30, 2019;
Published: February 2, 2019. **doi:** 10.12732/caa.v23i2.5
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION AND PRELIMINARIES

Let H be an infinite dimensional complex Hilbert space and $B(H)$ denote the algebra of all bounded linear operators acting on H. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = \sqrt{T^*T}$. In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernal condition $N(U) = N(|T|)$. An operator T is said to be positive if $(Tx, x) \geq 0$ for all $x \in H$. An operator T is said to be a p-hyponormal operator if and only if $(T^*T)^p \geq (TT^*)^p$ for a positive number p.

In [22], the class of log-hyponormal operators is defined as follows: T is called log-hyponormal if it is invertible and satisfies $log(T^*T)^p \geq log(TT^*)^p$. Class of p-hyponormal operators and class of lg hyponormal operators were defined as extension class of hyponrmal operators, i.e., $(T^*T) \geq (TT^*)$. It is well known that every P-hyponormal operator is a q-hyponormal operator for $p \geq q > 0$, by the Lowner-Heinz theorem $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$, and every invertible p-hyponormal operator is a log- hyponormal operator since log(.) is an operator monotone function. An operator T is called paranormal if $\|Tx\|^2 \leq \|T^2x\| \|x\|$ for all $x \in H$. It is also well known that there exists a hyponormal operator T such that T^2 is not hyponormal$(see[12])$.

Furuta, Ito and Yamazaki[7] introduced class$A(K)$ and absolute - k- paranormal operators for $k > 0$ as generaliztions of class A and paranormal operators, respectively. An operator T belongs to class $A(K)$ if $(T^*|T|^{2K}T)^{\frac{1}{k+1}} \geq |T|^2$ and T is said to be absolute -k- paranormal operator if $\left\| |T|^k Tx \right\| \leq \|T\|^{k+1}$ for every unite vector x. An operator T is called quasi class A if $T^*|T|^2T \geq T^*|T|^2T$.

Fuji, Izumino and Nakamoto[9] introduced p-paranormal operators for $p > 0$ as a generalization of paranormal operators. An operator $T \in classA(p, r)$ for $p > 0$ and $r > 0$ if $\left(|T^*||T|^{2p}|T^*|\right)^{\frac{r}{p+r}} \geq |T^*|^{2r}$ and class$AI(p, r)$ is class of all invertible operators which belong to class $A(p, r)$. Yamazaki[24]introduced absolute-(p, r)-paranormal operator. It is a further generalization of the classes of both absolute-k-paranormal operators and p-paranormal operators as a parallel concept of class$A(p, r)$. An operator T is said to be paranormal operator if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector x. Paranormal operators have been studied by many authors [2],[8]and [16].

In [2], Ando showed that T is paranormal if and only if

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0 \text{ for all } \lambda > 0.$$

In order to extend the class of paranormal operators and class of quasi-class A operators, Mecheri introduced a new class of operators called k-

quasi-paranormal operators. An operator T is called k-quasi-paranormal if $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\| \, \|T^k x\|$ for all $x \in H$ where k is a natural number. A 1-quasi-paranormal operator is quasi paranormal.

The following implication give us relations among the class of operators.

Hyponormal $\Rightarrow$ p-hyponormal $\Rightarrow$ class$A$ $\Rightarrow$ paranormal
$\Rightarrow$quasi-paranormal $\Rightarrow$ k-quasi-paranormal.

Hyponormal class$A$ $\Rightarrow$ paranormal $\Rightarrow$quasi-paranormal
$\Rightarrow$ k-quasi-paranormal.

An operator $T \in B(H)$ is said to be $*$-paranormal operator if $\|T^*x\|^2 \leq \|T^2x\|^2$ for every unit vector x. Hyponormal operators are paranormal and $*$ - paranormal. The clas of $*$ - paranormal operators was defined by S. M. Patel. In order to extend the class of paranormal and $*$ - paranormal operators Mecheri introduced the class of quasi $*$ - paranormal operators. An operator T is called quasi $*$ - paranormal operator if $\|T^*Tx\|^2 \leq \|T^3x\| \, \|Tx\|$ for every $x \in H$. An operator T is quasi $*$ - paranormal if and only if $T^{*3}T^3 - 2\lambda(T^*T)^2 + \lambda^2 T^*T \geq 0$ for all $\lambda > 0$.Every $*$ - paranormal operator is quasi $*$ - paranormal and we have the following implication:

Hyponormal $\Rightarrow$$*$ - paranormal $\Rightarrow$ quasi $*$ - paranormal.

Let $(X, \sum, \lambda)$ be a sigma - finite measure space and let $T : X \rightarrow X$ be a non singular measurable transformation. A bounded linear operator $Cf = f \circ T$ on $L^2(X, \sum, \lambda)$ is said to be a composition operator induced by T, when the measure λT^{-1} is absolutely continuous with respect to the measure λ and the Radon - Nikodym derivative $d\lambda T^{-1}/d\lambda = f_0$ is essentially bounded. The Radon - Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to λ is denoted by f_0^k, where T^k is obtained by composing T - k times.

Theorem 1. *Let $T \in B(H)$ be an operator of quasi $*$ - paranormal. Then if T is unitarily equivalent to S. The S is of quasi $*$ - paranormal.*

Proof. Since T is unitarily equivalent to S. There is an unitarily operator U such that $S = U^*TU$. We must show that $S^{*3}S^3 - 2\lambda(S^*S)^2 + \lambda^2 S^*S \geq 0$.Since T is of quasi $*$ - paranormal, we have $T^{*3}T^3 - 2\lambda(T^*T)^2 + \lambda^2 T^*T \geq 0$.

Hence $S^{*3}S^3 - 2\lambda(S^*S)^2 + \lambda^2 S^*S \geq 0$

$$\{(U^*T^*U)(U^*T^*U)(U^*T^*U)(U^*TU)(U^*TU)(U^*TU)\}$$
$$- 2\lambda\{(U^*T^*U)(U^*TU)\}^2 + \lambda^2\{(U^*T^*U)(U^*TU)\} \geq 0,$$

$$U^*\left\{T^{*3}T^3 - 2\lambda(T^*T)^2 + \lambda^2 T^*T\right\}U \geq 0.$$

Therefore S is quasi $*$ - paranormal. □

Theorem 2. *Let $T \in B(H)$ be a weighted shifted with with non zero weight $\{\alpha_n\}\,(n = 0, 1, 2, 3, ...)$. Then T is of quasi $*$ - paranormal if and only if $|\alpha_n|^2 \leq |\alpha_{n+1}|\,|\alpha_{n+2}|$ for $n = 1, 2, 3, ..$*

Proof. Let $\{\alpha_n\}_{n=0}^{\infty}$ be an orthogonal basis of a Hilbert space H. Since $Te_n = \alpha_n e_{n+1}$ and $T^*e_n = \overline{\alpha}_{n-1}e_{n-1}$, we have

$$\begin{aligned}
\|T^*Te_n\| &= \|T^*(\alpha_n e_{n+1})\|^2 \\
&= |\alpha_n|^2 \|T^*e_{n+1}\|^2 \\
&= |\alpha_n|^2 |\overline{\alpha}_{n-1}|^2 \|e_n\|^2 \\
&= |\alpha_n|^4 .
\end{aligned}$$

and $\left\|T^3 e_n\right\| \|Te_n\| = |\alpha_n|^2 |\alpha_{n+1}|\,|\alpha_{n+2}|$. We see that T is of quasi $*$ - paranormal if and only if $|T^*Tx|^2 \leq \left\|T^3x\right\| \|Tx\|$ for each vector $x \in H$ if and only $\|T^*Te_n\|^2 \leq \left\|T^3e_n\right\| \|Tx\|$ for each $n = 1, 2, 3, ...$ Hence T is of quasi $*$ - paranormal if and only if $|\alpha_n|^2 \leq |\alpha_{n+1}|\,|\alpha_{n+2}|$ for $n = 1, 2, 3, ..$ □

2. QUSI $*$ - PARANORMAL COMPOSITION OPERATORS

In this section, we characterize quasi $*$ - paranormal composition operator. Every essentially bounded complex valued measurable function $f_0$induces the bounded operator M_{f_0} on $L^2(\lambda)$, which is defined $M_{f_0}f = f_0 f$ for every $f \in L^2(\lambda)$. Further $C^*C = M_{f_0}$ and $C^{*2}C^2 = M_{f_0^{(2)}}$.

The following Theorem due to Harrington and Whitely is well known.

Theorem 3. *If P denote the projection of L^2 on $\overline{R(C)}$, then $C^*Cf = f_0 f$ and $C^*Cf = (f_0 \circ T)Pf$ for all $f \in L^2(\lambda)$ where P denote the projection L^2 onto $\overline{R(C)}$ and $\overline{R(C)} = \left\{f \in L^2 : f is T^{-1} \sum measurable\right\}$.*

The following theorem characterize quasi $*$ - paranormal composition operators on L^2 space.

Theorem 4. *Let $C \in B\left(L^2(\lambda)\right)$. Then C is of quasi $*$ - paranormal operator if and only if $f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0 \geq 0$, a.e, where P denote the projection L^2 onto $\overline{R(C)}$.*

Proof. Let $C \in B\left(L^2(\lambda)\right)$. Then C is of quasi $*$ - paranormal operator if and only if

$$C^{*3}C^3 - 2\lambda(C^*C)^2 + \lambda^2 C^*C \geq 0$$

.

Thus,

$$\left\langle \left(C^{*3}C^3 - 2\lambda(C^*C)^2 + \lambda^2 C^*C\right)\chi_E, \chi_E \right\rangle \geq 0$$

. for every characteristic function χ_E in $\sum$ such that $\lambda(E) < \infty$. Since $C^{*2}C^2 = M_{f_0^{(2)}}$, $C^*C = M_{f_0}$, we have

$$\left\langle (M_{f_0^{(3)}} - 2\lambda M_{f_0^{(2)}} + \lambda^2 M_{f_0})\chi_E, \chi_E \right\rangle \geq 0,$$

$$\int_E (f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0) \geq 0,$$

for every E in $\sum$. Hence C is of quasi $*$ - paranormal operator if and only if $f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0 \geq 0$ a.e. □

Corollary 5. *Let $C \in B\left(L^2(\lambda)\right)$ with dense range. Then C is of quasi $*$ - paranormal operator if and only if $f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0 \geq 0$, a.e.*

Example 6. Let $X = N$, theset of all natural number and λ be the counting measure on it. Define $T : N \to N$by $T(1) = 1, T(n+m+1) = n$, $m = 0, 1, 2, ..,$ and $n \in N$. Since $f_0^{(3)} - 2\lambda f_0^{(2)} + \lambda^2 f_0 \geq 0$, C is of quasi $*$ - paranormal composition operator.

Theorem 7. *Let $C \in B\left(L^2(\lambda)\right)$. Then $C^* \in$ quasi $*$ - paranormal operator if and only if $[(f_0 \circ T)^{(3)}P] - 2\lambda[(f_0 \circ T)^{(2)}P] + \lambda^2[(f_0 \circ T)P] \geq 0$, a.e, where P denote the projection L^2 onto $\overline{R(C)}$.*

Proof. Then C^* is of quasi $*$ - paranormal operator if and only if

$$C^3C^{*3} - 2\lambda(CC^*)^2 + \lambda^2 CC^* \geq 0,$$

i.e.

$$\left\langle \left(C^{*3}C^3 - 2\lambda(C^*C)^2 + \lambda^2 C^*C\right) f, f\right\rangle \geq 0.$$

We have $\langle CC^*f, f\rangle = \langle (f_0)pf, f\rangle$. where P denote the projection L^2 onto $\overline{R(C)}$. Thus $C^* \in$ quasi $*$ - paranormal operator if and only if

$$(\left\langle [(f_0 \circ T)^{(3)}P] - 2\lambda[(f_0 \circ T)^{(2)}P] + \lambda^2[(f_0 \circ T)P])f, f\right\rangle \geq 0,$$

a.e, for every $f \in L^2$, i.e.

$$[(f_0 \circ T)^{(3)}P] - 2\lambda[(f_0 \circ T)^{(2)}P] + \lambda^2[(f_0 \circ T)P] \geq 0, \text{ a.e.} \qquad \square$$

Theorem 8. *Let $C^* \in B\left(L^2(\lambda)\right)$ with dense range. Then $C^* \in$ quasi $*$ - paranormal operator if and only if $[(f_0 \circ T)^{(3)}P] - 2\lambda[(f_0 \circ T)^{(2)}P] + \lambda^2[(f_0 \circ T)P] \geq 0$, a.e.*

3. WEIGHTED QUASI $*$ - PARANORMAL COMPOSITION OPERATORS

A weighted composition operator(w.c.o)induced by T is a linear transformation acting on the set of complex valued $\sum$ measurable function f, defined as $Wf = w(f \circ T)$, w is a complex valued $\sum$ measurable function. when $w = 1$, we say that W is a composition operator. Let w_k denote $w(w \circ T)(w \circ T^2)........(w \circ T^{k-1})$ so that $W^k f = w_k(f \circ T)^k$. To examine the weighted quasi $*$ - paranormal composition operators effectively Alan Lambert associated conditional expectation operator E with T as $E(./T^{-1}\sum) = E(.)$. $E(f)$ is defined for each non-negative measurable function $f \in L^p(1 \leq p)$ and is uniquely determined by the conditions:

(i) $E(f)$ is $T^{-1}\sum$ measurable.

(ii) If B is any $T^{-1}\sum$ set for which $\int_B f d\lambda$, we have $\int_B f d\lambda = \int_B E(f) d\lambda$.

The projection operator E on L^p is identity if and only if $T^{-1}\sum = \sum$.

Proposition 9. *For $w \geq 0$:*

(i) $W^*Wf = f_0[E(w^2)] \circ T^{-1}f$.

(ii) $WW^*f = w(f_0 \circ T)E(wT)$.

Now we characterize weighted quasi $*$ - paranormal composition operators as follows.

Theorem 10. *W is quasi $*$ - paranormal if and only if*

$$f_0^{(3)}[E(w_3^2)]\circ T^{-3}-2\lambda\left\{f_0[E(w^2)]\circ T^{-1}\right\}^2+\lambda^2 f_0[E(w^2)]\circ T^{-1}\geq 0,\ \ a.e.$$

Proof. W is of quasi $*$ - paranormal operator if and only if

$$W^{*3}W^3-2\lambda(W^*W)^2+\lambda^2W^*W\geq 0$$

and hence,

$$\left\langle (W^{*3}W^3-2\lambda(W^*W)^2+\lambda^2W^*W)f,f\right\rangle\geq 0$$

for all $f\in L^2$ Since $W^kf=w_k(f_0\circ T^k)$ and $W^kf=f_0^k[E(w_k^2)]\circ T^{-k}$, $W^kW^k=f_0^k[E(w_k^2)]\circ T^{-1}f$ and we have $W^*Wf=f_0[E(w^2)]\circ T^{-1}f$ for $w\geq 0$, and hence

$$\int_E\left\{f_0^{(3)}[E(w_3^2)]\circ T^{-3}-2\lambda\left\{f_0[E(w^2)]\circ T^{-1}\right\}^2+\lambda^2 f_0[E(w^2)]\circ T^{-1}\right\}d\lambda \geq 0,$$

for every $E\in\sum$. And so

$$f_0^{(3)}[E(w_3^2)]\circ T^{-3}-2\lambda\left\{f_0[E(w^2)]\circ T^{-1}\right\}^2+\lambda^2 f_0[E(w^2)]\circ T^{-1}\geq 0 \text{ a.e.} \quad \square$$

Corollary 11. *Let $L^{-1}\sum=\sum$.W is of quasi $*$ - paranormal operator if and only if $f_0^{(3)}(w_3^2)\circ T^{-3}-2\lambda\left\{f_0(w^2)\circ T^{-1}\right\}^2+\lambda^2 f_0(w^2)\circ T^{-1}\geq 0$ a.e.*

The Aluthge transform of T is the operator $\widetilde{T}$given $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ introduced in [1] by Aluthge is the Aluthge transform of T. The idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one some spectral properties but it is closed to being a normal operator. More generally we may have family of operators $T_s:0<s\leq 1$ where $T_s=|T|^sU|T|^{1-s}$. For the composite operator C, the polar decomposition is given $C=U|C|$ where $|C|f=\sqrt{f_0}f$ and $Uf=\frac{1}{\sqrt{f_0\circ T}}f\circ T$. [2] Lambert has given general Aluthge transformation for composition operator as $C_s=|C|^sU|C|^{1-s}$ and $C_sf=\left(\frac{f_0}{f_0\circ T}\right)^{\frac{s}{2}}f\circ T$. That is C_s is the weighted composite multiplicative operator with weight $\pi=\left(\frac{1}{\sqrt{f_0\circ T}}\right)^{\frac{s}{2}}$ where $0<s<1$.

Since C_r is weighted composite multiplication operator it is easy to show that $|C_s| = \sqrt{f_0[E(\pi)^2 \circ T^{-1}]}f$ and $|C_s^*|\, f = \nu E[\nu f]$ where $\nu = \frac{\pi\sqrt{f_0 \circ T}}{[E(\pi\sqrt{f_0 \circ T})^2]^{\frac{1}{2}}}$.

Proposition 12. *For every $n \in N$:*

(i) $C_r^k f = \pi_k \cdot (f \circ T^k)$.

(ii) $C_r^{*k} f = f_0^k \cdot (E_{\pi k} f) \circ T^{-k}$.

(iii) $C_r^{*k} C_r^k f = f_0^k \cdot E(\pi_k^2) \circ T^{-k} f$.

Corollary 13. *Let $L^{-1}\sum = \sum$, $C_s \in B(L^2(\lambda))$. Then C_s is of quasi $*$ - paranormal operator if and only if $f_0^{(3)}(\pi_3^2) \circ T^{-3} - 2\lambda \left\{f_0(\pi^2) \circ T^{-1}\right\}^2 + \lambda^2 f_0(\pi^2) \circ T^{-1} \geq 0$ a.e.*

Proof. Since C_s is of quasi $*$ - paranormal operator with weight $\pi = \left(\frac{f_0}{f_0 \circ T}\right)^{\frac{s}{2}}$ it follows that C_s is of quasi $*$ - paranormal operator if and only if $f_0^{(3)}(\pi_3^2) \circ T^{-3} - 2\lambda \left\{f_0(\pi^2) \circ T^{-1}\right\}^2 + \lambda^2 f_0(\pi^2) \circ T^{-1} \geq 0$ a.e. □

The second Aluthge Transformation T described by B.P. Duggle is given by $\widehat{T} = \left|\widehat{T}\right|^{\frac{1}{2}} V \left|\widehat{T}\right|^{\frac{1}{2}}$ and $\left|\widehat{T}\right| = V\left|\widehat{T}\right|$ is the polar decomposition of $\left|\widehat{T}\right|$.

Senthilkmar and Prasad studied that the operator $\widehat{C} = \left|\widehat{C_s}\right|^{\frac{1}{2}} V \left|\widehat{C_s}\right|^{\frac{1}{2}}$ and $\left|\widehat{C_s}\right| = V\left|\widehat{C_s}\right|$ is the polar decomposition of $\left|\widehat{C_s}\right|$: $0 < s < 1$ is weighted composition operator with weight $w' = J^{\frac{1}{4}}\pi\left(\frac{\chi_{supJ}}{J^{\frac{1}{4}}} \circ T\right)$ where $J = f_0 E(\pi^2) \circ T^{-1}$

Corollary 14. *Let $L^{-1}\sum = \sum$, $\widehat{C} \in B(L^2(\lambda))$. Then $\widehat{C}$ is of quasi $*$ - paranormal operator if and only if*

$$f_0^{(3)}(w_3^{'2}) \circ T^{-3} - 2\lambda \left\{f_0(w^{'2}) \circ T^{-1}\right\}^2 + \lambda^2 f_0(w^{'2}) \circ T^{-1} \geq 0,$$

a.e.

Proof. Since $\widehat{C}$ is of quasi $*$ - paranormal operator with weight

$$w' = J^{\frac{1}{4}}\pi\left(\frac{\chi_{supJ}}{J^{\frac{1}{4}}} \circ T\right),$$

it follows that $\widehat{C}$ is of quasi $*$ - paranormal operator if and only if $f_0^{(3)}(w_3^{'2}) \circ T^{-3} - 2\lambda \left\{f_0(w^{'2}) \circ T^{-1}\right\}^2 + \lambda^2 f_0(w^{'2}) \circ T^{-1} \geq 0$ a.e. □

4. QUASI $*$ - PARANORMAL COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

The set $H^2(\gamma)$ of formal complex power series $f(z)=\sum_{n=0}^{\infty} a_n Z^n$ such that $\|f\|_\gamma^2=\sum_{n=0}^{2}|a_n|^2\gamma_n^2<\infty$ is the general Hardy space of functions analytic in the unit disc with inner product $\langle f,g\rangle_\gamma=\sum_{n=0}^{\infty} a_n\overline{b_n}\gamma_n^2$ for f as above and $g(z)=\sum_{n=0}^{\infty} b_n Z^n$ and $\gamma=\{\gamma_n\}_{n=0}^{\infty}$ be sequence of positive number with $\gamma_0=1$ and $\frac{\gamma_{n+1}}{\gamma_n}\to 1$ as $n\to\infty$.

If ϕ is n analytic function mapping the unit disc D into itself, we define the composition operator C_ϕ on the spaces $H^2(\gamma)$ by $C_\phi=f_0\phi$. Though the operator C_ϕ are defined everywhere on the classical Hardy space H^2, they are not necessarily defined on all of $H^2(\beta)$. the composition operator C_ϕ is defined $H^2(\gamma)$ only when the function ϕ is analytic function on some open set containing the closed unit disc having supremum norm strickly smaller than one.

The properties of composition operator on the general Hardy spaces $H^2(\gamma)$ are studied. In this section, we investigate the properties of k- quasi - paranormal composition operators on general Hardy space $H^2(\gamma)$.

For a sequence γ as above and a point w in D, let $k_w\gamma(z)=\sum_{n=0}^{2}\frac{1}{\gamma_2^n}(\overline{w_z})^n$. Then the function $k_w\gamma$ is a point evaluation $H^2(\gamma)$ i.e, for $f\in H^2(\gamma)$, $(f,k_w\gamma)_\gamma=f(w)$.

Then $k_0\gamma=1$ and $C^*k_w\gamma=k_{\phi(w)}\gamma$.

Theorem 15. *If C_ϕ is quasi $*$ - paranormal on $H^2(\gamma)$ then* $\left\|k_{\phi(0)}^{\gamma}\right\|_\gamma^2-1\geq 0$

Proof. Let C_ϕ is quasi $*$ - paranormal on $H^2(\gamma)$. By the definition of quasi $*$ - paranorma,

$$C_\phi^{*3}C_\phi^3-2\lambda(C_\phi^*C_\phi)^2+\lambda^2C_\phi^*C_\phi\geq 0,$$
$$\left\langle\left(C_\phi^{*3}C_\phi^3-2\lambda(C_\phi^*C_\phi)^2+\lambda^2C_\phi^*C_\phi\right)f,f\right\rangle\geq 0,$$
$$\left\langle\left(C_\phi^{*3}C_\phi^3f,f\right)\right\rangle-2\left\langle\left(\lambda(C_\phi^*C_\phi)^2f,f\right)\right\rangle+\lambda^2\left\langle\left(C_\phi^*C_\phi\right)f,f\right\rangle\geq 0,$$
$$\left\|C_\phi^3f\right\|^2-2\lambda\left\|C_\phi^*C_\phi f\right\|^2+\lambda^2\left\|C_\phi f\right\|^2\geq 0,$$
$$\left\|C_\phi^2(C_\phi f)\right\|^2-2\lambda\left\|C_\phi^*(C_\phi f)\right\|^2+\lambda^2\left\|C_\phi f\right\|^2\geq 0,$$
$$\left\|C_\phi^2(C_\phi k_0\gamma)\right\|^2-2\lambda\left\|C_\phi^*(C_\phi k_0\gamma)\right\|^2+\lambda^2\left\|C_\phi k_0\gamma\right\|^2\geq 0,$$

$$\left\|C_\phi^2(k_0\gamma)\right\|^2 - 2\lambda\left\|C_\phi^*(k_0\gamma)\right\|^2 + \lambda^2\left\|k_0\gamma\right\|^2 \geq 0.$$

Repeating the steps for 2 more we get

$$\begin{aligned}\|(k_0\gamma)\|^2 - 2\lambda\|(k_0\gamma)\|^2 + \lambda^2\|k_0\gamma\|^2 \geq 0,\\ 1 - 2\lambda\left\|k^\gamma_{\phi(0)}\right\|^2_\gamma + \lambda^2 \geq 0.\end{aligned}$$

By elementary properties of real quadratic form we get $\left\|k^\gamma_{\phi(0)}\right\|^2_\gamma - 1 \geq 0.$ □

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