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GREATEST STRONGLY CONNECTED SUBSPACES OF A NETWORK IN PRETOPOLOGY

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ABSTRACT: In this paper, we present properties of greatest strongly connected subspaces in the case of a network (which is defined as a family of pretopologies). The network can be analyse by the union or by the intersection or by the composition of the different pretopologies.

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1. INTRODUCTION

In Pretopology (see [1][2][3][4]), a network is defined as a family of pretopologies. Most often, it is studied by the union or by the intersection or by the composition of the different pretopologies constituting it (see [5], [3]).

The aim of this paper is to give results concerning the relationship between the decomposition of a network studied by the union (or by the intersection or by the composition) of the pretopologies of which it is formed and the decomposition of each pretopological space constituting it.

We highlight algorithms for searching the greatest strongly connected subspaces of a network given the strongly connected subspaces of each pretopological space of the network.

2. DIFFERENT TYPES OF PRETOPOLOGICAL SPACES (SEE [1], [2], [4])

Definition 1. Let X be a non empty set. P(X) denotes the family of subsets of X. We call pseudoclosure on X any mapping a from P(X) onto P(X) such as :

$$a(\emptyset) = \emptyset$$
$$\forall\ \mathrm{A} \subset \mathrm{X},\ \mathrm{A} \subset a(\mathrm{A})$$

(X, a) is then called pretopological space.

We can define 4 different types of pretopological spaces.

1- (X, a) is a V type pretopological space if and only if

$$\forall\ \mathrm{A} \subset \mathrm{X}, \forall\ \mathrm{B} \subset \mathrm{X},\ \mathrm{A} \subset \mathrm{B} \Rightarrow a(\mathrm{A}) \subset a(\mathrm{B}).$$

2- (X, a) is a V_D type pretopological space if and only if

$$\forall\ \mathrm{A} \subset \mathrm{X}, \forall\ \mathrm{B} \subset \mathrm{X},\ a(\mathrm{A} \cup \mathrm{B}) = a(\mathrm{A}) \cup a(\mathrm{B}).$$

3- (X, a) is a V_S type pretopological space if and only if

$$\forall\ \mathrm{A} \subset \mathrm{X},\ a(A) = \bigcup_{x\in A} a(\{x\}).$$

4- (X, a) a V_D type pretopological space, is a topological space if and only if

$$\forall\ \mathrm{A} \subset \mathrm{X},\ a(a(\mathrm{A})) = a(\mathrm{A}).$$

Property 2. *If (X, a) is a V_S space then (X, a) is a V_D space. If (X, a) is a V_D space then (X, a) is a V space.*

Example 3. Let X be a non empty set and R be a binary relationship defined on X.

The pretopology of descendants, noted a_d is defined by:

$\forall$ A $\subset$ X, a_d(A) = { x$\in$ X/ R(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R(x)= { y $\in$ X/ x R y }.

The pretopology of ascendants, noted a_a is defined by:
$\forall$ A $\subset$ X, a_a(A) = { x$\in$ X/ R^{-1}(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R^{-1}(x) = { y $\in$ X/ y R x }.

These prétopologies are V_S ones.

3. DIFFERENT PRETOPOLOGICAL SPACES DEFINED FROM A SPACE (X, $\mathcal{A}$) AND CLOSURES (SEE [1], [2] ,[5])

Definition 4. Let (X, a) be a V pretopological space. Let A $\subset$ X. A is a closed subset if and only if a(A) = A.

We note $\forall$ A $\subset$ X, a^0(A) = A and $\forall$n, n$\geq$ 1, a^n(A) = $a(a^{n-1})$(A).

We name closure of A the subset of X, denoted F_a(A), which is the smallest closed subset which contains A.

Remark 5. F_a(A) is the intersection of all closed subsets which contain A. In the case where (X, a) is a "general" pretopological space (i.e. is not a V space, nor a V_D space, nor a V_S space, nor a topological space), the closure may not exist.

Proposition 6. *Let (X, a) be a V space. Let A $\subset$ X. If one of the two following conditions is fulfilled :*

- X is a finite set

- a is of V_S type

then $F_a(A) = \bigcup_{n\geq 0} a^n(A)$.

Remark 7. If a is of V type then a^n and F_a also are of V type. If a is of V_S type then a^n and F_a are also of V_S type.

Definition 8. Let (X, a) be a V pretopological space. Let A $\subset$ X. We define the induced pretopology on A by a, denoted a_A, by :
$\forall$ C $\subset$ A, a_A(C) = a(C) $\cap$ A.

(A, a_A) (or more simply A) is said pretopological subspace of (X, a).

We note F_{aA} the closure according to a_A.

We note $(F_a)_A$ the closing obtained by restriction of closing F_a on A. $(F_a)_A$ is such as $\forall$ C $\subset$ A, $(F_a)_A$(C) = F_a(C) $\cap$ A.

4. STRONG CONNECTIVITY IN (X, A) (SEE [1], [2], [4], [6], [7], [8], [9], [10])

Definition 9. Let (X, a) be a V pretopological space. Let A a non empty subset of X. Let B a non empty subset of X. There exists a path in (X, a) from B to A if and only if B $\subset$ F_a(A).

Definition 10. Let (X, a) be a V pretopological space.

(X, a) is strongly connected if and only if $\forall$ C $\subset$ X, C $\neq \emptyset$, F_a(C) = X.

Proposition 11 (see [1][6]). *Let (X, a) be a V pretopological space.*

(X, a) is strongly connected $\Leftrightarrow$ $\forall$ $x \in X$ and $\forall$ $y \in X$, there exists a path in (X, a) from { y } to { x }.

Definition 12. Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.

A is a strongly connected subset of (X, a) if and only if A endowed with $(F_a)_A$ is strongly connected.

A is a strongly connected component of (X, a) if and only if A is a strongly connected subset of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X with A $\neq$ B, B is not a strongly connected subset of (X, a).

A is a strongly connected subspace of (X, a) if and only if (A, a_A), as a pretopological space, is strongly connected.

A is a greatest strongly connected subspace of (X, a) if and only if (A, a_A) is a strongly connected subspace of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X and A $\neq$ B, (B, a_B) is not a strongly connected subspace of (X, a).

Proposition 13 (see [4]). *Let (X, a) be a V pretopological space. Let $A \subset X$ with A non empty.*

A is a strongly connected subset of (X, a) if and only if $\forall$ $x \in A$ and $\forall$ $y \in A$, there exists a path in (X, a) from { y } to { x }.

5. DEFINITION OF A NETWORK AND DIFFERENT CLOSURES (SEE [5])

Definition 14. Let X a non empty set. Let I a countable family of indices. The family { (X, a_i), $i \in$I } of pretopological spaces is a network on X.

Definition 15. Let X a non empty set. For any pretopologies a_1 and a_2 defined on X, for any subset A of X, we define the three following mappings :

$(a_1 \cup a_2)(A) = a_1(A) \cup a_2(A)$ [union of pretopologies]

$(a_1 \cap a_2)(A) = a_1(A) \cap a_2(A)$ [intersection of pretopologies]

$(a_1 \odot a_2)(A) = a_1(a_2(A))$ [composition of pretopologies].

More generally, in a network { (X, a_i), i∈I } such as for any i∈I, a_i is of V type, we note F_{ai} the closure according to a_i, $F_{\cup}$ (respectively $F_{\cap}$) the closure according to $\bigcup_{i\in I} a_i$ (respectively $\bigcap_{i\in I} a_i$), $F_{\cup F}$ (respectively $F_{\cap F}$) the closure according to $\bigcup_{i\in I} F_{ai}$ (respectively $\bigcap_{i\in I} F_{ai}$).

We define the mapping, denoted $\prod_{i\in I} a_i$, from P(X) onto P(X) by:
$\forall$ A $\subset$ X, $\prod_{i\in I} a_i(A) = \{$ x∈X/ there exists n∈I such as x $\in a_n(a_{n-1}(\ldots (a_1(A))\ldots)) \}$.

And we denote $F_{\prod}$ the closure according to $\prod_{i\in I} a_i$ and $F_{\prod F}$ the closure according to $\prod_{i\in I} F_{ai}$.

Let σ a permutation of I. We denoted $\prod_{i\in I} a_{\sigma(i)}$, the mapping from P(X) onto P(X) by :
$\forall$ A $\subset$ X, $\prod_{i\in I} a_{\sigma(i)}(A) = \{$ x∈X/ there exists n∈I such as x $\in a_{\sigma(n)}(a_{\sigma(n-1)} (\ldots (a_{\sigma(1)}(A))\ldots)) \}$.

And we denote $F_{\prod \sigma}$ the closure according to $\prod_{i\in I} a_{\sigma(i)}$ and $F_{\prod F\sigma}$ the closure according to $\prod_{i\in I} F_{a\sigma(i)}$.

For any subset A of X, we note F_{aiA} the closure according to a_{iA}, for any i∈I, $F_{(\cup)A}$ (respectively $F_{(\cap)A}$) the closure according to $(\bigcup_{i\in I} a_i)_A$ (respectively $(\bigcap_{i\in I} a_i)_A$), $F_{(\cup F)A}$ (respectively $F_{(\cap F)A}$) the closure according to $(\bigcup_{i\in I} F_{ai})_A$ (respectively $(\bigcap_{i\in I} F_{ai})_A$), $F_{(\prod)A}$ (respectively $F_{(\prod F)A}$) the closure according to $(\prod_{i\in I} a_i)_A$ (respectively $(\prod_{i\in I} F_{ai})_A$), $F_{\cup A}$ (respectively $F_{\cap A}$) the closure according to $\bigcup_{i\in I} a_{iA}$ (respectively $\bigcap_{i\in I} a_{iA}$), $F_{\cup FA}$ (respectively $F_{\cap FA}$) the closure according to $\bigcup_{i\in I} F_{aiA}$ (respectively $\bigcap_{i\in I} F_{aiA}$), $F_{\prod A}$ (respectively $F_{\prod FA}$) the closure according to $\prod_{i\in I} a_{iA}$ (respectively $\prod_{i\in I} F_{aiA}$), $F_{\prod \sigma A}$ (respectively $F_{\prod F\sigma A}$) the closure according to $\prod_{i\in I} a_{\sigma(i)A}$ (respectively $\prod_{i\in I} F_{a\sigma(i)A}$), $F_{(\prod \sigma)A}$ (respectively $F_{(\prod F\sigma)A}$) the closure according to $(\prod_{i\in I} a_{\sigma(i)})_A$ (respectively $(\prod_{i\in I} F_{a\sigma(i)})_A$).

Proposition 16 (see [5]). *Let* $\{ (X, a_i), i \in I \}$ *a network such as for any* $i \in I$, a_i *is of* V *type.*

i- $F_{\cup} = F_{\cup F} = F_{\prod F} = F_{\prod} = F_{\prod \sigma} = F_{\prod F\sigma}$.

ii- $\forall\, A \subset X$, $F_{\cap}(A) \subset F_{\cap F}(A)$.

Proposition 17. *Let* $\{ (X, a_i), i \in I \}$ *a network such as for any* $i \in I$, a_i *is of* V *type. Let* $A \subset X$ *and* $C \subset A$.

i- $\bigcup_{i\in I} a_{iA} = (\bigcup_{i\in I} a_i)_A$ *and* $\bigcup_{i\in I} F_{aiA}(C) \subset F_{(\cup)A}(C) = F_{\cup A}(C)$.

ii- $(\bigcap_{i\in I} a_i)_A = \bigcap_{i\in I} a_{iA}$ *and* $F_{\cap A}(C) = F_{(\cap)A}(C) \subset \bigcap_{i\in I} F_{aiA}(C)$.

Proof. i- $(\bigcup_{i\in I} a_{iA})(\mathrm{C}) = \bigcup_{i\in I} a_i(\mathrm{C}) \cap \mathrm{A} = (\bigcup_{i\in I} a_i)(\mathrm{C}) \cap \mathrm{A} = (\bigcup_{i\in I} a_i)_A(\mathrm{C})$. This result implies $\mathrm{F}_{\cup A}(\mathrm{C}) = \mathrm{F}_{(\cup)A}(\mathrm{C})$. We have also $\bigcup_{i\in I} F_{aiA}(C) \subset \mathrm{F}_{\cup A}(\mathrm{C})$ (see Proposition 2-i in [5]).

ii- $(\bigcap_{i\in I} a_i)_A(\mathrm{C}) = (\bigcap_{i\in I} a_i)(\mathrm{C}) \cap \mathrm{A} = \bigcap_{i\in I} a_i(\mathrm{C}) \cap \mathrm{A} = \bigcap_{i\in I} a_{iA}(\mathrm{C})$. This result implies $\mathrm{F}_{\cap A}(\mathrm{C}) = \mathrm{F}_{(\cap)A}(\mathrm{C})$. We have also $\mathrm{F}_{\cap A}(\mathrm{C}) \subset \bigcap_{i\in I} F_{aiA}(C)$ (see Proposition 2-ii in [5]).

Proposition 18. *Let* $\{ (X, a_i), i \in I \}$ *a network such as for any* $i \in I$, a_i *is of* V *type. Let* $A \subset X$ *and* $C \subset A$.

i- $F_{(\cup)A}(C) = F_{\cup A}(C) = F_{\cup FA}(C) \subset F_{(\cup F)A}(C)$.

ii- $F_{\cap A}(C) = F_{(\cap)A}(C) \subset F_{\cap FA}(C) \subset F_{(\cap F)A}(C)$.

Proof. i- $\mathrm{F}_{(\cup)A}(\mathrm{C}) = \mathrm{F}_{\cup A}(\mathrm{C})$ (Proposition 17-i) and $\mathrm{F}_{\cup A} = \mathrm{F}_{\cup FA}$(Proposition 16-i).

We have also $(\bigcup_{i\in I} F_{ai})_A(\mathrm{C}) = (\bigcup_{i\in I} F_{ai})(\mathrm{C}) \cap \mathrm{A} = \bigcup_{i\in I} F_{ai}(\mathrm{C}) \cap \mathrm{A} = \bigcup_{i\in I}(F_{ai})_A(\mathrm{C})$.

Moreover, for any $i \in \mathrm{I}$, $\mathrm{F}_{aiA}(\mathrm{C}) \subset \mathrm{A}$ and $\mathrm{F}_{aiA}(\mathrm{C}) \subset \mathrm{F}_{ai}(\mathrm{C})$ then $\mathrm{F}_{aiA}(\mathrm{C}) \subset \mathrm{F}_{ai}(\mathrm{C}) \cap \mathrm{A} = (\mathrm{F}_{ai})_A(\mathrm{C})$ and then

$$\bigcup_{i\in I} F_{aiA}(C) \subset \bigcup_{i\in I}(F_{ai})_A(C) = (\bigcup_{i\in I} F_{ai})_A(C).$$

So we have $\mathrm{F}_{\cup FA}(\mathrm{C}) \subset \mathrm{F}_{(\cup F)A}(\mathrm{C})$ (see remark of paragraph 5 in [5]).

ii- $\mathrm{F}_{\cap A} = \mathrm{F}_{(\cap)A}$ (Proposition 17-ii) and $\mathrm{F}_{\cap A}(\mathrm{C}) \subset \mathrm{F}_{\cap FA}(\mathrm{C})$ (Proposition 16-ii).

Moreover

$$(\bigcap_{i\in I} F_{ai})_A(C) = (\bigcap_{i\in I} F_{ai})(C) \cap A = \bigcap_{i\in I} F_{ai}(C) \cap A = \bigcap_{i\in I}(F_{ai})_A(C).$$

We have also for any $i \in I$, $F_{aiA}(C) \subset A$ and $F_{aiA}(C) \subset F_{ai}(C)$ then

$$F_{aiA}(C) \subset F_{ai}(C) \cap A = (F_{ai})_A(C)$$

and then

$$\bigcap_{i \in I} F_{aiA}(C) \subset \bigcap_{i \in I} (F_{ai})_A(C) = (\bigcap_{i \in I} F_{ai})_A(C).$$

So we have $F_{\cap FA}(C) \subset F_{(\cap F)A}(C)$ (see remark of paragraph 5 in [5]).

Remark 19. The inclusions of i- and ii- may be strict.

Examples 20. i- Let { (X, a_i), $i \in I$ } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 pretopologies of descendants defined respectively by the following graphs 1 and 2 :

x	R(x)
a	{ b }
b	ø
c	ø

Graph 1

x	R(x)
a	{ b }
b	{ c }
c	ø

Graph 2

Let A = { a, c } and C = { c }.

$\bigcup_{i \in I} F_{aiA}(C) = C$ then $F_{\cup FA}(C) = C$. Moreover,

$$(\bigcup_{i \in I} F_{ai})_A(C) = (\bigcup_{i \in I} F_{ai})(C) \cap A = X \cap A = A$$

and then

$$F_{(\cup F)A}(C) = A.$$

ii- $F_{\cap A}(C) \subset F_{\cap FA}(C)$ (see paragraph 5 in [5]).

Let { (X, a_i), $i \in I$ } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 respectively pretopology of ascendants and pretopology of descendants defined by the following graph 3 :

x	R(x)
a	{ b }
b	{ c }
c	{ a }

Graph 3

Let A = { a, c } and C = { c }. $\bigcap_{i\in I} F_{aiA}$(C) = C then F$_{\cap FA}$(C) = C. Moreover,

$$(\bigcap_{i\in I} F_{ai})_A(C) = (\bigcap_{i\in I} F_{ai})(C) \cap A = X \cap A = A$$

then F$_{(\cap F)A}$(C) = A.

Proposition 21. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let $A \subset X$. Let σ a permutation of I.*

i- $F_{\prod A} = F_{\prod \sigma A}$.

ii- $F_{\prod FA} = F_{\prod F\sigma A}$.

Proof. i- and ii- See Proposition 16-i.

Remark 22. Generally speaking, F$_{(\prod)A}$ ≠ F$_{(\prod\sigma)A}$ and F$_{(\prod F)A}$ ≠ F$_{(\prod F\sigma)A}$.

Example 23. Let { (X, a_i), i∈I } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 two pretopologies of descendants defined by the following graphs 4 and 5 :

x	R(x)
a	∅
b	{ a }
c	∅

Graph 4

x	R(x)
a	∅
b	∅
c	{ b }

Graph 5

Let A = { a, c } and C = { a }.

$(\prod_{i\in I} a_i)_A$(C) = $a_2(a_1$(C)) ∩ A = a_2({ a, b }) ∩ A = X ∩ A = A then $F_{(\prod)A}$(C) = A. Moreover, $(\prod_{i\in I} a_{\sigma(i)})_A$(C) = $a_1(a_2$(C)) ∩ A = a_1({ a }) ∩ A = { a, b } ∩ A = { a } then $F_{(\prod\sigma)A}$(C) = C.

In the same way, $(\prod_{i\in I} F_{ai})_A$(C) = $F_{a2}(F_{a1}$(C)) ∩ A = A then $F_{(\prod F)A}$(C) = A. Moreover, $(\prod_{i\in I} F_{a\sigma(i)})_A$(C) = $F_{a1}(F_{a2}$(C)) ∩ A = C then $F_{(\prod F\sigma)A}$(C) = C.

Proposition 24. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let A ⊂ X. Let C ⊂ A.*

i- $\prod_{i\in I} a_{iA}(C) \subset (\prod_{i\in I} a_i)_A(C)$.

ii- $F_{\prod A}(C) \subset F_{(\prod)A}(C)$.

Proof. i- For any i∈I, a_{iA}(C) ⊂ a_i(C) then $\prod_{i\in I} a_{iA}$(C) ⊂ $\prod_{i\in I} a_i$(C).

Moreover, for any i∈I, a_{iA}(C) ⊂ A then $\prod_{i\in I} a_{iA}$(C) ⊂ A.

So we have

$$\prod_{i\in I} a_{iA}(C) \subset (\prod_{i\in I} a_i(C)) \cap A$$

then

$$\prod_{i\in I} a_{iA}(C) \subset (\prod_{i\in I} a_i)_A(C).$$

ii- $\prod_{i\in I} a_{iA}$(C) ⊂ $(\prod_{i\in I} a_i)_A$(C) then $F_{\prod A}$(C) ⊂ $F_{(\prod)A}$(C) (see remark of paragraph 5 in [5]).

Remark 25. The inclusions of i- and ii- may be strict.

Example 26. Let { (X, a_i), i∈I } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 pretopologies of descendants defined respectively by the previous graphs 4 and 5. Let A = { a, c } and C = { a }.

$(\prod_{i\in I} a_{iA})$(C) = $a_2(a_1$(C) ∩ A) ∩ A = a_2({ a }) ∩ A = { a } = C then $F_{\prod A}$(C) = C. Moreover, $(\prod_{i\in I} a_i)_A$(C) = $a_2(a_1$(C)) ∩ A = a_2({ a, b }) ∩ A = X ∩ A = A then $F_{(\prod)A}$(C) = A.

Proposition 27. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let A ⊂ X. Let C ⊂ A.*

$F_{\prod FA}(C) \subset F_{(\prod F)A}(C)$.

Proof. For any i∈I, a_{iA}(C) ⊂ a_i(C) then for any i∈I, $\prod_{i\in I} F_{aiA}$(C) ⊂ $\prod_{i\in I} F_{ai}$(C).

Moreover, $\prod_{i\in I} F_{aiA}$(C) ⊂ A then $\prod_{i\in I} F_{aiA}$(C) ⊂ $(\prod_{i\in I} F_{ai}$(C)) ∩ A.

So we have $\prod_{i\in I} F_{aiA}$(C) $\subset$ $(\prod_{i\in I} F_{ai})_A$(C) and then $F_{\prod FA}$(C) $\subset$ $F_{(\prod F)A}$(C) (see remark of paragraph 5 in [5]).

Remark 28. The inclusion can be strict.

Example 29. Let { (X, a_i), $i\in$I } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 pretopologies of descendants defined respectively by the previous graphs 1 and 2. Let A = { a, c } and C = { c }.

$\prod_{i\in I} F_{aiA}$(C) = C then $F_{\prod FA}$(C) = C. Moreover,

$$(\prod_{i\in I} F_{ai})_A(C) = (\prod_{i\in I} F_{ai}(C)) \cap A = X \cap A = A$$

and then $F_{(\prod F)A}$(C) = A.

Proposition 30. *Let { (X, a_i), $i\in I$ } a network such as for any $i\in I$, a_i is of V type. Let $A \subset X$.*

i- $F_{\prod FA} = F_{\prod A}$.

ii- $\forall$ $C \subset A$, $F_{(\prod)A}(C) \subset F_{(\prod F)A}(C)$.

Proof. i- See proposition 16-i.

ii- $\forall$ C $\subset$ X, $\forall$ $i\in$I, a_i(C) $\subset$ F_{ai}(C) then $\forall$ C $\subset$ X, $(\prod_{i\in I} a_i)$(C) $\subset$ $(\prod_{i\in I} F_{ai})$(C) and then $\forall$ C $\subset$ A, $(\prod_{i\in I} a_i)$(C) $\subset$ $(\prod_{i\in I} F_{ai})$(C).

So we have $\forall$ C $\subset$ A, $(\prod_{i\in I} a_i)$(C) $\cap$ A $\subset$ $(\prod_{i\in I} F_{ai})$(C) $\cap$ A.

And then $\forall$ C $\subset$ A, $(\prod_{i\in I} a_i)_A$(C) $\subset$ $(\prod_{i\in I} F_{ai})_A$(C).

So $\forall$ C $\subset$ A, $F_{(\prod)A}$(C) $\subset$ $F_{(\prod F)A}$(C) (see remark of paragraph 5 in [5]).

. **Remark 31.** The inclusion can be strict in ii-.

Example 32. Let { (X, a_i), $i\in$I } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 pretopologies of descendants defined respectively by the following graphs 6 and 7.

x	R(x)
a	ø
b	{ a }
c	{ b }

Graph 6

x	R(x)
a	{ a }
b	{ b }
c	ø

Graph 7

Let A = { a, c } and C = { a }.

$(\prod_{i\in I} a_i)_A$(C) = $a_2(a_1$(C)) ∩ A = a_2({ a, b }) ∩ A = C then $F_{(\prod)A}$(C) = C.

Moreover, $(\prod_{i\in I} F_{ai})_A$(C) = $F_{a2}(F_{a1}$(C)) ∩ A = F_{a2}(X) ∩ A = A then $F_{(\prod F)A}$(C) = A.

Proposition 33. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let A ⊂ X.*

i- $F_{\prod FA} = F_{\cup A}$.

ii- ∀ $C \subset A$, $F_{(\cup)A}(C) = F_{\cup A}(C) \subset F_{(\prod F)A}(C)$.

Proof. i- See proposition 16-i.

ii-$F_{(\cup)A}$(C) = $F_{\cup A}$(C) (Proposition 17-i)

then $F_{(\cup)A}$(C) = $F_{\prod FA}$(C) (see i-)

and then $F_{(\cup)A}$(C) ⊂ $F_{(\prod F)A}$(C) (Proposition 27).

Remark 34. The inclusion can be strict in ii-.

Example 35. Let { (X, a_i), i∈I } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 pretopologies of descendants defined respectively by the previous graphs 4 and 5. Let A = { a, c } and C = { a }.

$(\bigcup_{i\in I} a_i)_A$(C) = ($\bigcup_{i\in I} a_i$)(C) ∩ A = C then $F_{(\cup)A}$(C) = C.

Moreover, $(\prod_{i\in I} F_{ai})_A$(C) = $F_{a2}(F_{a1}$(C)) ∩ A = A then $F_{(\prod F)A}$(C) = A.

Consequence. Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let A ⊂ X. Let C ⊂ A.

$F_{(\cup)A}$(C) = $F_{\cup A}$(C) = $F_{\cup FA}$(C) = $F_{\prod A}$(C) = $F_{\prod \sigma A}$(C) = $F_{\prod FA}$(C) = $F_{\prod F\sigma A}$(C) ⊂ $F_{(\prod)A}$(C) ⊂ $F_{(\prod F)A}$(C) (Propositions 17-i, 18-i, 21, 24-ii, 27, 30, 33).

6. GREATEST STRONGLY CONNECTED SUBSPACES IN A NETWORK

Definition 36 (see [1][3]). Let X a non empty set. Let a_1 and a_2 two pretopologies on X.

a_1 is thinner than a_2 if and only if $\forall$ C $\subset$ X, a_1(C) $\subset$ a_2(C).

Proposition 37. *Let X a non empty set. Let a_1 and a_2 two V type pretopologies on X such as a_1 thinner than a_2. Let $A \subset X$ with A non empty.*

i- *a_{1A} is thinner than a_{2A}.*

ii- *If A is a strongly connected subspace of (X, a_1) then A is a strongly connected subspace of (X, a_2).*

Proof. i- If a_1 is thinner than a_2

Then $\forall$ C $\subset$ X, a_1(C) $\subset$ a_2(C) (by definition)

And then $\forall$ C $\subset$ X, a_1(C) $\cap$ A $\subset$ a_2(C) $\cap$ A

So we have $\forall$ C $\subset$ A, a_{1A}(C) $\subset$ a_{2A}(C).

ii- If A is a strongly connected subspace of (X, a_1)

then $\forall$ C $\subset$ A, C $\neq \emptyset$, F_{a1A}(C) = A (by définition)

then $\forall$ C $\subset$ A, C $\neq \emptyset$, A $\subset$ F_{a1A}(C)

and then $\forall$ C $\subset$ A, C $\neq \emptyset$, A $\subset$ F_{a2A}(C) $\subset$ A (see i- and remark of paragraph 5 in [5]).

So we have $\forall$ C $\subset$ A, C $\neq \emptyset$, F_{a2A}(C) = A

Then A is a strongly connected subspace of (X, a_2).

Proposition 38. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let $A \subset X$ with A non empty.*

i- *If A is a strongly connected subspace of (X, $\bigcup_{i\in I} a_i$) then A is a strongly connected subspace of (X, $\bigcup_{i\in I} F_{ai}$).*

ii- *If A is a strongly connected subspace of (X, $\bigcup_{i\in I} a_i$) then A is a strongly connected subspace of (X, $\prod_{i\in I} a_i$).*

iii- *If A is a strongly connected subspace of (X, $\prod_{i\in I} a_i$) then A is a strongly connected subspace of (X, $\prod_{i\in I} F_{ai}$).*

iv- *If A is a strongly connected subspace of (X, $\bigcap_{i\in I} a_i$) then A is strongly connected subspace of (X, $\bigcap_{i\in I} F_{ai}$).*

Proof. i- Let's show that $\bigcup_{i\in I} a_i$ is thinner than $\bigcup_{i\in I} F_{ai}$:

$\forall$ C $\subset$ X, $\forall$ i∈I, a_i(C) $\subset$ F_{ai}(C) then $\bigcup_{i\in I} a_i(C) \subset \bigcup_{i\in I} F_{ai}(C)$.

So we can apply the result of Proposition 37-ii.

ii- A is a strongly connected subspace of (X, $\bigcup_{i\in I} a_i$)
So $\forall$ C $\subset$ A, C $\neq \emptyset$, $F_{(\cup)A}$(C) = A (by definition)

Then $\forall$ C $\subset$ A, C $\neq \emptyset$, A $\subset$ $F_{(\cup)A}$(C)

And then $\forall$ C $\subset$ A, C $\neq \emptyset$, A $\subset$ $F_{(\prod)A}$(C) $\subset$ A (Propositions 17-i, 30-i, 33-i, 24-ii).

So we have $\forall$ C $\subset$ A, C $\neq \emptyset$, $F_{(\prod)A}$(C) = A

Then A is a strongly connected subspace of (X, $\prod_{i\in I} a_i$).

iii- A is a strongly connected subspace of (X, $\prod_{i\in I} a_i$)
So $\forall$ C $\subset$ A, C $\neq \emptyset$, $F_{(\prod)A}$(C) = A (by definition)

Then $\forall$ C $\subset$ A, C $\neq \emptyset$, A $\subset$ $F_{(\prod)A}$(C)

And then $\forall$ C $\subset$ A, C $\neq \emptyset$, A $\subset$ $F_{(\prod F)A}$(C) $\subset$ A (Proposition 30-ii).

So we have $\forall$ C $\subset$ A, C $\neq \emptyset$, $F_{(\prod F)A}$(C) = A

Then A is a strongly connected subspace of (X, $\prod_{i\in I} F_{ai}$).

iv- Let's show that $\bigcap_{i\in I} a_i$ is thinner than $\bigcap_{i\in I} F_{ai}$:

$\forall$ C $\subset$ X, $\forall$ $i$$\in$I, a_i(C) $\subset$ F_{ai}(C) then $\bigcap_{i\in I} a_i(C) \subset \bigcap_{i\in I} F_{ai}(C)$.

So we can apply the result of Proposition 37-ii.

Consequence. The greatest strongly connected subspaces of (X, $\bigcup_{i\in I} a_i$) are strongly connected subspaces of (X, $\prod_{i\in I} a_i$) but, generally speaking, they are not greatest strongly connected subspaces of (X, $\prod_{i\in I} a_i$). Moreover, if A is greatest strongly connected subspace of (X, $\prod_{i\in I} a_i$) then A isn't, generally speaking, greatest strongly connected subspace of (X, $\prod_{i\in I} a_{\sigma(i)}$) (see Remark 22).

Proposition 39. *Let { (X, a_i), i$\in$I } a network such as for any i$\in$I, a_i is of V type. Let A $\subset$ X with A non empty.*

i- If there exists i$\in$I such as A strongly connected subspace of (X, a_i) then A is strongly connected subspace of (X, $\bigcup_{i\in I} a_i$).

ii- If A is strongly connected subspace of (X, $\bigcap_{i\in I} a_i$) then for any i$\in$I, A is strongly connected subspace of de (X, a_i).

Proof. i- $\forall$ $i$$\in$I, a_i is thinner than $\bigcup_{i\in I} a_i$. We get the result according to Proposition 37-ii.

ii- $\forall$ $i$$\in$I, $\bigcap_{i\in I} a_i$ is thinner than a_i. We get the result according to Proposition 37-ii.

Remark 40. The converses of i- and ii- are not true generally speaking.

Examples 41.

i- Let { (X, a_i), i∈I } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 pretopologies of descendants defined respectively by the following graphs 8 and 9 :

x	R(x)
a	{ b }
b	{ a }
c	∅

Graph 8

x	R(x)
a	∅
b	{ c }
c	{ b }

Graph 9

X is strongly connected subspace of (X, $\bigcup_{i\in I} a_i$) but X is not strongly connected subspace of (X, a_1) and X is not strongly connected subspace of (X, a_2).

ii- Let { (X, a_i), i∈I } a network with X = { a, b, c }, I = { 1, 2 }, a_1 and a_2 respectively pretopology of ascendants and pretopology of descendants defined by the previous graph 3. X is strongly connected subspace of (X, a_1) and strongly connected subspace of (X, a_2) but X is not strongly connected subspace of (X, $\bigcap_{i\in I} a_i$).

Consequence. Given Proposition 39-ii, as studying the strongly connected components (see [3]), it does not seem possible to find a more judicious algorithm from the study of each a_i for the study of $\bigcap_{i\in I} a_i$. We will take into account only the case of $\bigcup_{i\in I} a_i$ and the case of $\prod_{i\in I} a_i$.

Proposition 42. *Let { (X, a_i), $i\in I$ } a network such as for any $i\in I$, a_i is of V type. Let { C_k, $k\in K$ } a family of subsets non empty of X such as :*

1- $\bigcup_{k\in K} C_k = X$

2- ∀ $k\in K$, there exists { A_j, $j\in J$ } a family of subsets non empty of X such as:

2-1- $C_k = \bigcup_{j\in J} A_j$

2-2- ∀ $j\in J$, there exists $i\in I$, A_j greatest strongly connected subspace of (X, a_i)

2-3- $\forall$ $j \in J$, $\forall j' \in J$, there exists a sequence $j_0 \ldots j_r$ of elements of J such as $j_0 = j$, $j_r = j'$ and $\forall$ $l = 0,\ldots,r\text{-}1$, $A_{jl} \cap A_{jl+1} \neq \emptyset$

2-4- $\forall$ $A' \subset X$, $A' \notin \{ A_j, j \in J \}$, if there exists $i \in I$ such as A' greatest strongly connected subspace of (X, a_i) then $A' \cap C_k = \emptyset$.

We have :

i- $\forall$ $k \in K$, C_k strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$.

ii- $\{ C_k, k \in K \}$ is a partition of X.

Proof. i- $\forall$ $j \in$J, there exists $i \in$I, A_j greatest strongly connected subspace of (X, a_i)

then $\forall$ $j \in$J, there exists $i \in$I, A_j strongly connected subspace of (X, a_i)

then $\forall$ $j \in$J, A_j is strongly connected subset of (X, $\bigcup_{i \in I} a_i$) (Proposition 39-i).

Morever, the union of two strongly connected subspaces with a non empty intersection is a strongly connected subspace (see [1]) hence the result.

ii- It is sufficient to show that $\forall$ $k \in$K, $\forall$ $k' \in$K with $k \neq k'$, $C_k \cap C_{k'} = \emptyset$ which is ensured by 2.

Remark 43. C_k is not greatest strongly connected subspace of (X, $\bigcup_{i \in I} a_i$) generally speaking.

Example 44. Let { (X, a_i), $i \in$I } a network with X = { a, b, c, d, e, f, g }, I = { 1, 2 }, a_1 and a_2 prétopologies of descendants defined respectively by the following graphs 10 and 11 :

x	R(x)
a	{ b }
b	{ a }
c	{ d }
d	{ c }
e	{ d }
f	ø
g	ø

Graph 10

x	R(x)
a	ø
b	{ e }
c	{ f }
d	ø
e	{ b }
f	{ g }
g	{ a, c }

Graph 11

Let A_1 = { a, b }. A_1 is greatest strongly connected subspace of (X, a_1). Let A_2 = { e, b }. A_2 is greatest strongly connected subspace of (X, a_2). Let C = $A_1 \cup A_2$. { A_1, A_2 } checks the conditions of the Proposition so C is strongly connected subspace of (X, $\bigcup_{i\in I} a_i$) but C is not greatest strongly connected subspace of (X, $\bigcup_{i\in I} a_i$). Indeed, X is greatest strongly connected subspace of (X, $\bigcup_{i\in I} a_i$).

Proposition 45. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. The same conditions apply as in Proposition 42.*

Let { F_q, q∈Q } the family of the strongly connected components of (X, $\bigcup_{i\in I} a_i$).

Let $A \subset X$ with A non empty.

A is a greatest strongly connected subspace of (X, $\bigcup_{i\in I} a_i$)

⇔ There exists q∈ Q, there exists Q_A and there exists Q_q such as $Q_A \subset Q_q \subset K$ with $F_q = \bigcup_{k\in Qq} C_k$ such as

1- $A = \bigcup_{k\in Q_A} C_k$

2- $F_{\cup A}(C_k) = A, \forall\ k \in Q_A$

3- ∀ B, $A \subset B \subset F_q$ and $A \neq B$, $B = \bigcup_{k\in Q_B} C_k$ with $Q_A \subset Q_B \subset Q_q$ and $Q_A \neq Q_B$, there exists k∈ Q_B, $F_{\cup B}(C_k) \neq B$.

Proof. See [2], [4] and Proposition 42.

Proposition 46. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. The same conditions apply as in Proposition 42.*

Let { F_q, q∈Q } the family of the strongly connected components of (X, $\bigcup_{i\in I} a_i$).

Let $A \subset X$ with A non empty.

A is a greatest strongly connected subspace of $(X, \prod_{i \in I} a_i)$

$\Leftrightarrow$ *There exists* $q \in Q$*, there exists* Q_A *and there exists* Q_q *such as* $Q_A \subset Q_q \subset K$ *with* $F_q = \bigcup_{k \in Qq} C_k$ *such as*

1- $A = \bigcup_{k \in Q_A} C_k$

2- $F_{(\prod)A}(C_k) = A, \forall\ k \in Q_A$

3- $\forall\ B,\ A \subset B \subset F_q$ *and* $A \neq B$, $B = \bigcup_{k \in Q_B} C_k$ *with* $Q_A \subset Q_B \subset Q_q$ *and* $Q_A \neq Q_B$*, there exists* $k \in Q_B$, $\mathrm{F}_{(\prod)B}(\mathrm{C}_k) \neq \mathrm{B}$.

Proof. See [2], [4], Propositions 42, 38-ii and Corollary 21 in [3].

7. CONCLUSION

Finally, if X is a finite set, we can give two algorithms to find greatest strongly connected subspaces of (X, $\bigcup_{i \in I} a_i$) (respectively (X, $\prod_{i \in I} a_i$)).

The first algorithm is to disregard the above results and therefore to consider $\bigcup_{i \in I} a_i$ (respectively $\prod_{i \in I} a_i$) as a pretopology (see [4]).

The following second algorithm uses the results developped in this paper (Propositions 42, 45 and 46). In particular, it will be used when greatest strongly connected subspaces are known for each pretopology of the network:

- Build the family of the strongly connected components { F_q, q∈Q } of (X, $\bigcup_{i \in I} a_i$) (see [3])
- For any i∈I, find the greatest strongly connected subspaces of (X, a_i)
- Build { C_k, k∈K } by joining step by step all greatest strongly connected subspaces with a non empty intersection (Proposition 42)
- All subset A defined as in Proposition 45 (respectively Proposition 46) is a greatest strongly connected subspace of (X, $\bigcup_{i \in I} a_i$) (respectively of (X, $\prod_{i \in I} a_i$)).

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