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EXISTENCE OF SOLUTIONS OF A FRACTIONAL COMPARTMENT MODEL WITH PERIODIC BOUNDARY CONDITION

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ABSTRACT: In this paper, the authors study a type of fractional order single compartment models with periodic boundary conditions. An explicit solution for the associated linear model is first derived. Then a series of conclusions on the existence and uniqueness of solutions for the nonlinear model are proved.

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1. INTRODUCTION

In this paper, we consider a boundary value problem (BVP) consisting of the nonlinear fractional differential equation (FDE)

$$u' + aD_{0+}^{1-\alpha}u = f(u,t),\ 0 < t < \omega, \tag{1.1}$$

and the periodic boundary condition (BC)

$$u(0) = u(\omega), \tag{1.2}$$

where $0 < \alpha < 1$, $a > 0$, $f \in C(\mathbb{R} \times [0,\omega], \mathbb{R})$, and $D_{0+}^{1-\alpha}u$ is the $(1-\alpha)$-th left Riemann-Liouville fractional derivative of $u : [0,\infty) \to \mathbb{R}$ defined by

$$(D_{0+}^{1-\alpha}u)(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided the right-hand side exists, where $\Gamma(\cdot)$ is the Gamma function. Note that the α-th left Riemann-Liouville fractional integral of u is defined by

$$(I_{0+}^{\alpha}u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds.$$

Therefore, $(D_{0+}^{1-\alpha}u) = \left(I_{0+}^{\alpha}u\right)'$.

FDEs have extensive applications in various fields of science and engineering, and have been a focus of research for decades; see [1, 2, 5, 6, 7, 10, 12, 13, 14] and the references therein. The involvement of fractional terms allows us to consider the impacts of the long-term memory in time or long-range spatial interactions to the problems, which often lead to interesting models with very good performance.

As a classic modeling approach, differential equations (DEs) have been extensively employed to describe the transition among various compartments of objects. This type of DEs are often referred as compartment models and have been widely applied to investigate both macro and micro level phenomena such as epidemics, in-host pathogen dynamics, and pharmacokinetics, etc.. One well-known example is the so-called susceptible, infected, removed (SIR) models and their variations. The reader is referred to [1, 2, 15, 3, 4, 9] and references therein for more information about compartment models and their applications.

Recently, Angstmann *et al.* [1] proposed a method to derive fractional order compartment models by analyzing the underlying stochastic processes. Both single compartment and multiple compartment models were investigated. In particular, a fractional order governing equation for a single compartment model was given in [1] by

$$\rho' = q^+(t) - w(t)\rho - \tau^{-\alpha}\Theta(t,0) D_{0+}^{1-\alpha}\left(\frac{\rho}{\Theta(t,0)}\right), \tag{1.3}$$

where q^+ denotes the arrival flux of the compartment, $\tau^{-\alpha}\Theta(t,0)D_{0+}^{1-\alpha}\left(\frac{\rho}{\Theta(t,0)}\right)$ and $w(t)\rho$ represent two removal processes of the compartment; see [1] for the details. It is easy to see that when $w(t) \equiv 0$ and $\Theta(t,0) \equiv 1$, Eq. (1.3) becomes a special case of FDE (1.1). In other words, FDE (1.1) may be interpreted as a single compartment model with one arrival flux and one removal process.

In practice, many problems modeled by compartment models will reoccur after a certain period. So it is natural to consider under what circumstance, the phenomena will repeat. Motivated by this question, we will investigate the existence and uniqueness of solutions of BVP (1.1), (1.2). If only the left (or right) Riemann-Liouville fractional derivatives are involved, the major feasible approach to study the existence of solutions of a fractional order BVP is to convert it to a fixed point problem for an appropriate operator. Many of these operators are constructed based on the establishment of the associated Green's functions. However, due to the unusual features of the fractional calculus, the Green's functions for various fractional BVPs are yet to be developed, especially when the linear FDEs involve multiple terms of the unknown function u and/or its derivative(s), e.g.

$$u' + aD_{0+}^{1-\alpha}u = 0. \tag{1.4}$$

Graef, Kong, Kong, and Wang proposed an approach based on the spectrum theory to construct the Green's functions for the FDEs involving multiple terms of u or its derivative(s). By this approach, the Green's functions may be constructed as series of functions, see for example [7, 8, 5]. One limitation of this approach is that it is difficult to prove the positivity of the Green's functions due to the complexity of the series of functions. This hurdle seriously restricts the application of such Green's functions.

In this paper, we will use an alternative method to construct the Green's function for the BVP consisting of Eq. (1.4) and BC (1.2). The Green's function will be given in terms of the Mittag-Leffler functions. The positivity and lower/upper bounds of the Green's function will also be investigated. These findings will offer more flexibility when fixed point theory is used to study the existence and multiplicity of solutions of the nonlinear BVP (1.1), (1.2).

The paper is organized as follows: after this introduction, the main results are stated in Section 2. One example is also given therein. All the proofs are given in Section 3.

2. MAIN RESULTS

We first consider an associated linear BVP consisting of the equation

$$u' + aD_{0+}^{1-\alpha}u = h(t),\ 0 < t < \omega, \tag{2.1}$$

and BC (1.2). Let $G : [0, \omega] \times [0, \omega]$ be defined by

$$G(t,s) = \begin{cases} \frac{\Lambda(t)\Lambda(\omega-s)}{1-\Lambda(\omega)}, & 0 \le t < s \le \omega, \\ \frac{\Lambda(t)\Lambda(\omega-s)}{1-\Lambda(\omega)} + \Lambda(t-s), & 0 \le s \le t \le \omega, \end{cases} \tag{2.2}$$

with

$$\Lambda(t) = E_\alpha(-at^\alpha), \tag{2.3}$$

where E_α is the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}.$$

Remark 2.1. (a) It is well-known that E_α is an entire function hence defined on $\mathbb{R}$. In addition when $\alpha \in (0, 1]$, it is clear that $\Lambda'(t) \le 0$ on $(0, \infty)$, $\Lambda(0) = 1$, $\lim_{t\to\infty} \Lambda(t) = 0$, and $0 < \Lambda(\omega) < 1$; see for example [11]. Therefore, G is a well defined piecewise function.

(b) When $\alpha = 1$, Eq. (2.1) becomes

$$u' + au = h(t).$$

By (2.2) and the fact that $E_1(-at) = e^{-at}$,

$$G(t,s) = \begin{cases} \frac{e^{a(s-t)}}{e^{a\omega}-1}, & 0 \le t < s \le \omega, \\ \frac{e^{a(\omega+s-t)}}{e^{a\omega}-1}, & 0 \le s \le t \le \omega. \end{cases}$$

Our first result is about the uniqueness of solutions of BVP (2.1), (1.2).

Theorem 2.1. *Assume $h \in C[0,\omega]$. BVP* (2.1), (1.2) *has a unique solution given by*

$$u(t) = \int_0^{\omega} G(t,s)h(s), \tag{2.4}$$

where G is defined by (2.2).

It is easy to see that G is the Green's function for BVP (1.4), (1.2). Now we are ready to study the nonlinear BVP (1.1), (1.2). Define

$$\gamma = \frac{\Lambda^2(\omega)}{2-\Lambda(\omega)} \quad \text{and} \quad \beta = \max_{t\in[0,\omega]} \left\{ \int_0^{\omega} G(t,s)ds \right\}. \tag{2.5}$$

Note from Remark 2.1 (a) and (2.2) that $0 < \gamma < 1$ and $0 < \beta < \infty$.

For $u \in C[0,\omega]$, the Banach space of continuous functions on $[0,\omega]$, we denote by $\|u\|$ the standard maximum norm of u. The next theorem is a result on the existence of positive solutions of BVP (1.1), (1.2).

Theorem 2.2. *If there exist $0 < r_* < r^*$ [respectively, $0 < r^* < r_*$], such that*

$$f(x,t) \le \beta^{-1} r_* \quad \text{for all } (x,t) \in [\gamma r_*, r_*] \times [0,\omega] \tag{2.6}$$

and

$$f(x,t) \ge \beta^{-1} r^* \quad \text{for all } (x,t) \in [\gamma r^*, r^*] \times [0,\omega]. \tag{2.7}$$

Then BVP (1.1), (1.2) *has at least one positive solution u with $r_* \le \|u\| \le r^*$ [respectively, $r^* \le \|u\| \le r_*$].*

By applying Theorem 2.2 multiple times, it is easy to see the following results hold.

Corollary 2.1. *Let $\{r_i\}_{i=1}^N \subset \mathbb{R}$ such that $0 < r_1 < r_2 < r_3 < \cdots < r_N$. Assume either*

(a) f satisfies (2.6) *with $r = r_i$ when i is odd, and satisfies* (2.7) *with $r = r_i$ when i is even; or*

(b) f satisfies (2.6) *with $r = r_i$ when i is even, and satisfies* (2.7) *with $r = r_i$ when i is odd.*

Then BVP (1.1), (1.2) *has at least $N-1$ positive solutions u_i with $r_i < \|u_i\| < r_{i+1}$, $i = 1, 2, \ldots, N-1$.*

Corollary 2.2. *Let $\{r_i\}_{i=1}^{\infty} \subset \mathbb{R}$ such that $0 < r_1 < r_2 < r_3 < \cdots$. Assume either*

(a) f satisfies (2.6) *with $r_* = r_i$ when i is odd, and satisfies* (2.7) *with $r^* = r_i$ when i is even; or*

(b) f satisfies (2.6) *with $r_* = r_i$ when i is even, and satisfies* (2.7) *with $r^* = r_i$ when i is odd.*

Then BVP (1.1), (1.2) *has an infinite number of positive solutions.*

The next result is about the uniqueness of solutions of BVP (2.1), (1.2).

Theorem 2.3. *Assume there exists $\lambda \in (0, 1/\beta)$ such that for any $x_1, x_2 \in \mathbb{R}$ and $t \in [0, \omega]$,*

$$|f(x_1, t) - f(x_2, t)| \leq \lambda |x_1 - x_2|. \tag{2.8}$$

Then BVP (1.1), (1.2) *has a unique solution.*

To demonstrate the application of our results, let us consider the following example.

Example 1. Let $f(x,t) = x^k$, $k > 0$ and $k \neq 1$. Then BVP (1.1), (1.2) has at least one positive solution.

In fact, if $k > 1$, then

$$\lim_{x \to 0+} \frac{x^k}{x} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{x^k}{x} = \infty.$$

There exist sufficiently small r_* and sufficiently large r^* such that (2.6) and (2.7) hold. If $0 < k < 1$, then

$$\lim_{x \to 0+} \frac{x^k}{x} = \infty \quad \text{and} \quad \lim_{x \to \infty} \frac{x^k}{x} = 0.$$

There exist sufficiently large r_* and sufficiently small r^* such that (2.6) and (2.7) hold. Then by Theorem 2.2, BVP (1.1), (1.2) has at least one positive solution for both cases.

3. PROOFS

We will need the following result to prove Theorem 2.1, see [13] for the details.

Lemma 3.1. *Let* $\Lambda(t)$ *be defined by* (2.3)*. Then the Laplace transform of* Λ *is*

$$\mathcal{L}\left\{\Lambda\right\}(s) = \frac{s^{\alpha-1}}{s^{\alpha}+a}. \tag{3.1}$$

Then Theorem 2.1 is proved by the Laplace transform method.

Proof of Theorem 2.1. Let $\hat{h}$ be a function defined on $[0, \infty)$ such that

(1) $\hat{h}(t) = h(t)$ on $[0, \omega]$;

(2) The Laplace transform $\mathcal{L}\left\{\hat{h}\right\}$ exists.

Consider the FDE

$$u' + aD_{0+}^{1-\alpha}u = \hat{h}(t), \ t > 0. \tag{3.2}$$

By taking the Laplace transform of (3.2), we have

$$s\mathcal{L}\left\{u\right\} - u(0) + as^{1-\alpha}\mathcal{L}\left\{u\right\} = \mathcal{L}\left\{\hat{h}\right\},$$

or

$$\mathcal{L}\left\{u\right\} = \frac{s^{\alpha-1}}{s^{\alpha}+a}\left(\mathcal{L}\left\{\hat{h}\right\} + u(0)\right).$$

By Lemma 3.1 and the convolution theorem of Laplace transform, we have

$$u(t) = \int_0^t \Lambda(t-s)\hat{h}(s)ds + u(0)\Lambda(t), \ t > 0. \tag{3.3}$$

By (1.2),

$$u(0) = u(\omega) = \int_0^{\omega} \Lambda(\omega - s)h(s)ds + u(0)\Lambda(\omega).$$

Then

$$u(0) = \int_0^{\omega} \frac{\Lambda(\omega - s)}{1 - \Lambda(\omega)}h(s)ds.$$

Then by (3.3) for any $t \in [0, \omega]$,

$$u(t) = \int_0^t \Lambda(t-s)h(s)ds + \int_0^\omega \frac{\Lambda(t)\Lambda(\omega-s)}{1-\Lambda(\omega)} h(s)ds$$
$$= \int_0^\omega G(t,s)h(s)ds,$$

where G is defined by (2.2). The uniqueness follows immediately. □

We will use the fixed point theory to prove the existence and uniqueness of solutions of the nonlinear BVP (1.1), (1.2). The next lemma plays an important role.

Lemma 3.2. *Let G be the Green's function defined by* (2.2). *Then $G(t,s) \geq 0$ on $[0,\omega] \times [0,\omega]$. Furthermore, for any $(t,s) \in [0,\omega] \times [0,\omega]$,*

$$\frac{\Lambda^2(\omega)}{1-\Lambda(\omega)} \leq G(t,s) \leq \frac{2-\Lambda(\omega)}{1-\Lambda(\omega)}. \tag{3.4}$$

Proof. For any (t,s) with $0 \leq t < s \leq \omega$, by (2.2), we have

$$\frac{\partial G}{\partial s} = -\frac{\Lambda(t)\Lambda'(\omega-s)}{1-\Lambda(\omega)} \geq 0.$$

Therefore,

$$\frac{\Lambda(t)\Lambda(\omega-t)}{1-\Lambda(\omega)} \leq G(t,s) \leq \frac{\Lambda(t)}{1-\Lambda(\omega)}. \tag{3.5}$$

Similarly, for any (t,s) with $0 \leq s \leq t \leq \omega$, by (2.2), we have

$$\frac{\partial G}{\partial s} = -\frac{\Lambda(t)\Lambda'(\omega-s)}{1-\Lambda(\omega)} - \Lambda'(t-s) \geq 0$$

and

$$\frac{\Lambda(t)}{1-\Lambda(\omega)} = \frac{\Lambda(t)\Lambda(\omega)}{1-\Lambda(\omega)} + \Lambda(t) \leq G(t,s) \leq \frac{\Lambda(t)\Lambda(\omega-t)}{1-\Lambda(\omega)} + 1. \tag{3.6}$$

By (3.5) and (3.6), for any $(t,s) \in [0,\omega] \times [0,\omega]$,

$$\frac{\Lambda(t)\Lambda(\omega-t)}{1-\Lambda(\omega)} < G(t,s) \leq \frac{\Lambda(t)\Lambda(\omega-t)}{1-\Lambda(\omega)} + 1.$$

Then the conclusions hold by the fact that

$$\frac{\Lambda(t)\Lambda(\omega-t)}{1-\Lambda(\omega)} \geq \frac{\Lambda^2(\omega)}{1-\Lambda(\omega)} > 0$$

and
$$\frac{\Lambda(t)\Lambda(\omega-t)}{1-\Lambda(\omega)}+1\le\frac{1}{1-\Lambda(\omega)}+1=\frac{2-\Lambda(\omega)}{1-\Lambda(\omega)}.$$
□

Let $(C[0,\omega],\|\cdot\|)$ be the Banach space of continuous functions on $[0,\omega]$ with the standard maximum norm and γ be defined by (2.5). Define a cone K in $C[0,\omega]$ by

$$K=\{u\in C[0,\omega]\mid u(t)\ge 0,\ t\in[0,\omega] \text{ and } \min_{t\in[0,\omega]}u(t)\ge\gamma\|u\|\} \tag{3.7}$$

and an operator $T:\ C[0,\omega]\to C[0,\omega]$ by

$$Tu=\int_0^\omega G(t,s)f(u(s),s)ds,\quad t\in[0,\omega]. \tag{3.8}$$

It is easy to see that u is a solution of BVP (1.1), (1.2) if and only if u is a fixed point of T.

Lemma 3.3. *$T(K)\subset K$ and T is completely continuous.*

Proof. For any $u\in K$, it is easy to see that $Tu\ge 0$ on $[0,\omega]$. By Lemma 3.2 and (3.8), for any $t\in[0,\omega]$

$$\begin{aligned}(Tu)(t) &= \int_0^\omega G(t,s)f(u(s),s)ds\\ &\ge \int_0^\omega \frac{\Lambda^2(\omega)}{1-\Lambda(\omega)}f(u(s),s)ds\\ &= \gamma\int_0^\omega \frac{2-\Lambda(\omega)}{1-\Lambda(\omega)}f(u(s),s)ds.\end{aligned}$$

Since $Tu\in C[0,\omega]$, there exists $t_1\in[0,\omega]$ such that $\|Tu\|=(Tu)(t_1)$. Hence

$$\|Tu\|=\int_0^\omega G(t_1,s)f(u(s),s)ds\le\int_0^\omega\frac{2-\Lambda(\omega)}{1-\Lambda(\omega)}f(u(s),s)ds.$$

Therefore, $\min_{t\in[0,\omega]}(Tu)(t)\ge\gamma\|Tu\|$ and this implies that $T(K)\subset K$. The complete continuity of T can be shown by a standard argument using the Arzela-Arscoli Theorem. We omit the details. □

In the following, for $r > 0$ we let

$$\Omega_r = \{u \in C[0,\omega] \mid \|u\| < r\}. \tag{3.9}$$

Proof of Theorem 2.2. Let K and Ω_r be defined by (3.7) and (3.9), respectively. For any $u \in K \cap \partial\Omega_{r_*}$, $\|u\| = r_*$ and $\gamma r_* \leq u(t) \leq r_*$ on $[0,\omega]$. From (2.6), $f(u(t),t) \leq \beta^{-1} r_*$ on $[0,\omega]$. For any $t \in [0,\omega]$

$$\begin{aligned} (Tu)(t) &= \int_0^\omega G(t,s) f(u(s),s) ds \\ &\leq \beta^{-1} r_* \int_0^\omega G(t,s) ds \leq \beta^{-1} r_* \beta = r_* = \|u\|. \end{aligned}$$

Thus $\|Tu\| \leq \|u\|$. For any $u \in K \cap \partial\Omega_{r^*}$, $\|u\| = r^*$ and $\gamma r^* \leq u(t) \leq r^*$ on $[0,\omega]$. From (2.7), $f(u(t),t) \geq \beta^{-1} r^*$ on $[0,\omega]$. Let $t_2 \in [0,\omega]$ with

$$\int_0^\omega G(t_2,s) ds = \max_{t \in [0,\omega]} \int_0^\omega G(t,s) ds.$$

Then

$$\begin{aligned} (Tu)(t_2) &= \int_0^\omega G(t_2,s) f(u(s),s) ds \\ &\geq \beta^{-1} r^* \int_0^\omega G(t_2,s) a(s) ds = \beta \beta^{-1} r^* = \|u\|. \end{aligned}$$

Thus $\|Tu\| \geq (Tu)(t_2) \geq \|u\|$. Therefore, the conclusion follows from the Krasnosel'skii's fixed point theorem. □

Theorem 2.3 is proved by the contraction mapping theorem. We omit the details.

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