Communications in Applied Analysis, **23**, No. 1 (2019), 115-124 ISSN: 1083-2564

BOUNDED SOLUTION OF A VOLTERRA INTEGRODIFFERENTIAL EQUATION

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ABSTRACT: In this article the existence of a continuous and bounded solution of a nonlinear Volterra integrodifferential equation is studied. In the analysis, Schaefer's fixed point theorem and Liapunov's direct method are employed. The existence of a continuous and bounded solution is shown using Schaefer's fixed point theorem, which requires an *a priori* bound on all such solutions of an auxiliary equation. Liapunov's direct method is then applied to obtain such an *a priori* bound.

AMS Subject Classification: 45D05, 45J05

Received: August 19, 2018; **Accepted:** December 6, 2018;
Published: December 15, 2018 **doi:** 10.12732/caa.v23i1.8
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION

We consider the introgrodifferential equation

$$x'(t) = a(t) - b(t)f(x(t)) + \int_0^t k(t,s)g(s,x(s))ds, \quad x(0) = 0, \quad t \geq 0, \qquad (1.1)$$

where $f : R \to R$, $g : [0,\infty) \times R \to R$, $a, b : [0,\infty) \to R$, are all continuous functions, where $R = (-\infty, \infty)$. Also, $k(t,s)$ is continuous on $0 \le s \le t < \infty$.

Many results are known about the existence of local solutions of (1.1). In particular, it is known that if the functions f and g satisfy certain local Lipschitz conditions, then a solution of (1.1) exists on an interval $[0,T]$ for some $T > 0$. In this paper, we study the existence of a bounded solution of (1.1) on $[0,\infty)$, a unbounded domain. We employ Schaefer's fixed point theorem, stated below, as the main mathematical tool in the analysis. The mapping H in the equation $x = \lambda Hx$ of the Schaefer's fixed point theorem needs to be completely continuous, i.e., it is continuous as a mapping and it maps bounded sets into relatively compact sets. The relative compactness property is normally obtained applying the Arzela-Ascoli theorem when the domain of the problem is bounded. Since the domain of (1.1) is unbounded, Arzela-Ascoli's theorem does not apply. To overcome this problem, we resort to a theorem, Theorem 2.1, which can be found in (see [1, Theorem 4.3.1, page 150]). Kaufmann, Kosmatov, and Raffoul [9] studied a problem on an unbounded domain employing Krasnoselskii's fixed point theorem, which has a contraction mapping and a completely continuous mapping. They used Theorem 2.1 to obtain the relative compactness property of the completely continuous mapping.

Schaefer's fixed point theorem requires an *a priori* bound on all solutions of an auxiliary equation. We apply a variant of Liapunov's direct method to obtain the *a priori* bound. In particular, in Theorem 2.4, we obtain such an *a priori* bound on solutions of auxiliary equation, (2.7), applying Liapunov's method. Then, in Theorem 2.5, we show that (1.1) has a continuous bounded solution on $[0,\infty)$ provided all such solutions of an auxiliary equation, (2.7), have an *a priori* bound for all λ, $0 < \lambda \le 1$. We refer the readers to Burton [3] for basic results on the existence of solutions of (1.1).

We remark that one could use the topological method of Granas [7] to prove Theorem 2.5. The method of Granas would require a compact homotopy mapping $h : [0,1] \times \mathbb{B} \to \mathbb{B}$, where $\mathbb{B}$ is the Banach space of continuous bounded functions on $[0,\infty)$ with the supremum norm. The mapping h can be defined by multiplying the right hand side of equation (2.2) by λ. The compactness of h follows from the similar arguments used in this article for the compactness of the mapping H. The topological method of Granas has been used by many researchers (cf. [4],[5],[6],[8]). In [4] and [5] the method is applied to Volterra integrodifferential equations, and in [6] and [8] it is applied to boundary value problems.

For convenience, we state Schaefer's fixed point theorem below.

Theorem 1.1 (Schaefer's fixed point theorem [10]). *Let $(S, \|\cdot\|)$ be a normed space and let H be a completely continuous mapping of S into S. Then either*

(i) the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

Note that a mapping is completely continuous if it is continuous and maps bounded sets into relatively compact sets. A set B is relatively compact if the closure of B is compact.

2. EXISTENCE OF BOUNDED SOLUTIONS USING SCHAEFER'S FIXED POINT THEOREM

Assume

(A1) (i) $|a(t)| \leq \bar{a} < \infty$,
and
(ii) $|b(t)| \leq \bar{b} < \infty$, for all $t \in [0, \infty)$;

(A2) there exists constants $\bar{f}$ and $\bar{g}$ such that
(i) $|f(x) - f(y)| \leq \bar{f}|x - y|$,
(ii) $|g(t, x) - g(t, y)| \leq \bar{g}|x - y|$;

(A3)

$$\sup_{t \geq 0} \int_0^t |k(t, s)| ds \leq k^* < \infty.$$

By adding $x(t)$ on both sides of (1.1) and using Leibniz method for the linear equation, we have that equation (1.1) can be transformed into the integral equation

$$x(t) = \int_0^t e^{-(t-s)} a(s) ds + \int_0^t e^{-(t-s)} x(s) ds \tag{2.1}$$

$$- \int_0^t e^{-(t-s)} b(s) f(x(s)) ds + \int_0^t e^{-(t-s)} \int_0^s k(s,u) g(u, x(u)) du \, ds.$$

Let $\mathbb{B}$ be the space of bounded continuous functions $\phi : [0, \infty) \to (-\infty, \infty)$ with the supremum norm. Define a mapping H by

$$\begin{aligned} H\phi(t) = & \int_0^t e^{-(t-s)} a(s) ds + \int_0^t e^{-(t-s)} \phi(s) ds \\ & - \int_0^t e^{-(t-s)} b(s) f(\phi(s)) ds + \int_0^t e^{-(t-s)} \int_0^s k(s,u) g(u, \phi(u)) du \, ds. \end{aligned} \tag{2.2}$$

Taking the derivative of this equation one obtains,

$$\begin{aligned} |(H\phi)'(t)| \leq & |a(t)| + |\phi(t)| + \int_0^t e^{-(t-s)} (|a(s)| + |\phi(s)|) ds + |b(t)||f(\phi(t))| \\ & + \int_0^t e^{-(t-s)} |b(s)||f(\phi(s))| ds + \int_0^t |k(t,u)||g(u, \phi(u))| du \\ & + \int_0^t e^{-(t-s)} \int_0^s |k(s,u)||g(u, \phi(u))| du \, ds. \end{aligned}$$

For any $\phi \in \mathbb{B}$, let $|\phi(t)| \leq \bar{\phi} < \infty$. Notice that the assumption (A2) (Lipschitz continuity of f and g) implies that there exists constants f^* and g^* such that $|f(0)| \leq f^* < \infty$ and $|g(t,0)| \leq g^* < \infty$. Now using assumptions (A1)–(A3) one obtains

$$|(H\phi)'(t)| \leq 2(\bar{a} + \bar{\phi}) + 2\bar{b}(\bar{f}\bar{\phi} + f^*) + 2k^*(\bar{g}\bar{\phi} + g^*) := L < \infty. \tag{2.3}$$

Pick any $\phi \in \mathbb{B}$. For any $t_1 \neq t_2 \in [0, \infty)$, one gets, by the Mean Value Theorem and (2.3),

$$\begin{aligned} |(H\phi)(t_1) - (H\phi)(t_2)| & \leq |(H\phi)'(\xi)||t_1 - t_2| \\ & \leq L|t_1 - t_2|. \end{aligned}$$

This shows that $(H\phi)(t)$ is continuous in t. For any $\phi \in \mathbb{B}$, with $|\phi(t)| \leq \bar{\phi} < \infty$, it follows from (2.2), and assumptions (A1)–(A3),

$$|(H\phi)(t)| \leq (\bar{a} + \bar{\phi}) + \bar{b}(\bar{f}\bar{\phi} + f^*) + k^*(\bar{g}\bar{\phi} + g^*) := M < \infty, \tag{2.4}$$

which shows that $(H\phi)(t)$ is bounded on $[0, \infty)$. Therefore, H maps from $\mathbb{B}$ into $\mathbb{B}$.

Now, to apply Schaefer's theorem, we need to show that H is completely continuous, i.e., H is a continuous mapping, and that H maps bounded sets into relatively compact sets.

To show the continuity of the mapping H, let $\phi, \psi \in \mathbb{B}$. Then (2.2), and assumptions (A1)–(A3) yield,

$$\begin{aligned} |(H\phi)(t) - (H\psi)(t)| &\leq ||\phi - \psi|| + \bar{b}(\bar{f}||\phi - \psi||) + k^*(\bar{g}||\phi - \psi||) \\ &\leq ||\phi - \psi||(1 + \bar{b}\bar{f} + k^*\bar{g}). \end{aligned} \tag{2.5}$$

Therefore H is continuous as a mapping.

Now, we'll show that H maps bounded sets into relatively compact sets. To show this, we use the following result [1] for the relative compactness.

Theorem 2.1 (see [1, Theorem 4.3.1, page 150])**.** *Let M be the space of all bounded continuous real-valued functions on $[0, \infty)$, and $S \subset M$. Then S is relatively compact in M if the following conditions hold.*

(i) S is bounded in M,

(ii) the functions in S are equicontinuous on any compact subinterval of $[0, \infty)$,

(iii) the functions in S are equiconvergent, i.e., given $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that $|\phi(t) - \phi(\infty)| < \epsilon$ for all $t > T$ and for all $\phi \in S$.

Let

$K = \{\phi \in \mathbb{B} : ||\phi|| \leq m, \phi(0) = x_0, \lim_{t\to\infty} \phi(t) = \eta\}$,

where $m > 0$, and η is an arbitrary but fixed real number.

Additional assumptions.

(A4) (i) $\lim_{t\to\infty} a(t) = \hat{a} < \infty$,
and
(ii) $\lim_{t\to\infty} b(t) = \hat{b} < \infty$,

(A5) $\lim_{t\to\infty} g(t, x) = g_k < \infty$ uniformly with respect to $x \in K$,

Lemma 2.2. *In addition to assumption (A3), we assume $k(t, s)$ satisfies the following conditions.*

(A6) $\lim_{t\to\infty} \int_0^T k(t, s)ds = 0$, *for each* $T > 0$,

(A7) $\lim_{t\to\infty}\int_0^t k(t,s)ds = \hat{k}$.

Then for every $x \in K$, $\lim_{t\to\infty}\int_0^t k(t,s)x(s)ds = \hat{k}\lim_{t\to\infty} x(t)$.

The proof of Lemma 2.2 follows from the proof of Lemma 5.2 [2].

Lemma 2.3. *The image of the cone* K *under* H, $H(K)$, *is relatively compact where* H *is defined by* (2.2).

To prove that $H(K)$ is relatively compact, we will use the above theorem and show that

(i) $H(K)$ is (uniformly) bounded,

(ii) $H(K)$ is equicontinuous on compact subintervals of $[0,\infty)$,

(iii) $H(K)$ is equiconvergent.

Proof. From (2.4) it follows that $(H\phi)(t)$ is bounded, which can be used to show that $H(K)$ is uniformly bounded. Likewise, from (2.5), it follows that $H(K)$ is equicontinuous on the whole interval $[0,\infty)$, hence on compact intervals of $[0,\infty)$.

Now, we show that $H(K)$ is equiconvergent.
It follows from the continuity of f that there exists a constant $f_\eta := f(\eta)$ such that $\lim_{t\to\infty} f(\phi(t)) = f_\eta$ uniformly with respect to $\phi \in K$.
Using Lemma 2.2, and assumptions (A4)–(A5), we get

$$(H\phi)(\infty) = \lim_{t\to\infty}(H\phi)(t) = \hat{a} + \eta + \hat{b}f_\eta + \hat{k}g_k \tag{2.6}$$

for all $\phi \in K$.

Therefore, for any $\epsilon > 0$, one can easily show that there exists a $T(\epsilon) > 0$ such that $|(H\phi)(t) - (H\phi)(\infty)| < \epsilon$ for all $t > T$, for all $\phi \in K$.

This proves that $H(K)$ is equiconvergent, and hence the proof of Lemma 2.3 is complete.

So, we have shown that the mapping $H : \mathbb{B} \to \mathbb{B}$ is completely continuous.

For the parameter λ used in Schaefer's theorem, we multiply (2.1) by λ and define the auxiliary equation

$$x_\lambda(t) = \lambda\Big[\int_0^t e^{-(t-s)}a(s)ds + \int_0^t e^{-(t-s)}x_\lambda(s)ds$$

$$- \int_0^t e^{-(t-s)} b(s) f(x_\lambda(s)) ds + \int_0^t e^{-(t-s)} \int_0^s k(s,u) g(u, x_\lambda(u)) du ds \Big].$$

Note that the above function is the unique solution of

$$\begin{aligned} x'_\lambda(t) =&(\lambda - 1) x_\lambda(t) + \lambda \Big[a(t) - b(t) f(x_\lambda(t)) + \int_0^t k(t,s) g(s, x_\lambda(s)) ds \Big], \\ x_\lambda(0) =&0, \end{aligned} \tag{2.7}$$

for all λ, $0 < \lambda \leq 1$. We obtain *a priori* bound for x_λ by applying a variant of Liapunov's method as the mathematical tool. □

Theorem 2.4. *In addition to assumptions (A1)–(A7), suppose the following conditions hold.*

(A8) (i) $a(t) \in L^1[0, \infty)$,
(ii) $b(t) > 0$ *for* $t \in [0, \infty)$,
(iii) $xf(x) > 0$ *for all* $x \neq 0$,
(iv) $|f(x)| \geq q|x|$, *for some* $q > 0$,
(v) $g(t, 0) = 0$,

(A9) there exists a constant $\alpha > 0$ *such that*

$$1 - qb(t) + \bar{g} \int_0^t |k(u + t, t)| du \leq -\alpha,$$

for all $t \geq 0$, *where* $\bar{g}$ *is the constant in (A2)(ii).*

Then there exists an a priori bound on all solutions $x_\lambda(t)$, *of* (2.7) *for all* λ, $0 < \lambda \leq 1$.

Proof. Let

$$V(t) = V(t, x_\lambda(\cdot)) = |x_\lambda(t)| + \lambda \int_0^t \int_{t-s}^\infty |k(u + s, s)| du |g(s, x_\lambda(s))| ds. \tag{2.8}$$

Now, differentiating $V(t)$, and using (2.7) along with assumptions (A8)–(A9), we obtain

$$\begin{aligned} V'(t) = &\frac{x_\lambda(t)}{|x_\lambda(t)|} x'_\lambda(t) + \lambda \int_0^\infty |k(u + t, t)| du |g(t, x_\lambda(t))| \\ &- \lambda \int_0^t |k(t,s)||g(s, x_\lambda(s))| ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{x_\lambda(t)}{|x_\lambda(t)|}\Big[(\lambda-1)x_\lambda(t) + \lambda\Big(a(t) - b(t)f(x_\lambda(t))\\
&\quad + \int_0^t k(t,s)g(s,x_\lambda(s))ds\Big)\Big]\\
&\quad + \lambda\int_0^\infty |k(u+t,t)|du|g(t,x_\lambda(t))| - \lambda\int_0^t |k(t,s)||g(s,x_\lambda(s))|ds\\
&\le (\lambda-1)|x_\lambda(t) + |a(t)| - \lambda b(t)|f(x_\lambda(t))| + \lambda\int_0^t |k(t,s)||g(s,x_\lambda(s))|ds\\
&\quad + \lambda\int_0^\infty |k(u+t,t)|du|g(t,x_\lambda(t))| - \lambda\int_0^t |k(t,s)||g(s,x_\lambda(s))|ds\\
&\le (\lambda-1)|x_\lambda(t)| + |a(t)| - \lambda b(t)q|x_\lambda(t)| + \lambda\int_0^\infty |k(u+t,t)|du\bar{g}|x_\lambda(t))|\\
&= |a(t)| + \lambda\Big[1 - qb(t) + \bar{g}\int_0^\infty |k(u+t,t)|du\Big]|x_\lambda(t)| - |x_\lambda(t)|\\
&\le |a(t)| - \lambda\alpha|x_\lambda(t)| - |x_\lambda(t)|
\end{aligned}
$$

Integrating the above inequality yields,

$$V(t) \le V(0) - (1+\lambda\alpha)\int_0^t |x_\lambda(s)|ds + \int_0^t |a(s)|ds.$$

Since $a(t) \in L^1[0,\infty)$, there exists a constant $B > 0$ such that

$$V(t) + (1+\lambda\alpha)\int_0^t |x_\lambda(s)|ds \le V(0) + \int_0^t |a(s)|ds \le B. \tag{2.9}$$

This implies $V(t) \le B$ for all λ, $0 < \lambda \le 1$. Therefore, from (2.8), we get $||x_\lambda|| \le B$ for all λ, $0 < \lambda \le 1$. This B is the required *a priori* bound on all solutions of (2.7). □

Theorem 2.5. *Let assumptions (A1)–(A7) hold. Suppose there exists a constant $B > 0$ such that $||x_\lambda|| \le B$ for all solution functions x_λ of* (**??**) *for all λ, $0 < \lambda \le 1$. Then* (1.1) *has a bounded continuous solution $x(t)$ on* $[0,\infty)$ *with $||x|| \le B$.*

Proof. In the above work, we have shown that when assumptions (A1)–(A7) hold then the mapping H is completely continuous. Since we assumed that there exists an *a priori* bound B for all solution functions x_λ of (**??**) for all λ, $0 < \lambda \le 1$, the conclusion of Theorem 2.5 follows from Schaefer's theorem. □

Remark 2.6. It follows from (2.9) that $x_\lambda(t) \in L^1[0,\infty)$ for all λ, $0 < \lambda \le 1$. Therefore when the assumptions of Theorems 2.4 and 2.5 hold then there exists a bounded continuous solution $x(t)$ of (1.1) on $[0,\infty)$, with $||x|| \le B$ and $x(t) \in L^1[0,\infty)$.

We note that under assumptions (A1)–(A9), the mapping H is not a contraction. Therefore, we have to use Schaefer's Fixed Point Theorem.

Example 2.7. Let $a(t) = \frac{1}{1+t^2}$, $b(t) = 1 + e^{-t}$, $k(t,s) = \frac{1}{1+(t-s)^2}$, $f(x) = qx + \frac{x^3}{1+x^2}$; $q > 0$, and $g(t,x) = \frac{x^3}{1+x^2} + xe^{-t}$.
These functions satisfy all the assumptions we used in this article.

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