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A CLASS OF WELL-POSED APPROXIMATIONS FOR ILL-POSED PROBLEMS IN BANACH SPACES

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ABSTRACT: We classify two well-posed approximations of the abstract Cauchy problem in Banach space, where the governing operator generates a holomorphic semigroup of angle θ. An advantage of the proposed approximations is that regularization for the original ill-posed problem may be established independently of the value of θ. Applications may be drawn for the classic backward heat equation in L^p spaces, $1 < p < \infty$ as well as additional problems with strongly elliptic differential operators of even order.

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1. INTRODUCTION

The abstract Cauchy problem

$$
\begin{aligned}
&u'(t) = Au, \quad 0 < t < T, \\
&u(0) = \varphi
\end{aligned}
\tag{1.1}
$$

where A is a closed linear operator in a Banach space $(X, \|\cdot\|)$ and $\varphi \in X$ has been studied extensively due to its applications in ill-posed problems. For different choices of the operator A, one may extract several different Cauchy problems which model ill-posed and inverse processes such as the backward heat equation, e.g. if $A = -\Delta$ defined in $X = L^p(\mathbb{R}^n)$, $1 < p < \infty$ (cf. [16, 19, 2, 3, 25]). Such problems lack existence, uniqueness, or continuous dependence of solutions on initial data.

In the case in which $-A$ generates a holomorphic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on X, authors such as Showalter [23], Mel'nikova [14], and Huang and Zheng [9, 10] provide techniques which regularize (1.1) in the following sense: there exists a family of bounded operators $\{R_\beta(t) : \beta > 0, t \in [0,T]\} \subseteq B(X)$ with approximation parameter $\beta > 0$ such that for any solution $u(t)$ of (1.1) and any change in initial data $\|\varphi - \varphi_\delta\| \leq \delta$, there exists $\beta = \beta(\delta) > 0$ such that

$$\beta \to 0 \quad \text{as} \quad \delta \to 0, \quad \text{and}$$

$$\|u(t) - R_\beta(t)\varphi_\delta\| \to 0 \quad \text{as} \quad \delta \to 0 \quad \text{for all} \quad t \in [0,T].$$

Continuous dependence on modeling techniques enable us to identify the operators $R_\beta(t)$ (cf. [2, 5]). Here, we define an approximate well-posed problem where A in (1.1) is replaced by a perturbation $f_\beta(A)$ which generates a C_0 semigroup $e^{tf_\beta(A)}$, $t \geq 0$ on X, and then show that the corresponding solutions are β-close. The operators $R_\beta(t)$ may then be thought of as solution operators, i.e. $R_\beta(t)\varphi = e^{tf_\beta(A)}\varphi$.

Historically, two of the most studied approximations are Lattes and Lions's $f_\beta(A) = A - \beta A^2$ [12] and Showalter's $f_\beta(A) = A(I + \beta A)^{-1}$ [23], although to allow for regularization these require the restriction $\theta \in (\frac{\pi}{4}, \frac{\pi}{2}]$. In [9], Huang and Zheng consider the modification $f_\beta(A) = A - \beta A^b$ with fractional power A^b, $1 < b < 2$, allowing for relaxation of the restriction on θ. In [8], Huang suggests a logarithmic approximation originally proposed by Boussetila and Rebbani [3] but Huang's results do not apply to the backward heat equation in Banach space. In this paper, we consider two more reasonable approximations

$$f_\beta(A) = Ae^{-\beta A}, \quad \beta > 0 \tag{1.2}$$

$$f_\beta(A) = (\ln 2)^{-1} A \operatorname{Log}(1 + e^{-\beta A}), \quad \beta > 0 \tag{1.3}$$

both operators being defined by the Dunford integral [18, 21], and e^{-tA} representing the holomorphic semigroup generated by $-A$. We show (Theorems

3.3 and 4.1) that problem (1.1) may be regularized using either of these two approximations for any $\theta \in (0, \pi/2]$, thus maintaining non-restrictive conditions on θ, and also allowing applications to the backward heat equation in $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

2. THE OPERATORS A AND $F_\beta(A)$

Let $\theta \in (0, \frac{\pi}{2}]$. Assume $-A$ is the infinitesimal generator of a bounded holomorphic semigroup e^{-tA} of angle θ on a Banach space X with $0 \in \rho(A) = \{w \in \mathbb{C} \ : \ (w - A)^{-1} \in B(X)\}$. By definition, e^{-tA} extends to an analytic function e^{-wA} defined in the open sector $\{w \in \mathbb{C} : |\arg w| < \frac{\pi}{2}\}$. Furthermore by [21, Theorem X.52], the spectrum $\sigma(A) = \mathbb{C} - \rho(A)$ of A is contained in the closed sector

$$S_{\frac{\pi}{2}-\theta} = \{w \in \mathbb{C} \ : \ |\arg w| \leq \frac{\pi}{2} - \theta\}$$

and for each $\alpha \in (0, \theta)$, there exists $M_\alpha > 0$ such that

$$\|(w - A)^{-1}\| \leq \frac{M_\alpha}{\operatorname{dist}(w, S_{\frac{\pi}{2}-\alpha})} \quad \text{for all} \quad w \in \mathbb{C} - S_{\frac{\pi}{2}-\alpha}. \tag{2.1}$$

First, consider the approximation $f_\beta(A) = Ae^{-\beta A}$ defined by

$$Ae^{-\beta A}x = \frac{1}{2\pi i}\int_{\Gamma_\phi} we^{-\beta w}(w - A)^{-1}x \, dw, \quad x \in X$$

where Γ_ϕ is the boundary of the sector S_ϕ where $\frac{\pi}{2} - \theta < \phi < \frac{\pi}{2}$, oriented so that it runs from $\infty e^{i\phi}$ to $\infty e^{-i\phi}$. By [20, Theorem 2.5.2 (d)], for each $x \in X$, one has $\|Ae^{-\beta A}x\| \leq C\|x\|/\beta$ where C is independent of x and β but dependent upon θ.

To prove similar properties for approximation (1.3), we require two technical lemmas.

Lemma 2.1. *Let $x \in \mathbb{R}, x \neq \frac{n\pi}{2}$ for $n \in \mathbb{Z}$. Then*

$$\sec x \arctan(\cot x) \in (-\frac{\pi}{2}, -1) \cup (1, \frac{\pi}{2}).$$

Proof. Define $g(x) = \sec x \arctan(\cot x)$, $x \in \mathbb{R}, x \neq \frac{n\pi}{2}$ for $n \in \mathbb{Z}$. Note that g is a periodic function with period 2π. Hence, it is sufficient to prove the result for $x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \cup (\pi, \frac{3\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$. First, consider $\theta \in (0, \pi)$. By the identity $\arctan(\cot\theta) = \frac{\pi}{2} - \theta$, we obtain $g(x) = \tilde{g}(x) := \sec x\,(\frac{\pi}{2} - x)$ on $(0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Using elementary calculus, one may show that $\tilde{g}(x)$ satisfies the properties

$$\lim_{x\to 0^+} \tilde{g}(x) = \frac{\pi}{2}, \qquad \lim_{x\to \frac{\pi}{2}^-} \tilde{g}(x) = 1,$$

$$\tilde{g}'(x) < 0 \quad \text{on} \quad (0, \frac{\pi}{2}),$$

which yield $1 < g(x) < \frac{\pi}{2}$ for $0 < x < \frac{\pi}{2}$. Next, since both $\sec x$ and $\frac{\pi}{2} - x$ restricted to $(0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ have odd symmetry with respect to the line $x = \frac{\pi}{2}$, we have that $\tilde{g}(x)$ has even symmetry with respect to $x = \frac{\pi}{2}$. This allows us to conclude that $1 < g(x) < \frac{\pi}{2}$ on $(0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. A similar argument shows that $-\frac{\pi}{2} < g(x) = \sec x\,(\frac{3\pi}{2} - x) < -1$ on $(\pi, \frac{3\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$ as desired. □

Lemma 2.2. *For all $x \geq 0$ and for all $a > 0$,*

$$\left| e^{ax} \arctan\left(\frac{\sin x}{e^{ax} + \cos x}\right)\right| < \frac{\pi}{2}.$$

Proof. Define for $x \geq 0$, $y > 0$,

$$h(x, y) = e^{xy} \arctan\left(\frac{\sin x}{e^{xy} + \cos x}\right).$$

First note $h(n\pi, y) = 0$ for all $y > 0$ and $n = 0, 1, 2 \ldots$. Also, for all $x \neq n\pi$,

$$\lim_{\substack{y\to 0^+ \\ x\neq n\pi}} h(x, y) = \arctan\left(\frac{\sin x}{1 + \cos x}\right) \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

Next, using L'Hospital's Rule,

$$\begin{aligned} \lim_{y\to\infty} h(x, y) &= \lim_{y\to\infty} \frac{e^{2xy} \sin x}{\sin^2 x + (e^{xy} + \cos x)^2} \\ &= \lim_{y\to\infty} \frac{\sin x}{1 + 2e^{-xy}\cos x + e^{-2xy}} = \sin x \in [-1, 1]. \end{aligned}$$

Finally, we have

$$h_x = e^{xy}\left(y \arctan\left(\frac{\sin x}{e^{xy} + \cos x}\right) + \frac{e^{xy}(\cos x - y \sin x) + 1}{\sin^2 x + (e^{xy} + \cos x)^2}\right),$$

$$h_y = e^{xy}\left(x \arctan\left(\frac{\sin x}{e^{xy} + \cos x}\right) - \frac{x e^{xy} \sin x}{\sin^2 x + (e^{xy} + \cos x)^2}\right).$$

For $x, y > 0$, it may be shown that $h_x = h_y = 0$ implies $e^{xy}\cos x + 1 = 0$. Hence $\cos x \neq 0$. Also, we can eliminate the case that $\sin x = 0$ since then $h(x, y) = 0$. Therefore $e^{xy}\cos x + 1 = 0$ and $x \neq n\pi/2$ for $n = 0, 1, 2, \ldots$ imply

$$h(x, y) = \sec x \arctan(\cot x)$$

which by Lemma 2.1 has range in $(-\pi/2, -1) \cup (1, \pi/2)$. Altogether, we have shown $|h(x, y)| < \pi/2$. □

Proposition 2.3. *Let Γ_ϕ be the contour that is the boundary of the sector S_ϕ where $\frac{\pi}{2} - \theta < \phi < \frac{\pi}{2}$, and oriented so that it runs from $\infty e^{i\phi}$ to $\infty e^{-i\phi}$. The operator defined by*

$$(\ln 2)^{-1} A\ Log(1 + e^{-\beta A}) = \frac{1}{2\ln 2\ \pi i}\int_{\Gamma_\phi} w\ Log(1 + e^{-\beta w})(w - A)^{-1} dw$$

is a bounded operator on X satisfying $\|(\ln 2)^{-1} A\ Log(1 + e^{-\beta A})x\| \leq C_\phi \|x\|/\beta$ for all $x \in X$ where C_ϕ is a constant depending on ϕ but independent of β.

Proof. For $w = re^{\pm i\phi} \in \Gamma_\phi$, we have

$$\begin{aligned}
&|\mathrm{Log}(1 + e^{-\beta w})|^2 \\
&= |\mathrm{Log}((1 + e^{-\beta r\cos\phi}\cos(\beta r\sin\phi)) \mp ie^{-\beta r\cos\phi}\sin(\beta r\sin\phi))|^2 \\
&= |\frac{1}{2}\ln(1 + 2e^{-\beta r\cos\phi}\cos(\beta r\sin\phi) + e^{-2\beta r\cos\phi}) \\
&\quad \mp i\arctan\left(\frac{\sin(\beta r\sin\phi)}{e^{\beta r\cos\phi} + \cos(\beta r\sin\phi)}\right)|^2.
\end{aligned}$$

Define the function $q(r) = 1 + 2e^{-\beta r\cos\phi}\cos(\beta r\sin\phi) + e^{-2\beta r\cos\phi}$, $r \geq 0$. We have the following estimate:

$$(1 - e^{-\beta r\cos\phi})^2 \leq q(r) \leq (1 + e^{-\beta r\cos\phi})^2.$$

Now, one can show that

$$1 < (1 - e^{-\beta r\cos\phi})^{-1} \leq (1 + e^{-\beta r\cos\phi})^2$$

for r satisfying $\beta r\cos\phi \geq \ln(\frac{1+\sqrt{5}}{2})$. Hence, for these r, $1 < (1 - e^{-\beta r\cos\phi})^{-2} \leq (1 + e^{-\beta r\cos\phi})^4$ which implies $0 < -2\ln(1 - e^{-\beta r\cos\phi}) \leq 4\ln(1 + e^{-\beta r\cos\phi})$. This shows that for $r \geq r_\phi := \beta^{-1}\sec\phi\ln(\frac{1+\sqrt{5}}{2})$,

$$|\ln(1 + 2e^{-\beta r\cos\phi}\cos(\beta r\sin\phi) + e^{-2\beta r\cos\phi})|$$

$$\begin{aligned}&\leq \max\{-2\ln(1-e^{-\beta r\cos\phi}), 2\ln(1+e^{-\beta r\cos\phi})\}\\&\leq 4\ln(1+e^{-\beta r\cos\phi})\\&\leq 4e^{-\beta r\cos\phi}\end{aligned}$$

where we have used the fact that $\ln(1+x)\leq x$ for all $x\geq 0$.

Next, by Lemma 2.2, we have

$$\left|\arctan\left(\frac{\sin(\beta r\sin\phi)}{e^{\beta r\cos\phi}+\cos(\beta r\sin\phi)}\right)\right| < \frac{\pi}{2}e^{-\beta r\cos\phi}.$$

Thus, for $w = re^{\pm i\phi}$ such that $r\geq r_\phi$ we have

$$\begin{aligned}&|\mathrm{Log}(1+e^{-\beta w})|^2\\&=\frac{1}{4}\ln^2(1+2e^{-\beta r\cos\phi}\cos(\beta r\sin\phi)+e^{-2\beta r\cos\phi})\\&\quad+\arctan^2\left(\frac{\sin(\beta r\sin\phi)}{e^{\beta r\cos\phi}+\cos(\beta r\sin\phi)}\right)\\&\leq 4e^{-2\beta r\cos\phi}+\frac{\pi^2}{4}e^{-2\beta r\cos\phi}.\end{aligned}$$

Meanwhile, by the substitution $s=\beta r$ we can see that the minimum of $q(r) = 1+2e^{-s\cos\phi}\cos(s\sin\phi)+e^{-2s\cos\phi}$ does not depend on β. Hence, if $m_\phi = \min_{r\geq 0} q(r)$, then $0 < m_\phi < q(r) \leq 4$ for all $r\geq 0$ with m_ϕ independent of β. Therefore,

$$\begin{aligned}&|\mathrm{Log}(1+e^{-\beta w})|^2\\&=\frac{1}{4}\ln^2(1+2e^{-\beta r\cos\phi}\cos(\beta r\sin\phi)+e^{-2\beta r\cos\phi})\\&\quad+\arctan^2\left(\frac{\sin(\beta r\sin\phi)}{e^{\beta r\cos\phi}+\cos(\beta r\sin\phi)}\right)\\&\leq \frac{1}{4}\max\{\ln^2(m_\phi),\ln^2(4)\}+\frac{\pi^2}{4} := M_\phi^2\end{aligned}$$

for $0\leq r\leq r_\phi$.

Thus for any $x\in X$, using property (2.1),

$$\begin{aligned}&\ln 2\,\pi\|(\ln 2)^{-1}A\,\mathrm{Log}(1+e^{-\beta A})x\|\\&\leq \frac{1}{2}\int_{\Gamma_\phi}|w|\,|\mathrm{Log}(1+e^{-\beta w})|\,\|(w-A)^{-1}\|\|x\|\,|dw|\\&\leq \frac{1}{2}C'\|x\|\int_{\Gamma_\phi}|\mathrm{Log}(1+e^{-\beta w})|\,|dw|\end{aligned}$$

$$\leq C'\|x\| \left(\int_0^{r_\phi} M_\phi \, dr + \int_{r_\phi}^{\infty} \sqrt{4 + \frac{\pi^2}{4}} \, e^{-\beta r \cos\phi} dr \right)$$
$$\leq C'\|x\| \left(M_\phi r_\phi + \sqrt{4 + \frac{\pi^2}{4}} \, \beta^{-1} \sec\phi \right)$$
$$= C'\|x\| \left(M_\phi \beta^{-1} \sec\phi \ln((1+\sqrt{5})/2) + \sqrt{4 + \frac{\pi^2}{4}} \, \beta^{-1} \sec\phi \right),$$

where C' is a constant independent of β. □

We close this section with an approximation property between the operators A and $f_\beta(A)$ that is motivated by Condition A of Ames and Hughes [2, Definition 1].

Lemma 2.4. *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on a Banach space X with $0 \in \rho(A)$. Let $0 < \beta < 1$. For either approximation* (1.2) *or* (1.3)*, there exists a constant R independent of β such that*

$$\|(-A + f_\beta(A))x\| \leq \beta R \|A^2 x\|$$

for all $x \in Dom(A^2)$.

Proof. Let $x \in Dom(A^2)$. Then since e^{-tA} is uniformly bounded,

$$\begin{aligned} \| - Ax + Ae^{-\beta A}x\| &= \|(I - e^{-\beta A})Ax\| \\ &= \| \int_0^\beta e^{-tA} A^2 x dt\| \\ &\leq \beta \left(\max_{t \geq 0} \|e^{-tA}\| \right) \|A^2 x\|. \end{aligned}$$

On the other hand, it is straightforward that $\mathrm{Log}(1+e^{-\beta A})$ commutes with A on $Dom(A)$. Adjust Γ_ϕ so that it has a small bump of radius δ to the left of the origin. That is define $\Gamma'_\phi = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with

$$\Gamma_1 = \{re^{i\phi} : r \geq \delta\}, \quad \Gamma_2 = \{\delta e^{i\gamma} : \phi \leq \gamma \leq 2\pi - \phi\}, \quad \Gamma_3 = \{re^{-i\phi} : r \geq \delta\}.$$

Then for $x \in Dom(A^2)$, by the substitution $z = \beta w$ together with Cauchy's Theorem and (2.1), we have

$$(\ln 2)\| - Ax + f_\beta(A)x\|$$

$$
\begin{aligned}
&= \|\mathrm{Log}(1+e^{-\beta A})Ax - \ln 2\; Ax\| \\
&= \|\frac{1}{2\pi\beta i}\int_{\beta\Gamma'_\phi} \mathrm{Log}(1+e^{-z})((\beta^{-1}z-A)^{-1} - \beta z^{-1})Ax\; dz\| \\
&= \|\frac{1}{2\pi\beta i}\int_{\Gamma'_\phi} \mathrm{Log}(1+e^{-z})((\beta^{-1}z-A)^{-1} - \beta z^{-1})Ax\; dz\| \\
&= \|\frac{1}{2\pi i}\int_{\Gamma'_\phi} z^{-1}\mathrm{Log}(1+e^{-z})(\beta^{-1}z-A)^{-1}A^2x\; dz\| \\
&\leq \beta\frac{C'}{2\pi}\int_{\Gamma'_\phi} |z|^{-2}\; |\mathrm{Log}(1+e^{-z})|\; \|A^2x\|\; |dz| \\
&\leq \beta\frac{C''}{2\pi}\left(\int_\delta^\infty \frac{dr}{r^2}\right)\|A^2x\| \\
&= \beta\frac{M'}{2\pi}\|A^2x\|
\end{aligned}
$$

where C'' and M' are constants independent of β. □

3. CONTINUOUS DEPENDENCE OF SOLUTIONS

In this section, we apply Lemma 2.4 to prove two estimates which demonstrate continuous dependence on modeling according to either definition (1.2) or (1.3) of $f_\beta(A)$. Let $u(t)$ be a classical solution of (1.1), that is a function $u \in C([0,T]:X)\cap C^1((0,T):X)$ with $u(t) \in Dom(A)$ for all $t \in (0,T)$, satisfying problem (1.1) in X. Further, let $v_\beta(t)$ be the solution of the corresponding problem with A replaced by $f_\beta(A)$. Note, from our discussion in Section 1, then $v_\beta(t) = R_\beta(t)\varphi = e^{tf_\beta(A)}\varphi$. As each operator e^{-tA}, $t \geq 0$ is a bounded, *injective* operator (cf. [4, Lemma 3.1]), we define e^{tA} by $e^{tA}x = (e^{-tA})^{-1}x$ for all $x \in Dom((e^{-tA})^{-1})$.

The following proposition demonstrates a basic estimate between $u(t)$ and $v_\beta(t)$ that may be established by choosing the initial data φ of (1.1) in a small enough domain. In particular, for each definition (1.2) or (1.3) we will find an appropriate Q such that $\varphi \in Dom((e^{-QA})^{-1})$.

Proposition 3.1. *Fix ϕ as in the definition of Γ_ϕ above and let $0 < \epsilon < 1$.*

Choose $Q > T$ *to satisfy the inequality* $b(Q-T)\cos\phi \geq T+\epsilon$ *by the rule*

$$b = \begin{cases} 1 & \text{if} \quad f_\beta(A) = Ae^{-\beta A} \\ \frac{2}{\pi}\ln 2 & \text{if} \quad f_\beta(A) = (\ln 2)^{-1} A \; Log(1+e^{-\beta A}). \end{cases}$$

Then, if $u(t)$ *is a classical solution of* (1.1) *with initial data* $\varphi \in Dom((e^{-QA})^{-1})$, *then*

$$\|u(t) - v_\beta(t)\| \leq \beta K (T-t+\epsilon)^{-1} \|e^{QA}\varphi\| \quad \text{for} \quad 0 \leq t \leq T$$

where K *is a constant independent of* β, ϵ, *and* t.

Proof. Set $\psi = e^{QA}\varphi$. By Lemma 2.4,

$$\begin{aligned} &\|u(t) - v_\beta(t)\| \\ &= \|(I - e^{tf_\beta(A)}e^{-tA})e^{tA}\varphi\| \\ &= \|(I - e^{tf_\beta(A)}e^{-tA})e^{tA}e^{-QA}\psi\| \\ &= \| - \int_0^t \left(\frac{\partial}{\partial\tau} e^{\tau f_\beta(A)} e^{-\tau A}\right) e^{tA}e^{-QA}\psi \, d\tau\| \\ &\| - \int_0^t (-A + f_\beta(A)) e^{\tau f_\beta(A)} e^{-\tau A} e^{tA} e^{-QA}\psi \, d\tau\| \\ &\leq \beta R \int_0^t \|A^2 e^{\tau f_\beta(A)} e^{-\tau A} e^{tA} e^{-QA}\psi\| \, d\tau. \end{aligned}$$

For the approximation (1.2),

$$\begin{aligned} &\|A^2 e^{\tau f_\beta(A)} e^{-\tau A} e^{tA} e^{-QA}\psi\| \\ &= \|\frac{1}{2\pi i}\int_{\Gamma_\phi} w^2 e^{-\tau w(1-e^{-\beta w})} e^{-(Q-t)w}(w-A)^{-1}\psi \, dw\| \\ &\leq \frac{1}{\pi}\int_0^\infty r^2 e^{-\tau r(\cos\phi - e^{-\beta r\cos\phi}(\cos(\phi-\beta r\sin\phi)))} e^{-(Q-t)r\cos\phi}\|(w-A)^{-1}\psi\| \, dr \\ &\leq \frac{1}{\pi}\int_0^\infty r^2 e^{\tau r e^{-\beta r\cos\phi}\cos(\phi-\beta r\sin\phi)} e^{-(Q-t)r\cos\phi}\|(w-A)^{-1}\psi\| \, dr \\ &\leq \frac{1}{\pi}\int_0^\infty r^2 e^{\tau r} e^{-(Q-T)r\cos\phi}\|(w-A)^{-1}\psi\| \, dr \\ &\leq \frac{1}{\pi}\int_0^\infty r^2 e^{(\tau-(T+\epsilon))r}\|(w-A)^{-1}\psi\| \, dr \\ &\leq \frac{C'}{\pi}\int_0^\infty r e^{(\tau-(T+\epsilon))r}\|\psi\| \, dr \\ &= \frac{C'}{\pi}(T-\tau+\epsilon)^{-2}\|e^{QA}\varphi\|. \end{aligned}$$

Next for the approximation (1.3), let $Q > T$ satisfy $\frac{2}{\pi}\ln 2\,(Q-T)\cos\phi \geq T+\epsilon$. Since $\frac{1}{2}\ln(1+2e^{-\beta r\cos\phi}\cos(\beta r\sin\phi)+e^{-2\beta r\cos\phi}) \leq \ln 2$, then

$$
\begin{aligned}
&\|A^2 e^{\tau f_\beta(A)} e^{-\tau A} e^{tA} e^{-QA}\psi\| \\
&= \|\frac{1}{2\pi i}\int_{\Gamma_\phi} w^2 e^{-(\ln 2)^{-1}\tau w(\ln 2 - Log(1+e^{-\beta w}))} e^{-(Q-t)w}(w-A)^{-1}\psi\, dw\| \\
&\leq \frac{1}{\pi}\int_0^\infty r^2 e^{\left\{\begin{array}{l} -(\ln 2)^{-1}\tau r(\cos\phi(\ln 2 - \frac{1}{2}\ln(1+2e^{-\beta r\cos\phi}\cos(\beta r\sin\phi)+e^{-2\beta r\cos\phi})) \\ -\sin\phi\arctan(\frac{\sin(\beta r\sin\phi)}{e^{\beta r\cos\phi}+\cos(\beta r\sin\phi)})) \end{array}\right\}} \\
&\times\ e^{-(Q-t)r\cos\phi}\|(w-A)^{-1}\psi\|\, dr \\
&\leq \frac{1}{\pi}\int_0^\infty r^2 e^{(\ln 2)^{-1}\tau r\sin\phi\arctan(\frac{\sin(\beta r\sin\phi)}{e^{\beta r\cos\phi}+\cos(\beta r\sin\phi)})} e^{-(Q-t)r\cos\phi}\|(w-A)^{-1}\psi\|\, dr \\
&\leq \frac{1}{\pi}\int_0^\infty r^2 e^{\frac{\pi}{2}(\ln 2)^{-1}\tau r} e^{-(Q-T)r\cos\phi}\|(w-A)^{-1}\psi\|\, dr \\
&\leq \frac{1}{\pi}\int_0^\infty r^2 e^{\frac{\pi}{2}(\ln 2)^{-1}(\tau-(T+\epsilon))r}\|(w-A)^{-1}\psi\|\, dr \\
&\leq \frac{C'}{\pi}\int_0^\infty r e^{\frac{\pi}{2}(\ln 2)^{-1}(\tau-(T+\epsilon))r}\|\psi\|\, dr \\
&= \frac{C'}{\pi}(\pi/2)^{-2}(\ln 2)^2(T-\tau+\epsilon)^{-2}\|e^{QA}\varphi\|.
\end{aligned}
$$

In either case, this yields

$$
\begin{aligned}
&\|u(t)-v_\beta(t)\| \\
&\leq \beta R K'\int_0^t (T-\tau+\epsilon)^{-2}\|e^{QA}\varphi\|\, d\tau \\
&= \beta R K'\frac{t}{(T+\epsilon-t)(T+\epsilon)}\|e^{QA}\varphi\| \\
&< \frac{\beta R K'}{T+\epsilon-t}\|e^{QA}\varphi\|
\end{aligned}
$$

where K' is a constant independent of β, ϵ, and t.

□

Next, we prove a second, heartier estimate between $u(t)$ and $v_\beta(t)$ via a Hölder-type inequality. Fix $\alpha \in (0,\theta)$. By the definition of the semigroup e^{-tA}, we know that e^{-wA} is uniformly bounded in S_α (cf. [21, Theorem X.52]). Let S denote the bent strip $S = \{t + re^{\pm i\alpha} \ :\ s \leq t \leq T,\ r \geq 0\}$ and define for $\zeta = t + re^{\pm i\alpha} \in S$,

$$\phi(\zeta) = e^{-(re^{\pm i\alpha})A}(u(t)-v_\beta(t)).$$

Following [6, Section 4], our goal is to apply Carleman's Inequality (cf. [17], [7, p. 346]) in the strip S. We will also use the following lemma.

Lemma 3.2. [1, p. 148] *Let $\phi(z)$ be a continuous and bounded complex function on the bent strip $S = \{z = x + \eta e^{\pm i\theta} \mid s \leq x \leq T,\ \eta \geq 0\}$. For $\zeta = t + re^{\pm i\theta} \in S$, define*

$$\Phi(\zeta) = -\frac{1}{\pi}\int\int_S \phi(z)\left(\frac{1}{z-\zeta} + \frac{1}{\bar{z}+1+\zeta}\right)dxd\eta.$$

Then $\Phi(\zeta)$ is absolutely convergent, $\bar{\partial}\Phi(\zeta) = \phi(\zeta)$ where $\bar{\partial}$ denotes the Cauchy-Riemann operator, and there exist constants $\tilde{K} > 0$ and $\tilde{L} > T$ such that

$$\int_{-\infty}^{\infty}\left|\frac{1}{z-\zeta} + \frac{1}{\bar{z}+1+\zeta}\right|d\eta \leq \tilde{K}\left(1 + \log\frac{\tilde{L}}{|x-t|}\right) \quad for \quad x \neq t.$$

Theorem 3.3. *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on a Banach space X with $0 \in \rho(A)$ and let $0 < \beta, \epsilon < 1$. Fix ϕ as in Proposition 3.1 and let $Q > T$ satisfy the conditions of Lemma 3.1. Assume that $u(t)$ is a classical solution of (1.1) with initial data $\varphi \in Dom((e^{-QA})^{-1})$ satisfying $\|e^{QA}\varphi\| \leq M'$. Then there exist constants $\tilde{C}$ and M each independent of β and ϵ such that for $0 \leq t < T$,*

$$\|u(t) - v_\beta(t)\| \leq \tilde{C}\epsilon^{-2}\beta^{1-h(t)}M^{h(t)} \tag{3.1}$$

where $h(\zeta)$ is a harmonic function which is bounded and continuous on the bent strip $S = \{\zeta = t + re^{\pm i\alpha} \mid s \leq t \leq T,\ r \geq 0\}$, $\alpha \in (0, \theta)$, and assumes the values 0 and 1 respectively on the left and right hand boundary curves of S.

Proof. We first determine $\bar{\partial}\phi(\zeta)$ where $\bar{\partial}$ denotes the Cauchy-Riemann operator (cf. [22]). Since $e^{-(re^{\pm i\alpha})A}$ is bounded for every $r \geq 0$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\phi(\zeta) &= \frac{\partial}{\partial t}e^{-(re^{\pm i\alpha})A}(u(t) - v_\beta(t))\\ &= e^{-(re^{\pm i\alpha})A}(u'(t) - v'_\beta(t))\\ &= e^{-(re^{\pm i\alpha})A}(Au(t) - f_\beta(A)v_\beta(t)).\end{aligned}$$

Also, since $-A$ generates e^{-tA} and since both $u(t)$ and $v_\beta(t)$ are in $Dom(A)$,

$$\frac{\partial}{\partial r}\phi(\zeta) = \frac{\partial}{\partial r}e^{-(re^{\pm i\alpha})A}(u(t) - v_\beta(t))$$

$$= e^{-(re^{\pm i\alpha})A}(-e^{\pm i\alpha}A)(u(t)-v_\beta(t)).$$

Hence,

$$\begin{aligned}\bar{\partial}\phi(\zeta) &= \frac{1}{2i\,\sin(\pm\alpha)}\left(e^{\pm i\alpha}\frac{\partial}{\partial t}\phi(\zeta)-\frac{\partial}{\partial r}\phi(\zeta)\right)\\ &= \frac{e^{\pm i\alpha}}{2i\,\sin(\pm\alpha)}\Big[e^{-(re^{\pm i\alpha})A}(Au(t)-f_\beta(A)v_\beta(t)) \\ &\quad +e^{-(re^{\pm i\alpha})A}(Au(t)-Av_\beta(t))\Big].\end{aligned} \tag{3.2}$$

Following [1], define

$$\Phi(\zeta) = -\frac{1}{\pi}\int\int_S \bar{\partial}\phi(z)\left(\frac{1}{z-\zeta}+\frac{1}{\bar{z}+1+\zeta}\right)dxd\eta,$$

where $z = x+\eta e^{\pm i\alpha}$ and $\zeta = t+re^{\pm i\alpha}$ are in S. Using the fact that the semigroup e^{-tA} is bounded, we have from (3.2),

$$\begin{aligned}\|\bar{\partial}\phi(z)\| \leq & \frac{1}{2|\sin\alpha|}\,\|e^{-(\eta e^{\pm i\alpha})A}\|\left(\|Au(x)-f_\beta(A)v_\beta(x)\|+\|Au(x)-Av_\beta(x)\|\right)\\ \leq & \frac{1}{2|\sin\alpha|}\left(\max_{\eta\geq 0}\|e^{-(\eta e^{\pm i\alpha})A}\|\right)(2\|Au(x)-Av_\beta(x)\|\\ & +\|Av_\beta(x)-f_\beta(A)v_\beta(x)\|).\end{aligned}$$

Now, similar to Proposition 3.1, it may be shown that

$$\|Au(x)-Av_\beta(x)\| \leq \beta L(T-x+\epsilon)^{-2}\|e^{QA}\varphi\|$$

for some constant L depending on ϵ but independent of β and x. Also, by Lemma 2.4 and the proof of Proposition 3.1,

$$\begin{aligned}& \|Av_\beta(x)-f_\beta(A)v_\beta(x)\|\\ = & \|(-A+f_\beta(A))v_\beta(x)\|\\ \leq & \beta R\|A^2v_\beta(x)\|\\ = & \beta R\|A^2e^{xf_\beta(A)}\varphi\|\\ = & \beta R\|A^2e^{xf_\beta(A)}e^{-xA}e^{-(Q-x)A}(e^{QA}\varphi)\|\\ \leq & \beta L'(T-x+\epsilon)^{-2}\|e^{QA}\varphi\|\end{aligned}$$

for another constant L' depending on ϵ but independent of β and x. By the assumption $\|e^{QA}\varphi\| \leq M'$, we have altogether

$$\|\bar{\partial}\phi(z)\| \;\leq\; \beta C'\epsilon^{-2}, \tag{3.3}$$

where C' is a constant depending on ϵ but independent of β and z.

We have shown that $\bar{\partial}\phi(z)$ is bounded on S. It follows easily that $\bar{\partial}\phi(z)$ is also continuous on S. By Lemma 3.2, then $\Phi(\zeta)$ is absolutely convergent, $\bar{\partial}\Phi(\zeta) = \bar{\partial}\phi(\zeta)$, and there exist constants $\tilde{K} > 0$ and $\tilde{L} > T$ such that, for $x \neq t$,

$$\int_{-\infty}^{\infty} \left| \frac{1}{z-\zeta} + \frac{1}{\bar{z}+1+\zeta} \right| d\eta \leq \tilde{K} \left(1 + \log \frac{\tilde{L}}{|x-t|} \right).$$

Let $x^* \in X^*$ be arbitrary where X^* denotes the dual space of X and define $\Psi : S \to \mathbb{C}$ by $\Psi(\zeta) = x^*(\phi(\zeta) - \Phi(\zeta))$. For ζ in the interior of S, then $\bar{\partial}\Psi(\zeta) = x^*(\bar{\partial}\phi(\zeta) - \bar{\partial}\Phi(\zeta)) = x^*(0) = 0$. Therefore, Ψ is analytic on the interior of S (cf. [22, Theorem 11.2]).

Next, by Proposition 3.1,

$$\begin{aligned} \|\phi(\zeta)\| &= \|e^{-(re^{\pm i\alpha})A}(u(t) - v_\beta(t))\| \\ &\leq \left(\max_{r \geq 0} \|e^{-(re^{\pm i\alpha})A}\| \right) \|u(t) - v_\beta(t)\| \\ &\leq \beta \left(\max_{r \geq 0} \|e^{-(re^{\pm i\alpha})A}\| \right) K(T - t + \epsilon)^{-1} \|e^{QA}\varphi\| \\ &\leq \beta C'' \epsilon^{-1} \end{aligned} \tag{3.4}$$

where C'' is a constant independent of β, ϵ, and ζ. Also, from (3.3) and Lemma 3.2,

$$\begin{aligned} \|\Phi(\zeta)\| &= \left\| -\frac{1}{\pi} \int\int_S \bar{\partial}\phi(z) \left(\frac{1}{z-\zeta} + \frac{1}{\bar{z}+1+\zeta} \right) dx d\eta \right\| \\ &\leq \frac{1}{\pi} \beta C' \epsilon^{-2} \int_s^T \left(\int_{-\infty}^{\infty} \left| \frac{1}{z-\zeta} + \frac{1}{\bar{z}+1+\zeta} \right| d\eta \right) dx \\ &\leq \beta \frac{\tilde{K}}{\pi} C' \epsilon^{-2} \int_s^T \left(1 + \log \frac{\tilde{L}}{|x-t|} \right) dx \\ &\leq \beta C' \epsilon^{-2} \end{aligned} \tag{3.5}$$

for a possibly different constant C' independent of β, ϵ, and ζ. Then from (3.4) and (3.5), we have for $\zeta = t + re^{\pm i\alpha} \in S$,

$$\begin{aligned} |\Psi(\zeta)| &= |x^*(\phi(\zeta) - \Phi(\zeta))| \\ &\leq \|x^*\| \left(\|\phi(\zeta)\| + \|\Phi(\zeta)\| \right) \\ &\leq \beta M \epsilon^{-2} \|x^*\| \end{aligned} \tag{3.6}$$

where M is a constant independent of β, ϵ, and ζ. Hence Carleman's Inequality (cf. [17], [7, p. 346]) implies

$$|\Psi(t)| \leq M(0)^{1-h(t)} M(T)^{h(t)}, \tag{3.7}$$

for $0 \leq t \leq T$, where $M(t) = \sup_{r\geq 0} |\Psi(t + re^{\pm i\alpha})|$ and h is a harmonic function which is bounded and continuous on S and assumes the values 0 and 1 respectively on the left and right hand boundary curves of S. Note,

$$\|\phi(re^{\pm i\alpha})\| = \|e^{-(re^{\pm i\alpha})A}(u(0) - v_\beta(0))\| = \|e^{-(re^{\pm i\alpha})A}(\varphi - \varphi)\| = 0.$$

Then from (3.5), we have

$$|\Psi(re^{\pm i\alpha})| \leq \|x^*\| \left(\|\phi(re^{\pm i\alpha})\| + \|\Phi(re^{\pm i\alpha})\|\right) \leq \|x^*\|\beta C'\epsilon^{-2},$$

and so

$$M(0) = \sup_{r\geq 0} |\Psi(re^{\pm i\alpha})| \leq \beta C'\epsilon^{-2}\|x^*\|. \tag{3.8}$$

Also, from (3.6) and the fact that $0 < \beta < 1$, we have

$$M(T) = \max_{r\geq 0} |\Psi(T + re^{\pm i\alpha})| \leq M\epsilon^{-2}\|x^*\|. \tag{3.9}$$

From (3.7), (3.8), and (3.9), it follows that for $0 \leq t \leq T$,

$$|\Psi(t)| \leq (\beta C')^{1-h(t)} M^{h(t)} \epsilon^{-2}\|x^*\|.$$

Taking the supremum over $x^* \in X^*$ with $\|x^*\| \leq 1$, we obtain $\|\phi(t) - \Phi(t)\| \leq \tilde{C}\epsilon^{-2}\beta^{1-h(t)}M^{h(t)}$ for $0 \leq t \leq T$ where $\tilde{C}$ and M are constants each independent of ϵ and β. Then by (3.5), for $0 \leq t \leq T$,

$$\begin{aligned}
\|u(t) - v_\beta(t)\| &= \|\phi(t)\| \\
&= \|(\phi(t) - \Phi(t)) + \Phi(t)\| \\
&\leq \tilde{C}\epsilon^{-2}\beta^{1-h(t)}M^{h(t)} + \beta C'\epsilon^{-2} \\
&= (\tilde{C} + \beta^{h(t)}M^{-h(t)}C')\epsilon^{-2}\beta^{1-h(t)}M^{h(t)} \\
&\leq \tilde{C}\epsilon^{-2}\beta^{1-h(t)}M^{h(t)}
\end{aligned}$$

for a possibly different constant $\tilde{C}$ independent of β and ϵ. □

4. REGULARIZATION AND APPLICATION

With the implication of Theorem 3.3, we prove regularization for the problem (1.1). Since $f_\beta(A)$ is a bounded operator in either (1.2) or (1.3) satisfying $\|f_\beta(A)\| \leq C/\beta$, then $v_\beta(t)$ in Theorem 3.3 satisfies $\|v_\beta(t)\| = \|e^{tf_\beta(A)}\varphi\| \leq e^{CT/\beta}\|\varphi\|$ for $0 \leq t \leq T$. Hence

$$\|\varphi - \varphi_\delta\| \leq \delta \quad \text{implies} \quad \|v_\beta(t) - v_\beta^\delta(t)\| \leq e^{CT/\beta}\delta \quad \text{for all} \quad t \in [0, T]$$

where $v_\beta^\delta(t)$ is the solution of the well-posed problem

$$\begin{aligned} v'(t) &= f_\beta(A)v, \quad 0 < t < T, \\ v(0) &= \varphi_\delta. \end{aligned}$$

Now, let $u(t)$ be a classical solution of (1.1) that satisfies the hypotheses of Theorem 3.3. Choosing $\beta = -2CT(\ln\delta)^{-1}$ we have $\beta \to 0$ as $\delta \to 0$, and

$$\begin{aligned} \|u(t) - v_\beta^\delta(t)\| &\leq \|u(t) - v_\beta(t)\| + \|v_\beta(t) - v_\beta^\delta(t)\| \\ &\leq \tilde{C}\epsilon^{-2}\beta^{1-h(t)}M^{h(t)} + e^{CT/\beta}\delta \\ &= \tilde{C}\epsilon^{-2}\beta^{1-h(t)}M^{h(t)} + \sqrt{\delta} \\ &\to 0 \quad \text{as} \quad \delta \to 0 \end{aligned}$$

for all $0 \leq t < T$ where we have used the fact that for such values of t, the harmonic function h from Theorem 3.3 must satisfy $0 \leq h(t) < 1$. Meanwhile, the case for $t = T$ may be addressed separately using the estimate of Proposition 3.1. Indeed, applying $\|e^{QA}\varphi\| \leq M'$,

$$\begin{aligned} \|u(T) - v_\beta^\delta(T)\| &\leq \|u(T) - v_\beta(T)\| + \|v_\beta(T) - v_\beta^\delta(T)\| \\ &\leq \beta K\epsilon^{-1}M' + e^{CT/\beta}\delta \\ &= \beta K\epsilon^{-1}M' + \sqrt{\delta} \\ &\to 0 \quad \text{as} \quad \delta \to 0. \end{aligned}$$

We have proved

Theorem 4.1. *Let $-A$ be the infinitesimal generator of a bounded holomorphic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on a Banach space X with $0 \in \rho(A)$.*

Let $0 < \beta < 1$, *and assume the hypotheses of Theorem* 3.3. *Then by either definition* (1.2) *or* (1.3) *of* $f_\beta(A)$,

$$\{R_\beta(t) = e^{tf_\beta(A)} : 0 < \beta < 1,\ t \in [0, T]\}$$

is a family of regularizing operators for the ill-posed problem (1.1).

Remark and application. One may obtain regularization results in the case that $-A$ generates a holomorphic, not necessarily bounded semigroup e^{-tA} of angle $\gamma \in (0, \pi/2]$. In this case, for any $\theta \in (0, \gamma)$, there exists $\lambda \in \mathbb{R}$ such that $-A + \lambda$ is the infinitesimal generator of a bounded holomorphic semigroup of angle θ with $0 \in \rho(A - \lambda)$ (cf. [21, Theorem X.53], [4]). With this in mind, the most salient application of the paper is to the backward heat equation since $A = -\Delta$ generates a holomorphic semigroup of angle $\theta = \frac{\pi}{2}$ on $X = L^p(\mathbb{R}^n)$, $1 < p < \infty$ (cf. [20]) and more generally to strongly elliptic differential operators of even order on X (cf. [9, Section 5]).

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