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EXISTENCE AND MULTIPLICITY OF THE SOLUTIONS FOR THE $p(x)$-KIRCHHOFF EQUATION VIA GENUS THEORY

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ABSTRACT: This paper is concerned with the existence and multiplicity of nontrivial weak solutions for a $p(x)$-Kirchhoff problem.

By using variational approach and Krasnoselskii's genus theory, we show the existence and multiplicity of the solutions for $p(x)$-Kirchhoff equation.

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ or $N \geq 2$ with smooth boundary $\partial\Omega$. In this work, we are interested with the following equation

$$-M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\Delta^2_{p(x)}u = f(x,u) \quad \text{in } \Omega, \tag{1.1}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2}\Delta u) = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$, is the $p(x)$-biharmonic operator, p is a continuous function on $\overline{\Omega}$.

We assume that $M(t)$ and $f(x,t)$ satisfy the following assumptions:

(M_1) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and satisfies the polynomial growth condition

$$m_1 t^{\beta-1} \leq M(t) \leq m_2 t^{\alpha-1},$$

for all $t > 0$, $(m_1, m_2) \in \mathbb{R}^2$ such that $0 < m_1 \leq m_2$ and $\alpha \geq \beta > 1$.

(f_1) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$C_1|t|^{s(x)-1} \leq f(x,t) \leq C_2|t|^{q(x)-1},$$

for all $(t,x) \in]0,+\infty[\times\overline{\Omega}$, where C_1 and C_2 are positive constants and $s, q \in C(\overline{\Omega})$ such that $1 < s(x) < q(x) < p^*(x) < \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$.

(f_2) f is an odd function with respect to the variable t,

$$f(x,t) = -f(x,-t) \quad \forall (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

The problem (1.1) is a generalization of a model introduced by Kirchhoff [10] . More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

which extends the classical D′Alembert′s wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of Eq. (1.3) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, and hence the equation is no longer a pointwise identity. The parameters in (1.3) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and ρ_0 is the initial tension.

The operator $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is said to be the $p(x)-$ Biharmonic, and becomes $p-$Biharmonic when $p(x) = p$ (a constant). The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics [14], electrorheological fluids [15] and image restoration [16] .

In recent years, elliptic problems involving p-Kirchhoff type operators have been studied in many papers, we refer to [17, 18], in which the authors have used different methods to get the existence of solutions for (1.1) in the case when $p(x) = p$ is a constant.

In the case of $p(x)$-Laplacian operator, the authors studied in [19] the Kirchhof type equation

$$-M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = f(x,u), \quad \text{in } \Omega, \tag{1.4}$$

with Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

by using the Krasnoselskii's Genus theory. The authors showed the existence and multiplicity of the solutions of the problem (1.4)-(1.5).

In the [24], the authors study the existence and multiplicity of solutions via Krasnoselskii's Genus, on the Sobolev space with variable exponent, to the following bi-nonlocal $p(x)$-Kirchhoff equation,

$$-M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = f(x,u)\left[\int_\Omega F(x,u)dx\right]^r \quad \text{in } \Omega, \tag{1.6}$$

with Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.7}$$

Motivated by the above works papers and the results in [11], [12] and [13], we study the existence and multiplicity of the solutions of the problem (1.1)-(1.2). Our paper is organized as follows. We first present some necessary preliminary results on variable exponent Sobolev spaces. Next, we give the main results and proofs about the existence and multiplicity of the solutions.

2. PRELIMINARIES

In order to deal with $p(x)-$ biharmonic operator problems, we need some results on spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ and some properties of $p(x)-$biharmonic operator, which we will use later.

Let

$$L^{p(x)}(\Omega)=\Big\{u:\Omega\longrightarrow\mathbb{R} \text{ measurable function} \quad \text{and satisfy} \quad \int_\Omega |u(x)|^{p(x)}dx<\infty\Big\},$$

be a generalized Lebesgue space, where $p\in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega})=\left\{h\in C(\overline{\Omega}):\;\; h(x)>1,\;\; \forall x\in\overline{\Omega}\right\}.$$

Let p^+ and p^- given by

$$p^+=\max_{x\in\overline{\Omega}}p(x),\quad p^-=\min_{x\in\overline{\Omega}}p(x),$$

and for all $x\in\overline{\Omega}$ and $k\geq 1$, we consider

$$p^*(x)=\begin{cases}\frac{Np(x)}{N-p(x)} & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\geq N,\end{cases}$$

and

$$p_k^*(x)=\begin{cases}\frac{Np(x)}{N-kp(x)} & \text{if } kp(x)<N,\\ +\infty & \text{if } kp(x)\geq N.\end{cases}$$

The space $L^{p(x)}(\Omega)$ endowed with the following norm

$$|u|_{p(x)}=\inf\left\{\mu>0;\;\; \int_\Omega\left|\frac{u(x)}{\mu}\right|^{p(x)}dx\leq 1\right\}$$

is a Banach space. The following results are proved in [25] :

Proposition 2.1. *The space $(L^{p(x)}(\Omega),|.|_{p(x)})$ is separable, uniformly convex and reflexive; its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$ is the sense*

$$\frac{1}{p(x)}+\frac{1}{q(x)}=1,\quad \forall x\in\Omega.$$

For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, then the following Hölder's inequality

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \tag{2.1}$$

holds.

Let $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ a real valued mapping given for every $u \in L^{p(x)}(\Omega)$ by

$$\rho(u) = \int_\Omega |u|^{p(x)}dx,$$

then the following properties holds.

Proposition 2.2. *([25])*

1. $|u|_{p(x)} < 1(=1, > 1)$ *if and only if* $\rho(u) < 1(=1, > 1)$.
2. *If* $|u|_{p(x)} > 1$, *then* $|u|^{p^-}_{p(x)} \leq \rho(u) \leq |u|^{p^+}_{p(x)}$.
3. *If* $|u|_{p(x)} < 1$, *then* $|u|^{p^+}_{p(x)} \leq \rho(u) \leq |u|^{p^-}_{p(x)}$.
4. $|u_n - u|_{p(x)} \longrightarrow 0$ *if and only if* $\rho(u_n - u) \longrightarrow 0$.

We recall also the following proposition, which will be needed later:

Proposition 2.3. *([21])*
Let p and q be measurable functions such that $p \in L^\infty(\Omega)$ and $1 < p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$ such that $u \neq 0$. Then

1. *if* $|u|_{p(x)q(x)} \leq 1$, *then*

$$|u|^{p^+}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)},$$

2. *if* $|u|_{p(x)q(x)} \geq 1$, *then*

$$|u|^{p^-}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \ D^\alpha u \in L^{p(x)}(\Omega), \ |\alpha| \leq k\},$$

where

$$D^{\alpha}u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_N^{\alpha_N}},$$

is the derivation in the distribution sense and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index such that $|\alpha| = \sum_{i=1}^{i=N} \alpha_i$.

The set $W^{k,p(x)}(\Omega)$ endowed with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^{\alpha}u|_{p(x)},$$

also becomes a Banach space which is separable and reflexive. For more details on the Sobolev space with a variable exponent, we refer to [5, 6].

Through this paper, we consider the following set

$$X = \{u \in W^{2,p(x)}(\Omega) : \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}$$

and we prove the following elementary result.

Proposition 2.4. *The norm* $\|u\|_{2,p(x)}$ *is equivalent to the norm* $\|u\| = |\Delta u|_{p(x)}$ *in the space* X.

Proof. Let $E = \left\{u \in C^{\infty}(\overline{\Omega}) : \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega\right\}$. Since $C^{\infty}(\overline{\Omega})$ is dense in $W^{2,p(x)}(\Omega)$ ([29], Theorem 9.1.7) and X is a closed subspace of $W^{2,p(x)}(\Omega)$, we can deduce that E is dense in X.

Now let $u \in E$, first we prove that $\Sigma_{|\alpha|=2}|D^{\alpha}u|_{p(x)} \leq C|\Delta u|_{p(x)}$.

From [28], we have

$$\frac{\partial^2 u}{\partial x_j \partial x_k} = -R_j R_k(\Delta u)$$

and $R_j u$ is called Riesz transform of u and

$$R_j u(x) = \lim_{\varepsilon\to 0} c_n \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} u(x-y)dy, \qquad j = 1, ..., n,$$

with $c_n = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}$.

Thus R_j is defined by the kernel $K_j(x) = c_n \frac{x_j}{|x|^{n+1}}$.

Now, we prove that $K_j(x)$ satisfies the conditions of Theorem 8.14 in [26].

$$\int_{|x|=1} K_j(x)dS = \int_{|x|=1} c_n \frac{x_j}{|x|^{n+1}} dS = 0, \tag{2.2}$$

and if we take $\sigma = 2$, we find

$$\int_{|x|=1} |K_j(x)|^\sigma dS = \int_{|x|=1} c_n \frac{x_j^2}{|x|^{2(n+1)}} dS = \frac{2\pi^{\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} \tag{2.3}$$

is bounded uniformly with respect to x. Due Theorem 8.14 [26], we have that the operators R_j with $j = 1, ..., n$, are bounded in $L^{p(\cdot)}(\Omega)$, thus for $u \in E$, we get

$$\left|\frac{\partial^2 u}{\partial x_j \partial x_k}\right|_{p(x)} = |-R_j R_k(\Delta u)|_{p(x)} \leq C_1 |\Delta u|_{p(x)},$$

for $j, k = 1, ...n$. Furthermore,

$$|\Delta u|_{p(x)} \leq \Sigma_{|\alpha|=2} |D^\alpha u|_{p(x)} \leq C_2 |\Delta u|_{p(x)}. \tag{2.4}$$

From [27], we have $|Du| = |D\Gamma * (\Delta u)| \leq CI_1(\Delta u)$ where $I_1(\Delta u) = \int_\Omega \frac{\Delta(x-y)}{|y|^{n-1}} dy$ and Γ is a Newtonian potential. We use the same proof of Theorem 3.1 in [20], we can find that

$$I_1(\Delta u) \leq C_3 M(\Delta u),$$

where M is the maximal function, see Proposition 2.3 [20].

Therefore

$$|Du|_{p(x)} \leq C_4 |\Delta u|_{p(x)}. \tag{2.5}$$

So, from Proposition 2.3 [20], we can get

$$|u|_{p(x)} \leq C_5 |\Delta u|_{p(x)}. \tag{2.6}$$

We combining (2.4),(2.5) and (2.6), we deduce

$$|\Delta u|_{p(x)} \leq \|u\|_{2,p(x)} \leq C_6 |\Delta u|_{p(x)}. \tag{2.7}$$

So, the set X is adopted in the papers [8] and [3]. They have proved that X is a nonempty, well defined and closed subspace of $W^{2,p(x)}(\Omega)$. For this they have showed the following boundary trace embedding theorem for Sobolev spaces with variable exponent.

Theorem 2.5. *([8])*

Let Ω be a bounded domain in $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary. If $2p(x) \geq N \geq 2$ for all $x \in \overline{\Omega}$, then for all $q \in C_+(\Omega)$ there is a continuous boundary trace embedding

$$W^{2,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega), \tag{2.8}$$

and

$$W^{2,p(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\partial\Omega). \tag{2.9}$$

Proposition 2.6. *([8])*

If $2p(x) \geq N$ for all $x \in \overline{\Omega}$, then the set

$$X = \{u \in W^{2,p(x)}(\Omega) : \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}$$

is a closed subspace of $W^{2,p(x)}(\Omega)$.

Also, as a direct consequence of proposition 2.6 becomes $(X; \|.\|)$ is a Banach space that is separable and reflexive.

Proposition 2.7. *([2])*

Consider the cost function J given by

$$J(u) = \int_\Omega |\Delta u|^{p(x)} dx,$$

then for all $(u, u_n) \in X^2$, the following properties holds

1. *$\|u\| < 1 \ (= 1; > 1)$ if and only if*
$$J(u) < 1 \ (= 1; > 1),$$
2. *if $\|u\| \leq 1$, then $\|u\|^{p^+} \leq J(u) \leq \|u\|^{p^-}$,*
3. *if$\|u\| \geq 1$, then $\|u\|^{p^-} \leq J(u) \leq \|u\|^{p^+}$,*
4. *if $\|u_n\| \longrightarrow 0$, then $J(u_n) \longrightarrow 0$,*
5. *if $\|u_n\| \longrightarrow \infty$, then $J(u_n) \longrightarrow \infty$.*

Proposition 2.8. *([3])*

Let $p \in C_+(\overline{\Omega})$ such that $2p(x) > N$, then for all $x \in \overline{\Omega}$ we have

(1) there exists a continuous and compact embedding of $W^{2,p(x)}(\Omega)$ *into* $L^{q(x)}(\Omega)$, *for all* $q \in C_+(\Omega)$.

(2) there exists a continuous embedding of $W^{2,p(x)}(\Omega)$ *into* $C(\overline{\Omega})$.

Proposition 2.9. *([9])*
Let Λ *be the cost function defined on the Banach space* X *by*

$$\Lambda(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx.$$

Then we have

1. *The functional* Λ *is convex.*
2. *The mapping* $\Lambda' : X \longrightarrow X'$ *is a strictly monotone, bounded homeomorphism and of* (S_+), *namely if* $u_n \rightharpoonup u$ *and* $\overline{\lim_{n\to\infty}} \langle \Lambda'(u_n), u_n - u\rangle \leq 0$, *then* $u_n \longrightarrow u$.

Definition 2.10. The function $u \in X$ be said a weak solution of the problem (1.1)-(1.2) if

$$M\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\right) \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \varphi dx = \int_\Omega f(x,u)\varphi dx, \quad \forall \varphi \in X.$$

This relation is called the weak variational formulation equivalent to the problem (1.1)-(1.2).

Let $I : X \longrightarrow \mathbb{R}$ be the associated cost function to the problem (1.1)-(1.2) given by

$$I(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\right) - \int_\Omega F(x,u)dx,$$

where $\widehat{M}(t) = \int_0^t M(s)ds$ and $F(x,u) = \int_0^u f(x,t)dt$.
The cost function is $C^1(X,\mathbb{R})$ class and for every $u, v \in X$, we have

$$\langle I'(u), v\rangle = M\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\right) \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx$$

$$- \int_{\Omega} f(x,u)v dx.$$

Hence, we can notice that the critical points of the functional I are the weak solutions for problem (1.1)-(1.2).

3. MAIN RESULTS AND PROOFS

In the first stage of this section, we present some notions on the Krasnoselskii's genus theory (see [23, 22]) that we use in the proof of our main result.

Let Y be a real Banach space. set

$$\mathcal{R} = \{E \subset Y \backslash \{0\} : E \quad \text{is compact and} \quad E = -E\}.$$

Definition 3.1. Let $E \in \mathcal{R}$ and $Y = \mathbb{R}^k$.
The genus $\gamma(E)$ of E is defined by

$$\gamma(E) = \min \Big\{ k \geq 1; \text{there exists an odd continuous mapping} \\ \phi : E \to \mathbb{R}^k \backslash \{0\} \Big\}. \quad (3.1)$$

If the mapping ϕ does not exist for any $k > 0$, we set $\gamma(E) = \infty$.

Note also that if E is a subset, which consists of finitely many pairs of points, then $\gamma(E) = 1$.
Moreover, from the Definition 3.1, $\gamma(\emptyset) = 0$. A typical example of a set of genus k is a set, which is homeomorphic to a $(k-1)$ dimensional sphere $\mathbb{S}^{k-1}$ via an odd map.

Now, the following Kranoselskii's genus results are necessary throughout the present paper.

Theorem 3.2. *Let $Y = \mathbb{R}^N$ and $\Omega \subset \mathbb{R}^N$ be an open, symmetric and bounded subset and $\partial\Omega$ with bound. We assume that $0 \in \Omega$, then $\gamma(\partial\Omega) = N$.*

Corollary 3.3. *The genus of unit sphere $\mathbb{S}^{N-1}$ of the space $\mathbb{R}^{N-1}$ is $\gamma(\mathbb{S}^{N-1}) = N$*

Remark 3.4. If Y is separable infinite dimensional space with unit sphere S, then $\gamma(S) = \infty$.

Definition 3.5. The functional I satisfies the Palais-Smale condition (PS) if for every sequence $(u_n) \subset Y$ such that

$$|I(u_n)| \leq C \quad \text{and} \quad I^{'}(u_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

then there is a subsequence of (u_n) which converges in the sense of the norm of Y.

The first result of the present paper is :

Theorem 3.6. *We assume that* (M_1), (f_1) *and* (f_2) *hold. If* $p(x) < q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *and* $q^+ < \beta p^-$, *then there are infinitely many solutions of the problem (1.1)-(1.2).*

The following result obtained by Clark in [7] is the main idea, which we use in the proof of Theorem 3.6.

Theorem 3.7. *Let* $J \in C^1(X, \mathbb{R})$ *be a cost function satisfying the (PS) condition. We assume that the following conditions*

(i) *J is bounded from below and even ;*

(ii) *There is a compact set* $K \in \mathcal{R}$ *such that* $\gamma(K) = k$ *and* $\sup_{x \in K} J(x) < J(0)$,

holds. Then J *possesses at least* k *pairs of distinct critical points, and their corresponding critical values are less than* $J(0)$.

Lemma 3.8. *We assume* (M_1), (f_1) *and* $q^+ < \beta p^-$ *hold. Then* J *is bounded from below.*

From (M_1) and (f_1), we have

$$\begin{aligned} I(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right) - \int_\Omega F(x,u)dx \\ &\geq m_1 \int_0^{\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx} \Psi^{\beta-1}d\Psi - \frac{C_2}{q^-}\int_\Omega |u|^{q(x)}dx \\ &= \frac{m_1}{\beta}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)^\beta - \frac{C_2}{q^-}\int_\Omega |u|^{q(x)}dx \end{aligned}$$

and considering proposition (2.2) and proposition (2.8), we get

$$I(u) \geq \frac{m_1}{\beta(p^+)^\beta}\|u\|^{\beta p^-} - \frac{C_2}{q^-}\max\left\{|u|^{q^-}_{q(x)}, |u|^{q^+}_{q(x)}\right\}$$

$$\begin{aligned} &\geq \frac{m_1}{\beta(p^+)^\beta}\|u\|^{\beta p^-} \\ &- \frac{C_2}{q^-}\max\left\{C^{q^-}\|u\|^{q^-}, C^{q^+}\|u\|^{q^+}\right\} \\ &\geq \frac{m_1}{\beta(p^+)^\beta}\|u\|^{\beta p^-} - \frac{C_2 C^{q^+}}{q^-}\|u\|^{q^+} \end{aligned} \tag{3.2}$$

for $\|u\|$ large enough. Hence, I is bounded from below.

Lemma 3.9. *We assume that (M_1), (f_1) and $q^+ < \beta p^-$ hold, then I satisfies the (PS) condition.*

Let us assume that there exists a sequence (u_n) in X such that

$$I(u_n) \longrightarrow c \quad \text{and} \quad I^{'}(u_n) \longrightarrow 0. \tag{3.3}$$

From (3.3), we have $|I(u_n)| \leq C_4$. This fact, combined with (3.2), implies that

$$C_4 \geq I(u_n) \geq \frac{m_1}{\alpha(p^+)^\alpha}\|u_n\|^{\beta p^-} - \frac{C_3}{q^-}\|u_n\|^{q^+} \geq C_5,$$

where $\|u_n\| > 1$. Because $q^+ < \beta p^-$, we obtain that $(\|u_n\|)$ is bounded in X. Hence, we may extract a subsequence $(u_n) \subset X$ and $u \in X$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad X.$$

By proposition 2.5, we obtain the following results:

$$\begin{aligned} u_n &\longrightarrow u \quad \text{in} \quad L^{q(x)}(\Omega), \\ u_n &\longrightarrow u \quad \text{a.e.} \quad \Omega. \end{aligned}$$

Then by (3.3), we have $\langle I^{'}(u_n), u_n - u\rangle \longrightarrow 0$. Thus

$$\begin{aligned} &\langle I^{'}(u_n), u_n - u\rangle = \\ &\quad M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n - \Delta u)dx \\ &\qquad - \int_\Omega f(x,u_n)(u_n - u)dx \longrightarrow 0 \end{aligned}$$

By (f_1) and proposition 2.1, it follows

$$\left|\int_\Omega f(x,u_n)(u_n - u)dx\right| \leq$$

$$C_2 \left| \int_\Omega |u_n|^{q(x)-2} u_n (u_n - u) dx \right| \\ \leq C_6 \left| |u_n|^{q(x)-1} \right|_{q'(x)} |u_n - u|_{q(x)}.$$

Because (u_n) converges strongly to u in $L^{q(x)}(\Omega)$, that is, $|u_n - u|_{q(x)} \to 0$ as $n \to \infty$, we get

$$\int_\Omega f(x, u_n)(u_n - u) dx \to 0.$$

Hence,

$$M \left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx \right) \int_\Omega |\Delta u_n|^{p(x)-2} \Delta u_n (\Delta u_n - \Delta u) \to 0.$$

From (M1), it follows

$$\int_\Omega |\Delta u_n|^{p(x)-2} \Delta u_n (\Delta u_n - \Delta u) \to 0.$$

Eventually, by proposition 2.9, we get $u_n \to u$ in X.

Proof of Theorem 3.6: Set (see [22])

$$\mathcal{R}_k = \{E \subset \mathcal{R} : \gamma(E) \geq k\}$$

$$c_k = \inf_{E \in \mathcal{R}_k} \sup_{u \in E} I(u), k = 1, 2, \ldots,$$

then we have

$$-\infty < c_1 \leq c_2 \leq \ldots \leq c_k \leq c_{k+1} \leq \ldots$$

Now, we will show that $c_k < 0$ for every $k \in \mathbb{N}$. Because X is a separable Banach space, for any $k \in \mathbb{N}$, we can choose a $k-$dimensional linear subspace X_k of X such that $X_k \subset C_0^\infty(\Omega)$. As the norms on X_k are equivalent, there exists $r_k \in (0,1)$ such that $u \in X_k$ with $\|u\| \leq r_k$ implies $|u|_{L^\infty} \leq \delta$.
Set $S_{r_k}^k = \{u \in X_k : \|u\| = r_k\}$. By the compactness of $S_{r_k}^k$ and condition (f_1), there exists a constant $\eta_k > 0$ such that

$$\int_\Omega F(x,u) dx \geq \frac{C_1}{s^+} \int_\Omega |u|^{s(x)} dx \geq \eta_k, \quad \forall u \in S_{r_k}^k. \tag{3.4}$$

By considering again (M_1) and (f_1), for $u \in S_{r_k}^k$ and $t \in (0,1)$, we have

$$I(tu) \quad = \widehat{M} \left(\int_\Omega \frac{|\Delta tu|^{p(x)}}{p(x)} dx \right) - \int_\Omega F(x, tu) dx$$

$$\leq m_2\left(\int_\Omega \frac{|\Delta tu|^{p(x)}}{p(x)}dx\right) - \frac{C_1}{s^+}\int_\Omega |tu|^{s(x)}dx$$
$$\leq \frac{m_2}{\alpha(p^-)^\alpha} t^{\alpha p^-} r_k^{\alpha p^-} - t^{s^+}\eta_k. \quad (3.5)$$

Because $s^+ < q^- \leq q^+ < \beta p^- \leq \alpha p^-$. we can find
$t_k \in (0,1)$ and $\varepsilon_k > 0$ such that

$$I(t_k u) \leq -\varepsilon_k < 0, \forall u \in S^k_{r_k},$$

that is,

$$I(u) \leq -\varepsilon_k < 0, \forall u \in S^k_{t_k r_k}.$$

It is clear that $\gamma(\mathbb{S}^k_{t_k r_k}) = k$, so $c_k \leq -\varepsilon_k < 0$. Finally, by Lemma 3.8 , Lemma 3.9 and above results, we can apply Theorem 3.7 to obtain that the functional I admits at least k pairs of distinct critical points, and since k is arbitrary, we obtain infinitely many critical points of I. The proof is completed.

Theorem 3.10. *Suppose* (M_1), (f_1) *and* (f_2) *hold. If* $q(x) < p(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$, *then the problem (1.1)-(1.2) has a sequence of solution* $\{\pm u_k : k = 1, 2, \ldots\}$ *such that* $I(\pm u_k) < 0$.

Firstly, we will prove that I is coercive. If we follow the same processes applied in the proof of the Lemma 3.8, and consider the fact $q^+ < p^-$, it is easy to get the coerciveness of I. Because I is weakly lower semi-continuous, I attains its minimum on X, that is, (1.1)-(1.2) has a solution. By help of coerciveness, we know that I satisfies the (PS) condition on X. Moreover, from condition (f_2), I is even.
In the rest of the proof, since we develop the same arguments which we used in the proof of Theorem 3.6, we omit the details.
Hence, if we follow the similar steps as we did in (3.4) and (3.5), and consider the fact $s^+ < q^- \leq q^+ < p^- < \alpha p^-$, we can find $t_k \in (0,1)$ and $\varepsilon_k > 0$ such that

$$I(u) \leq -\varepsilon_k < 0, \quad \forall u \in S^k_{t_k r_k}.$$

Obviously, $\gamma(S^k_{t_k r_k}) = k$, so $c_k \leq -\varepsilon_k < 0$. By Krasnoselskii's genus, each c_k is a critical value of I, hence there is a sequence of solutions $\{\pm u_k : k = 1, 2, ...\}$ such that $I(\pm u_k) < 0$.

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