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PREY-PREDATOR TRIDIAGONAL 4-DIMENSIONAL MODELS

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ABSTRACT: The prey-predator Lotka-Volterra models are some of the most popular mathematical models in biology and chemistry and they are in fact the first abstract models to analyze cooperativity, oscillatory behavior, and spaces synchronization at large scale of biochemistry, biomolecular, and medical interactions models.

In the article we will consider 4-dimensional tridiagonal Lotka-Volterra models. We determine some criteria for existence of first integrals of the systems.

We also discuss some differences of some properties of tridiagonal Lotka-Volterra models based on the parity of dimensions.

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1. INTRODUCTION AND STATEMENT

We consider the prey-predator periodic tridiagonal Lotka-Volterra models in the form

$$\begin{aligned}
\dot{x}_1 &= x_1(\qquad a_{12}x_2 \qquad\qquad + a_{14}x_4 + b_1),\\
\dot{x}_2 &= x_2(a_{21}x_1 \qquad + a_{23}x_3 \qquad\qquad + b_2),\\
\dot{x}_3 &= x_3(\qquad a_{32}x_2 \qquad\qquad + a_{34}x_4 + b_3),\\
\dot{x}_4 &= x_4(a_{41}x_1 \qquad + a_{43}x_3 \qquad\qquad + b_4),
\end{aligned} \tag{1.1}$$

where a_{ij} and b_i are real numbers.

The main purpose of this article is to establish some criteria for existence of first integrals of the system (1.1), based on Theorem 2 from [1]. We will also discuss some differences of the properties of system (1.1) based on the parity of dimension: $d = 3, 4$.

It is natural to rewrite the system (1.1) in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{D}(\boldsymbol{x})\left(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\right), \tag{1.2}$$

where: $\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^t \in \mathbb{R}^4$;

$$\boldsymbol{D}(\boldsymbol{x}) = \begin{pmatrix} x_1 & 0 & 0 & 0\\ 0 & x_2 & 0 & 0\\ 0 & 0 & x_3 & 0\\ 0 & 0 & 0 & x_4 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 0 & a_{12} & 0 & a_{14}\\ a_{21} & 0 & a_{23} & 0\\ 0 & a_{32} & 0 & a_{34}\\ a_{41} & 0 & a_{43} & 0 \end{pmatrix};$$

$\boldsymbol{b} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \end{pmatrix}^t \in \mathbb{R}^4$. Also, let us set

$$\boldsymbol{B} = \begin{pmatrix} a_{12} & a_{21} & 0 & 0\\ 0 & a_{23} & a_{32} & 0\\ 0 & 0 & a_{34} & a_{43}\\ a_{14} & 0 & 0 & a_{41} \end{pmatrix}$$

In [1], a result has been proved for existence of first integral of system (1.1) for any d. Here, we will formulate the algorithm in cited theorem in the case of four-dimensional spaces: $d = 4$.

Let:

$$a_{i\,i+1} \neq 0, \quad a_{i+1\,i} \neq 0, \quad i = 1, \ldots, 3, \quad a_{14} \neq 0 \neq a_{41}, \tag{1.3}$$

$$\det(\boldsymbol{B}) = a_{12}a_{23}a_{34}a_{41} - a_{14}a_{21}a_{32}a_{43} = 0. \tag{1.4}$$

Hence the PLU-decomposition of matrix $\boldsymbol{B}$ is

$$\boldsymbol{B} = \boldsymbol{L_B}\boldsymbol{U_B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{a_{14}}{a_{12}} & -\dfrac{a_{14}a_{21}}{a_{12}a_{23}} & \dfrac{a_{41}}{a_{43}} & 1 \end{pmatrix} \begin{pmatrix} a_{12} & a_{21} & 0 & 0 \\ 0 & a_{23} & a_{32} & 0 \\ 0 & 0 & a_{34} & a_{43} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore (see the first conditions in (1.3)), $\operatorname{rank}(\boldsymbol{B}) = \operatorname{rank}(\boldsymbol{U_B}) = 3$ and all solutions $\boldsymbol{\lambda}^*$ of the linear equation $\boldsymbol{B}\boldsymbol{\lambda} = \boldsymbol{0}$ are

$$\boldsymbol{\lambda}^* = \begin{pmatrix} -a_{21}a_{32}a_{43}t \\ a_{12}a_{32}a_{43}t \\ -a_{23}a_{12}a_{43}t \\ a_{34}a_{23}a_{12}t \end{pmatrix}, \quad t \in \mathbb{R}. \tag{1.5}$$

Analogously, for the determinant and PLU-decomposition of matrix $\boldsymbol{A}$, we receive:

$$\det(\boldsymbol{A}) = \frac{a_{12}a_{34}\left(a_{21}a_{43} - a_{23}a_{41}\right)^2}{a_{21}a_{43}},$$

$$\boldsymbol{A} = \boldsymbol{P_A}\boldsymbol{L_A}\boldsymbol{U_A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{a_{41}}{a_{21}} & 0 & 1 & 0 \\ 0 & \frac{a_{32}}{a_{12}} & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} a_{21} & 0 & a_{23} & 0 \\ 0 & a_{12} & 0 & a_{14} \\ 0 & 0 & \dfrac{a_{21}a_{43} - a_{23}a_{41}}{a_{21}} & 0 \\ 0 & 0 & 0 & a_{34}\dfrac{a_{21}a_{43} - a_{23}a_{41}}{a_{21}a_{43}}. \end{pmatrix}$$

Hence, in the general case, we have $\det(\boldsymbol{A}) \neq 0$.

This is the basic difference between odd and even cases: If the dimension d is an odd number, then $\det(\boldsymbol{B}) = 0$ implies $\det(\boldsymbol{A}) = 0$ (see Section 2, Example 4 in [1], for the case $d = 3$). But this implication does not hold true in even dimensions.

2. REGULAR CASES

Let

$$\det(\boldsymbol{A}) \neq 0, \text{ i.e. } a_{21}a_{43} \neq a_{23}a_{41}.$$

In this case, we have a unique solution of the system $\boldsymbol{A}^t\boldsymbol{\mu} = -\boldsymbol{c}$, where $\boldsymbol{c} = \begin{pmatrix} b_1\lambda_1 & b_2\lambda_2 & b_3\lambda_3 & b_4\lambda_4 \end{pmatrix}^t$:

$$\boldsymbol{\mu}^* = \begin{pmatrix} a_{3,2}a_{4,3}a_{2,1}(a_{2,3}b_4 - a_{4,3}b_2)t \\ a_{4,3}(-a_{1,2}a_{2,3}a_{4,1}b_3 + a_{2,1}a_{3,2}a_{4,3}b_1)t \\ a_{1,2}a_{2,3}a_{4,3}(-a_{2,1}b_4 + a_{4,1}b_2)t \\ a_{2,3}a_{2,1}a_{4,3}(b_3a_{1,2} - a_{3,2}b_1)t \end{pmatrix}, \qquad t \in \mathbb{R}. \tag{2.1}$$

In this way, we obtained the following result.

Theorem 1. *Let* (1.2) *be periodic tridiagonal system. Moreover, let:*

1. *Assumptions* (1.3) *and* (1.4) *hold true.*
2. $a_{21}a_{43} \neq a_{23}a_{41}$, *i.e.* $\det(\boldsymbol{A}) \neq 0$.

Then (setting $t = 1$ *in* (1.5) *and* (2.1)*)*

$$\Psi(\boldsymbol{x}) = \langle \boldsymbol{\lambda}^*, \boldsymbol{x} \rangle + \langle \boldsymbol{\mu}^*, \ln(\boldsymbol{x}) \rangle,$$

is a first integral of the periodic Lotka-Volterra system.

Moreover, if $\boldsymbol{\mu}^* = \boldsymbol{0}$ *and* $t \neq 0$, *then* $\boldsymbol{b} = \boldsymbol{0}$; *if* $\boldsymbol{\lambda}^* = \boldsymbol{0}$ *(i.e. if* $t = 0$*), then* $\boldsymbol{\mu}^* = \boldsymbol{b} = \boldsymbol{0}$. *Hence, if* $\boldsymbol{b} \neq \boldsymbol{0}$, *then* $\boldsymbol{\mu}^* \neq \boldsymbol{0}$.

Proof. The proof of Theorem 1 follows directly form Theorem 2 in [1].

Let us only mark that in this case, we receive $\langle \boldsymbol{\mu}^*, \boldsymbol{b} \rangle = 0$ after slight simplifications. □

3. SINGULAR CASES

Let

$$\det(\boldsymbol{A}) = 0, \text{ i.e. } a_{21}a_{43} = a_{23}a_{41}.$$

Hence, the U-component in PLU-decomposition of the matrix $\boldsymbol{A}^t$ is

$$\boldsymbol{U_A} = \begin{pmatrix} a_{21} & 0 & a_{23} & 0 \\ 0 & a_{12} & 0 & \dfrac{a_{12}a_{34}}{a_{32}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, the system $\boldsymbol{A}^t\boldsymbol{\mu} = -\boldsymbol{c}$, where $\boldsymbol{c} = \begin{pmatrix} b_1\lambda_1 & b_2\lambda_2 & b_3\lambda_3 & b_4\lambda_4 \end{pmatrix}^t$ has solutions if and only if:

$$\textbf{Subcase 1}: t = 0, \text{ i.e. } \boldsymbol{c} = \boldsymbol{0}$$

or

$$\textbf{Subcase 2}: t \neq 0, \ a_{12}b_3 = a_{32}b_1, \text{ and } a_{23}b_4 = a_{43}b_2.$$

Let $t = 0$, all solutions of $\boldsymbol{A\mu} = \boldsymbol{c}$ are

$$\boldsymbol{\mu}^* = \begin{pmatrix} -\dfrac{a_{32}}{a_{12}}u \\ -\dfrac{a_{43}}{a_{23}}v \\ u \\ v \end{pmatrix}, \quad u, v \in \mathbb{R}. \tag{3.1}$$

Assuming Subcase 2, the solutions are

$$\boldsymbol{\mu}^* = \begin{pmatrix} -a_{23}a_{32}(a_{12}a_{43}b_2t + u) \\ a_{12}a_{43}(a_{23}a_{32}b_1t - v) \\ a_{12}a_{23}u \\ a_{12}a_{23}v \end{pmatrix}, \quad u, v \in \mathbb{R}. \tag{3.2}$$

Obviously, in both cases, we may calculate all solutions of $\boldsymbol{A\mu} = \boldsymbol{c}$ by (3.2).

Theorem 2. *Let* (1.2) *be periodic tridiagonal system. Moreover, let:*

1. *Assumptions* (1.3) *and* (1.4) *hold true.*
2. $a_{21}a_{43} = a_{23}a_{41}$, *i.e.* $\det(\boldsymbol{A}) = 0$.

Then there exist two vectors $\boldsymbol{\lambda}^*$ *and* $\boldsymbol{\mu}^*$ *such that*

$$\Psi(\boldsymbol{x}) = \langle \boldsymbol{\lambda}^*, \boldsymbol{x} \rangle + \langle \boldsymbol{\mu}^*, \ln(\boldsymbol{x}) \rangle ,$$

is a first integral of the periodic Lotka-Volterra system.

Proof. Let

$$b_3 a_{12} = b_1 a_{32} \text{ and } b_4 a_{23} = b_2 a_{43}. \tag{3.3}$$

Then the system $\boldsymbol{A}^t \boldsymbol{\mu} = \boldsymbol{c}$ has a solution $\boldsymbol{\mu}^*$, given by (3.2).

Moreover, the equality $\langle \boldsymbol{\mu}^*, \boldsymbol{b} \rangle = 0$ is obvious.

Using Theorem 2 in [1], we received a three-parametric family of first integrals in this case:

$$\begin{aligned}\Psi(\boldsymbol{x}) = {} & a_{12}a_{23}a_{34}tx_4 - a_{12}a_{23}a_{43}tx_3 + a_{12}a_{32}a_{43}tx_2 - a_{21}a_{32}a_{43}tx_1 \\ & - a_{23}a_{32}(a_{12}a_{43}b_2 t + u)\ln(x_1) + a_{12}a_{43}(a_{23}a_{32}b_1 t - v)\ln(x_2) \\ & + a_{12}a_{23}u\ln(x_3) + a_{12}a_{23}v\ln(x_4).\end{aligned}$$

If one of the equalities in (3.3) does not hold true, then the system $\boldsymbol{A}^t \boldsymbol{\mu} = \boldsymbol{c}$ is an inconsistent system, if $\boldsymbol{c} \neq \boldsymbol{0}$. Hence in this case, we have to choose $t = 0$.

Let us suppose, the first equality in (3.3) is not valid.

Let $t = 0$. Then $\boldsymbol{c} = \boldsymbol{\lambda}^* = \boldsymbol{0}$.

The system $\boldsymbol{A}^t \boldsymbol{\mu} = \boldsymbol{0}$ has solutions $\boldsymbol{\mu}^*$, see (3.1). In this case, the equation $\langle \boldsymbol{\mu}^*, \boldsymbol{b} \rangle = 0$ has the following form

$$((b_3 u + b_4 v)a_{23} - a_{43}b_2 v)a_{12} - a_{32}b_1 a_{23}u = 0.$$

Hence, setting

$$u = -\frac{(a_{23}b_4 - a_{43}b_2)a_{12}}{a_{23}(a_{12}b_3 - a_{32}b_1)}v$$

we received the first integral in the form (here, we set $v = 1$)

$$\begin{aligned}\Psi(\boldsymbol{x}) &= \langle \boldsymbol{\mu}^*, \ln(\boldsymbol{x}) \rangle \\ &= -\frac{a_{32}}{a_{12}}\ln(x_1) - \frac{a_{43}}{a_{23}}\ln(x_2) \\ &\quad - \frac{(a_{23}b_4 - a_{43}b_2)a_{12}}{a_{23}(a_{12}b_3 - a_{32}b_1)}\ln(x_3) + \ln(x_4).\end{aligned}$$

□

4. COMMENTS AND REMARKS

Remark 1. In [1] (see also [3]), we proved that any Lotka-Volterra system

$$\begin{aligned}\dot{x}_1 =& x_1(& a_{12} x_2 + & x_3 & + b_1),\\ \dot{x}_2 =& x_2(x_1 & + & a_{23} x_3 & + b_2),\\ \dot{x}_3 =& x_3(a_{31} x_1 + & x_2 & & + b_3).\end{aligned} \tag{4.1}$$

has a two-parametric family of first integrals in the form

$$\Psi_3(\boldsymbol{x}) = x_1 t - a_{12} x_2 t + a_{12} a_{23} x_3 t - (a_{12} a_{23} b_3 t + a_{23} u) \ln(x_1) \\ + u \ln(x_2) + (a_{12} a_{23} b_1 t + a_{12} a_{23} u) \ln(x_3).$$

if $\det(\boldsymbol{A}) = \det(\boldsymbol{B}) = a_{12} a_{23} a_{31} + 1 = 0$ and $b_2 = a_{23} b_1 - a_{12} a_{23} b_3$.

Let us mark, if $t = 0$, then $\Psi_3(\boldsymbol{x})$ does not contains any non-logarithmic terms, i.e. there exists a first integral containing only linear combination of $\ln(x_i)$, $i = 1, 2, 3$.

If 4-dimensional case: the first integral is a linear combination of $\ln(x_i)$, $i = 1, 2, 3, 4$ in singular case (and if equalities (3.3) do not valid). In the regular case, such a presentation of the first integral is impossible.

It is not difficult to generalize these results in even and odd dimensions.

In the following two remarks, we will discus the case $\det(\boldsymbol{B}) \neq 0$, if $d = 3$ and $d = 4$, respectively.

Remark 2. Consider the system (4.1) and let us suppose now $\det(\boldsymbol{B}) \neq 0$. Let us also suppose that there exist a first integral of (4.1) in the form $\Psi(\boldsymbol{x}) = \langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle + \langle \boldsymbol{\mu}, \ln(\boldsymbol{x}) \rangle$. Then

$$\begin{aligned} 0 =& \dot{\Psi}(\boldsymbol{x}) = \langle \boldsymbol{\lambda}, (\boldsymbol{D}(\boldsymbol{x})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})) \rangle + \langle \boldsymbol{\mu}, (\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}) \rangle \\ =& \langle \boldsymbol{\lambda}, \boldsymbol{D}(\boldsymbol{x})\boldsymbol{A}\boldsymbol{x} \rangle + \langle \boldsymbol{\lambda}, \boldsymbol{D}(\boldsymbol{x})\boldsymbol{b} \rangle + \langle \boldsymbol{\mu}, \boldsymbol{A}\boldsymbol{x} \rangle + \langle \boldsymbol{\mu}, \boldsymbol{b} \rangle. \end{aligned}$$

This equality is possible if and only if

$$\begin{aligned} \langle \boldsymbol{\lambda}, \boldsymbol{D}(\boldsymbol{x})\boldsymbol{A}\boldsymbol{x} \rangle =& 0, \\ \langle \boldsymbol{\lambda}, \boldsymbol{D}(\boldsymbol{x})\boldsymbol{b} \rangle + \langle \boldsymbol{\mu}, \boldsymbol{A}\boldsymbol{x} \rangle =& 0, \\ \langle \boldsymbol{\mu}, \boldsymbol{b} \rangle =& 0. \end{aligned} \tag{4.2}$$

Rewriting the system (4.2) in coordinate form, we have

$$\begin{aligned}
&(a_{12}\lambda_1 + \lambda_2)\, x_1x_2 + (a_{31}\lambda_3 + \lambda_1)\, x_1x_3 \\
&\qquad + (a_{23}\lambda_2 + \lambda_3)\, x_2x_3 = 0, \\
&(\mu_2 + a_{31}\mu_3 + \lambda_1 b_1)\, x_1 + (a_{12}\mu_1 + \mu_3 + \lambda_2 b_2)\, x_2 \\
&\qquad + (\mu_1 + a_{23}\mu_2 + b_3\lambda_3)\, x_3 = 0, \\
&b_1\mu_1 + b_2\mu_2 + b_3\mu_3 = 0.
\end{aligned} \tag{4.3}$$

It is natural to use the matrices

$$\boldsymbol{B} = \begin{pmatrix} a_{12} & 1 & 0 \\ 0 & a_{23} & 1 \\ 1 & 0 & a_{31} \end{pmatrix} \text{ and } \boldsymbol{A} = \begin{pmatrix} 0 & a_{12} & 1 \\ 1 & 0 & a_{23} \\ a_{31} & 1 & 0 \end{pmatrix}.$$

The first equation in (4.3) is equivalent to the following homogeneous system

$$\boldsymbol{B\lambda} = \boldsymbol{0}.$$

Obviously ($\det(\boldsymbol{B}) \neq 0$) this system has only trivial solution, i.e. $\boldsymbol{\lambda}^* = \boldsymbol{0}$.

The second equation in (4.3) is equivalent to the following homogeneous system

$$\boldsymbol{A}^t\boldsymbol{\mu} = \boldsymbol{D}(\boldsymbol{b})\boldsymbol{\lambda}^* = \boldsymbol{0}, \tag{4.4}$$

where $\boldsymbol{D}(\boldsymbol{b}) = \left(\begin{smallmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{smallmatrix}\right)$. Here, again, the homogeneous system (4.4) has only trivial solution $\boldsymbol{\mu}^* = \boldsymbol{0}$, because $\det(\boldsymbol{A}) = \det(\boldsymbol{B}) \neq 0$.

Therefore, the system (4.1) does not posses a first integral in the form of Ψ.

It is not hard to generalize the arguments above for any odd dimension d, i.e. for the tridiagonal periodic d-dimensional Lotka-Volterra systems.

Contrary to the previous remark, in the next remark we will show that in even dimensions there exist first integrals in the form of Ψ (in the general case).

Remark 3. Consider the system

$$\dot{x}_1 = x_1(\quad a_{12}x_2 \quad + \quad x_4 + b_1),$$

$$\begin{aligned}
\dot{x}_2 =& x_2(\quad x_1 \quad +a_{23}x_3 \quad +b_2),\\
\dot{x}_3 =& x_3(\quad x_2 \quad +a_{34}x_4+b_3),\\
\dot{x}_4 =& x_4(a_{41}x_1 \quad + \quad x_3 \quad +b_4),
\end{aligned} \tag{4.5}$$

where a_{ij} and b_i are real numbers. Let $\det(\boldsymbol{B}) = a_{12}a_{23}a_{34}a_{41} + 1 \neq 0$.

Let us suppose that there exist a first integral of (4.5) in the form $\Psi(\boldsymbol{x}) = \langle \boldsymbol{\lambda}, \boldsymbol{x}\rangle + \langle \boldsymbol{\mu}, \ln(\boldsymbol{x})\rangle$. Using similar arguments as in the previous remark, we obtain the system

$$\begin{aligned}
&(\lambda_1 a_{12} + \lambda_2)\, x_2 x_1 + (\lambda_3 a_{31} + \lambda_1)\, x_3 x_1\\
&\qquad + (\lambda_2 a_{23} + \lambda_3)\, x_3 x_2 = 0,\\
&(\mu_3 a_{31} + \lambda_1 b_1 + \mu_2)\, x_1 + (\mu_1 a_{12} + \lambda_2 b_2 + \mu_3)\, x_2\\
&\qquad + (a_2 3\mu_2 + b_3\lambda_3 + \mu_1)\, x_3 = 0,\\
&\mu_1 b_1 + \mu_2 b_2 + \mu_3 b_3 = 0.
\end{aligned} \tag{4.6}$$

Again, we introduce the matrices

$$\boldsymbol{B} = \begin{pmatrix} a_{12} & 1 & 0 & 0\\ 0 & a_{23} & 1 & 0\\ 0 & 0 & a_{34} & 1\\ 1 & 0 & 0 & a_{41}\end{pmatrix} \text{ and } \boldsymbol{A} = \begin{pmatrix} 0 & a_{12} & 0 & 1\\ 1 & 0 & a_{23} & 0\\ 0 & 1 & 0 & a_{34}\\ a_{41} & 0 & 1 & 0\end{pmatrix}$$

and again it is possible to rewrite (4.6) in matrix form

$$\begin{aligned}
\boldsymbol{B\lambda} =& \boldsymbol{0},\\
\boldsymbol{A}^t\boldsymbol{\mu} =& \boldsymbol{D}(\boldsymbol{b})\boldsymbol{\lambda}^*,\\
\mu_1 b_1 + \mu_2 b_2 + \mu_3 b_3 =& 0
\end{aligned} \tag{4.7}$$

It follows from $\det(\boldsymbol{B}) \neq 0$ that the unique solution of the first equation in (4.7) is $\boldsymbol{\lambda}^* = \boldsymbol{0}$. Hence, the second equation has the form

$$\boldsymbol{A}^t\boldsymbol{\mu} = \boldsymbol{0}. \tag{4.8}$$

Now, if $\det(\boldsymbol{A}) \neq 0$, i.e. if $a_{12}a_{34}+a_{23}a_{41} = 1+a_{12}a_{23}a_{34}a_{41}$, then the equation (4.8) has nontrivial solution.

Let for example $a_{41}a_{23} = 1$, then

$$\boldsymbol{\mu}^* = \begin{pmatrix} 0\\ u\\ 0\\ -a_{23}u\end{pmatrix}, \quad u \in \mathbb{R}.$$

Setting $u = 1$ and supposing $b_4 a_{23} = b_2$, we receive a first integral of the system (4.5):

$$\Psi(\boldsymbol{x}) = \ln(x_2) - a_{23} \ln(x_4).$$

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