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HYPERBOLIC-PARABOLIC BALANCE LAWS: ASYMPTOTIC BEHAVIOR AND A CHEMOTAXIS MODEL

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ABSTRACT: We consider Cauchy problem of a general system of hyperbolic-parabolic balance laws. The author has proposed a set of structural conditions, which lead to global existence, L^p $(p \geq 2)$ decay rates and asymptotic behavior of solution when the Cauchy data are small perturbations of a constant equilibrium state. After surveying these recent results, we focus on a Keller-Segel type chemotaxis model with a logistic growth term. We use this model as an example to illustrate the difference between one space dimension and multi space dimensions in time asymptotic behavior. For the model in one space dimension we construct a time asymptotic solution in explicit formulation, and obtain faster decay rates of the solution to the asymptotic solution, in comparison with the rates of the solution to the constant equilibrium state. This shows that in one space dimension, a time asymptotic solution needs to include solutions of Burgers equation if the system is nonlinear in nature. By contrast, for multi space dimensions asymptotic solutions can be solutions to linear systems.

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1. INTRODUCTION

We are interested in an important class of partial differential equations:

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} \left[B_{jk}(w) w_{x_k}\right]_{x_j} + r(w), \quad m \geq 1, \tag{1.1}$$

where $w, f_j, r \in \mathbb{R}^n$ and $B_{jk} \in \mathbb{R}^{n\times n}$. Here w is the unknown density function, representing mass density, momentum density, etc; f_j, $1 \leq j \leq m$, are the flux functions; and r represents external force, relaxation, chemical reactions and so forth. The matrices B_{jk}, $1 \leq j, k \leq m$, are known as viscosity matrices. They describe viscosity, heat conduction, species diffusion, etc. Usually, the flux functions satisfy an entropy condition so that the corresponding inviscid system is hyperbolic, [1]. The viscosity matrices, however, are rank deficient as dictated by physics. Therefore, (1.1) is hyperbolic-parabolic rather than uniformly parabolic. The appearance of the lower order term makes (1.1) a system of balance laws rather than conservation laws.

Many interesting models from continuum physics and life sciences take the from (1.1). Among them are the familiar Navier-Stokes equations for compressible flows and the full system of magnetohydrodynamics, which are examples of the special case of hyperbolic-parabolic conservation laws:

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} [B_{jk}(w) w_{x_k}]_{x_j}, \quad m \geq 1, \tag{1.2}$$

setting $r = 0$ in (1.1). Other examples include Euler equations with damping and polyatomic gas flows in thermal non-equilibrium, which are of the special case of hyperbolic balance laws:

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = r(w), \quad m \geq 1, \tag{1.3}$$

with $B_{jk} = 0$ in (1.1). For the most general form (1.1), with nontrivial B_{jk} and r, we have polyatomic gas flows in both translational and vibrational non-equilibrium as an important example. We also have Keller-Segel equations with logistic growth in chemotaxis as an interesting application.

For (1.1) we consider the Cauchy problem with initial condition:

$$w(x, 0) = w_0(x), \tag{1.4}$$

where the Cauchy data w_0 is a small perturbation of a constant equilibrium state $\bar{w}$, $r(\bar{w}) = 0$. The author has proposed a set of structural conditions for (1.1), under which the existence of global solution to (1.1), (1.4) is established if w_0 is near $\bar{w}$ [8]. The result applies to all space dimensions, $m \geq 1$. The same set of structural conditions also imply optimal L^p ($p \geq 2$) convergence rates of w to $\bar{w}$. The rates are obtained for multi space dimensions ($m \geq 2$) in [9], and for one space dimension ($m = 1$) in [11]. Under the set of structural conditions, we may further obtain a time asymptotic solution of (1.1), (1.4) for multi space dimensions. It is the solution of the corresponding linear system of (1.1), linearized around the constant equilibrium state $\bar{w}$, with the same initial condition (1.4). Convergence rates of w to the asymptotic solution is faster than the rates of the asymptotic solution to $\bar{w}$, [10].

In this paper we first survey the structural conditions and the results on global existence and L^p decay rates. Then we focus on the discussion of asymptotic behavior of solution. In particular, there is an intrinsic difference between one space dimension and multi space dimensions. Due to the slow decay of solution, in one space dimension the solution to the corresponding linear system is not a time asymptotic solution if (1.1) in nonlinear in nature (i.e., when the reduced system is genuinely nonlinear). To illustrate such a difference we use Keller-Segel chemotaxis model with logistic growth in one space dimension as an example. For this model we construct in explicit form a time asymptotic solution, which is the self similar solution to the Burgers equation (a nonlinear equation). The convergence rate of the solution of Keller-Segel equations to the solution of Burgers equation obtained in this paper shows two things: Firstly, the L^p decay rates obtained in [11] are indeed optimal for generic perturbations. Secondly, the solution to the linear system, linearized around the constant equilibrium state, cannot be an asymptotic solution.

Next, we formulate our assumptions for (1.1). We consider a neighborhood $\mathbb{O}$ of a constant equilibrium state $\bar{w}$, $r(\bar{w}) = 0$. We define the equilibrium manifold $\mathbb{E}$ in $\mathbb{O}$ as

$$\mathbb{E} = \{w \in \mathbb{O} | r(w) = 0\}. \tag{1.5}$$

The functions $f_j(w)$, $B_{jk}(w)$ and $r(w)$ are assumed to be smooth in $\mathbb{O}$.

Assumption 1.1. *1. There exists a strictly convex entropy function η (a scalar function of w in $\mathbb{O}$) satisfying the following properties:*

(i) $\eta'' f_j'$, $1 \le j \le m$, *are symmetric in* $\mathbb{O}$.

(ii) In $\mathbb{O}$, $(\eta'' B_{jk})^t = \eta'' B_{kj}$, $1 \le j,k \le m$, *and* $\eta'' \sum_{j,k=1}^m B_{jk}\xi_k\xi_j$ *is symmetric, semi-positive definite for all unit vectors* $\xi = (\xi_1, \dots, \xi_m)^t \in \mathbb{S}^{m-1}$.

(iii) On $\mathbb{E}$, $\eta'' r'$ *is symmetric, semi-negative definite.*

2. *Equation* (1.1) *has* n_1 *conservation laws. That is, there is a partition* $n = n_1 + n_2$, $n_1, n_2 \ge 0$, *such that*

$$r(w) = \begin{pmatrix} 0_{n_1\times 1} \\ r_2(w) \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{1.6}$$

with $w_1 \in \mathbb{R}^{n_1}$, $r_2, w_2 \in \mathbb{R}^{n_2}$, *and* $(r_2)_{w_2} \in \mathbb{R}^{n_2\times n_2}$ *is nonsingular (if* $n_2 > 0$*).*

3. *There is a diffeomorphism* $\varphi \to w$ *from an open set of* $\mathbb{R}^n$ *to* $\mathbb{O}$ *and a constant orthogonal matrix* $P \in \mathbb{R}^{n\times n}$ *such that*

$$P^t B_{jk}(w(\varphi)) w_\varphi(\varphi) P = \begin{pmatrix} 0_{n_3\times n_3} & 0_{n_3\times n_4} \\ 0_{n_4\times n_3} & B_{jk}^* \end{pmatrix}, \qquad 1 \le j,k \le m. \tag{1.7}$$

Here $n_3, n_4 \ge 0$ *are two constants such that* $n_3 + n_4 = n$, *and* $\sum_{j,k=1}^m B_{jk}^*\xi_k\xi_j \in \mathbb{R}^{n_4\times n_4}$ *is nonsingular (if* $n_4 > 0$*) for all values of* φ *and all* $\xi = (\xi_1, \dots, \xi_m)^t \in \mathbb{S}^{m-1}$.

4. *[7] Let* $\xi = (\xi_1, \dots, \xi_m)^t \in \mathbb{S}^{m-1}$ *and*

$$A(\xi) = \sum_{j=1}^m f_j'(\bar w)\xi_j, \qquad B(\xi) = \sum_{j,k=1}^m B_{jk}(\bar w)\xi_k\xi_j. \tag{1.8}$$

Let N_1 *be the null space of* $B(\xi)$ *and* N_2 *be the null space of* $r'(\bar w)$. *Then for each* ξ, $N_1 \cap N_2$ *contains no eigenvectors of* $A(\xi)$.

A discussion on each of these conditions can be found in [8]. Here we only comment on the partitions $n = n_1 + n_2$ and $n = n_3 + n_4$. First, these two partitions are independent. The number n_1 is the number of conservation laws in (1.1), including viscous and inviscid ones. On the other hand, n_3 is the number of equations without a viscosity term, or being "hyperbolic type", including both conservation laws and non-conservation laws. The locations of the conservation laws and rate equations are also independent to the locations of the "hyperbolic" equations and "parabolic" equations. Matrix P in condition (3) of Assumption 1.1 is usually a permutation to accommodate such independence. In next section we use Keller-Segel model with logistic growth to illustrate the two partitions.

We introduce the following notations to abbreviate the norms of Sobolev spaces with respect to x:

$$\|\cdot\|_s = \|\cdot\|_{H^s(\mathbb{R}^m)}, \quad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^m)}. \tag{1.9}$$

With φ and P given in condition (3) of Assumption 1.1 we define

$$\tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} \equiv P^t \varphi(w), \tag{1.10}$$

where $\tilde{w}_1 \in \mathbb{R}^{n_3}$ and $\tilde{w}_2 \in \mathbb{R}^{n_4}$. Our first theorem is on global existence.

Theorem 1.2. *[8] Let $\bar{w}$ be a constant equilibrium state, Assumption 1.1 be satisfied, $s > \frac{m}{2} + 1$ $(m \geq 1)$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}^m)$. Then there exists a constant $\delta > 0$ such that if $\|w_0 - \bar{w}\|_s \leq \delta$, the Cauchy problem* (1.1), (1.4) *has a unique global solution w. The solution satisfies $w - \bar{w} \in C([0,\infty); H^s(\mathbb{R}^m))$, $D_x w \in L^2([0,\infty); H^{s-1}(\mathbb{R}^m))$, $D_x \tilde{w}_2(w) \in L^2([0,\infty); H^s(\mathbb{R}^m))$, $r(w) \in L^2([0,\infty); H^s(\mathbb{R}^m))$, and*

$$\sup_{t\geq 0} \|w - \bar{w}\|_s^2(t) + \int_0^\infty [\|D_x w\|_{s-1}^2(t) + \|D_x \tilde{w}_2(w)\|_s^2(t) + \|r_2(w)\|_s^2(t)]\, dt \leq C\|w_0 - \bar{w}\|_s^2, \tag{1.11}$$

where $C > 0$ is a constant. Here $D_x w$ denotes first partial derivatives of w with respect to x, etc.

Let D_x^l be partial derivatives $(\partial/\partial x)^\alpha$ with a multi index α such that $|\alpha| = l$. Our next theorem is on L^2 decay rates, cited from [9] for multi space dimensions, and from [11] for one space dimension.

Theorem 1.3. *[9, 11] Let $\bar{w}$ be a constant equilibrium state of* (1.1)*, and Assumption 1.1 be true. Let s be an integer, $s > \frac{m}{2} + 1$ if $m \geq 2$, $s \geq 4$ if $m = 1$, and $w_0 - \bar{w} \in H^s(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$. Then there exists a constant $\delta > 0$*

such that if $\delta_0 \equiv \|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1} \le \delta$, *the solution of* (1.1), (1.4) *given in Theorem 1.2 has the following estimates for* $t \ge 0$:

$$\|D_x^l(w-\bar{w})\|(t) \le C\delta_0(t+1)^{-\frac{m}{4}-\frac{l}{2}}, \quad 0 \le l \le s-2, \tag{1.12}$$

$$\|D_x^l r_2(w)\|(t) \le C\delta_0(t+1)^{-\frac{m}{4}-\frac{l+1}{2}}, \quad 0 \le l \le s-4. \tag{1.13}$$

Here $C > 0$ *in* (1.12) *and* (1.13) *is a constant.*

Recall Gagliardo-Nirenberg inequality [6]: There is a constant $C > 0$ such that for $g \in H^k(\mathbb{R}^m)$,

$$\|D_x^l g\|_{L^p} \le C\|D_x^k g\|^\theta \|g\|^{1-\theta}, \tag{1.14}$$

where $0 \le l \le k$, $p \in [2, \infty]$, and $\theta = [l + m(1/2 - 1/p)]/k \le 1$ ($\theta < 1$ if $p = \infty$). Applying (1.14) to $g = w - \bar{w}$ with $k = s - 2$, and to $g = r_2(w)$ with $k = s - 4$, we have the following corollary of Theorem 1.3:

Corollary 1.4. *[9, 11] Under the assumptions and notations of Theorem 1.3, the solution of* (1.1), (1.4) *has the following* L^p *estimates with* $p \ge 2$: *For* $t \ge 0$,

$$\|D_x^l(w-\bar{w})\|_{L^p}(t) \le C\delta_0(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l}{2}}, \quad 0 \le l \le s-2-m(\frac{1}{2}-\frac{1}{p}), \tag{1.15}$$

$$\|D_x^l r_2(w)\|_{L^p}(t) \le C\delta_0(t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l+1}{2}}, \quad 0 \le l \le s-4-m(\frac{1}{2}-\frac{1}{p}). \tag{1.16}$$

If $p = \infty$, *we further require* $l \ne s - 2 - m/2$ *for* (1.15), *and* $l \ne s - 4 - m/2$ *for* (1.16). *Here* $C > 0$ *is a constant.*

For multi space dimensions we further discuss asymptotic behavior of solution. The time asymptotic solution w^* of (1.1), (1.4) is the solution of the corresponding linear system with the same initial data:

$$w_t^* + \sum_{j=1}^m f_j'(\bar{w})w_{x_j}^* = \sum_{j,k=1}^m B_{jk}(\bar{w})w_{x_k x_j}^* + r'(\bar{w})(w^* - \bar{w}), \quad m \ge 2, \tag{1.17}$$

$$w^*(x, 0) = w_0(x). \tag{1.18}$$

Theorem 1.5. *[10] Under the assumptions and notations of Theorem 1.3, the solution of* (1.1), (1.4) *is time-asymptotically approximated by the solution of* (1.17), (1.18), *with the following L^p estimates for $p \geq 2$:*

$$\|D_x^l(w - w^*)\|_{L^p}(t) \leq C\delta_0^2 (t+1)^{-\frac{m}{2}(1-\frac{1}{p})-\frac{l+1}{2}} \begin{cases} [1+\ln(t+1)] & \text{if } m = 2 \\ 1 & \text{if } m > 2 \end{cases}, \quad t \geq 0, \quad (1.19)$$

where $0 \leq l \leq s - 2 - m(1/2 - 1/p)$, ($l \neq s - 2 - m/2$ if $p = \infty$,) and $C > 0$ is a constant.

Remark 1.6. Comparing (1.19) with (1.15), not only the convergence rate is faster by $(t+1)^{-\frac{1}{2}}$ (or by $(t+1)^{-\frac{1}{2}}[1+\ln(t+1)]$ if $m = 2$), the amplitude of the error is in the order of δ_0^2, smaller than δ_0.

We comment that Theorem 1.5 does not apply to one space dimension. This is clear for the special cases (1.2) and (1.3). For hyperbolic-parabolic conservation laws (1.2), time asymptotic solutions in one space dimension and pointwise error estimates are given in [5]. For hyperbolic balance laws (1.3), similar results are obtained in [12]. In both cases, Burgers waves (nonlinear waves) are an indispensable part of the asymptotic solution as long as the system has genuinely nonlinear characteristic fields. For the general system (1.1), we have not had a similar general result yet. In nest section we focus on a Keller-Segel type chemotaxis model in one space dimension, which is in the form of (1.1), with nontrivial viscosity terms and a lower order term. We find in explicit form a time asymptotic solution, which is a Burgers wave, with L^2 convergence rates. This settles the claim that in one space dimension, Theorem 1.5 does not apply, and a time asymptotic solution necessarily includes nonlinear waves for the general setting of (1.1).

2. KELLER-SEGEL MODEL WITH LOGISTIC GROWTH

In this section we study a Keller-Segel type chemotaxis model in one space dimension. The model can be converted into the form of (1.1). We give the main result of this paper on the asymptotic behavior of solution to the model.

The following model was proposed by Keller and Segel [2] to describe the oriented movement of cells toward a chemical concentration gradient:

$$\begin{cases} c_t = \varepsilon c_{xx} - \mu u c^p, \\ u_t = (Du_x - \chi u c^{-1} c_x)_x, \end{cases} \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.1}$$

where the unknown functions $c(x,t)$ and $u(x,t)$ denote the chemical concentration and cell density, respectively. The constants $\varepsilon \geq 0$ and $D \geq 0$ are, respectively, diffusion coefficients of the chemical and cells. The constants $\mu > 0$ and $\chi > 0$ are the coefficients of density-dependent degradation rate and of chemotactic sensitivity, respectively, while $p \geq 0$ is the degradation rate.

In our discussion we set $p = 1$, and the degradation term in (2.1) is $-\mu uc$. This implies that the chemical (oxygen) is consumed only when calls (bacteria) encounter the chemical. We also consider that cells undergo logistic growth. Therefore, our model reads

$$\begin{cases} c_t = \varepsilon c_{xx} - \mu u c, \\ u_t = (Du_x - \chi u c^{-1} c_x)_x + au(1 - \frac{u}{K}), \end{cases} \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.2}$$

where the constants $a \geq 0$ and $K > 0$ are the natural growth rate and the typical carrying capacity.

The singularity in the chemotactic sensitivity in (2.1) or (2.2) can be removed by the inverse Hopf-Cole transformation [3]:

$$v = (\ln c)_x. \tag{2.3}$$

Under the new variables v and u, we write (2.2) as

$$\begin{cases} v_t + (\mu u - \varepsilon v^2)_x = \varepsilon v_{xx}, \\ u_t + \chi(uv)_x = Du_{xx} + au(1 - \frac{u}{K}). \end{cases} \tag{2.4}$$

Using the positive parameters μ, χ and K we simplify (2.4) by re-scaling:

$$\tilde{t} = \mu K t, \quad \tilde{x} = \sqrt{\frac{\mu K}{\chi}} x, \quad \tilde{u} = \frac{u}{K}, \quad \tilde{v} = \sqrt{\frac{\chi}{\mu K}} v. \tag{2.5}$$

This converts (2.4) into

$$\begin{cases} \tilde{v}_{\tilde{t}} + (\tilde{u} - \tilde{\varepsilon}\tilde{v}^2)_{\tilde{x}} = \tilde{\varepsilon}\tilde{v}_{\tilde{x}\tilde{x}}, \\ \tilde{u}_{\tilde{t}} + (\tilde{u}\tilde{v})_{\tilde{x}} = \tilde{D}\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{a}\tilde{u}(1 - \tilde{u}), \end{cases} \tag{2.6}$$

where

$$\tilde{\varepsilon} = \frac{\varepsilon}{\chi}, \qquad \tilde{D} = \frac{D}{\chi}, \qquad \tilde{a} = \frac{a}{\mu K}. \tag{2.7}$$

Dropping the tilde accent we write (2.6) as

$$\begin{cases} v_t + (u - \varepsilon v^2)_x = \varepsilon v_{xx}, \\ u_t + (uv)_x = D u_{xx} + a u(1-u), \end{cases} \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.8}$$

where $\varepsilon \geq 0$, $D \geq 0$ and $a \geq 0$ are constant parameters.

We consider Cauchy problem of (2.8) with initial data

$$(v, u)(x, 0) = (v_0, u_0)(x), \tag{2.9}$$

where (v_0, u_0) is a perturbation of a constant equilibrium state $(\bar{v}, \bar{u})$. Here to be equilibrium, $\bar{u} = 0$ or 1, and to be stable equilibrium $\bar{u} = 1$. Also, setting $\bar{v} = 0$ is more physically meaningful when converting back to the variable c, see a discussion in [13]. Therefore, we take the constant equilibrium state as $(0, 1)$. Now we take a neighborhood $\mathbb{O}$ of $(0, 1)$. The equilibrium manifold is

$$\mathbb{E} = \{(v, 1)\} \cap \mathbb{O}.$$

Equation (2.8) is in the form of (1.1), which is reduced to

$$w_t + f(w)_x = (B(w) w_x)_x + r(w)$$

in one space dimension. Here

$$w = \begin{pmatrix} v \\ u \end{pmatrix}, \quad f(w) = \begin{pmatrix} u - \varepsilon v^2 \\ uv \end{pmatrix}, \quad B = \begin{pmatrix} \varepsilon & 0 \\ 0 & u \end{pmatrix}, \quad r(w) = \begin{pmatrix} 0 \\ au(1-u) \end{pmatrix}.$$

It is straightforward to verify that Assumption 1.1 is satisfied by (2.8) if at lest one of ε, D and a is positive. A more detailed discussion is given in [11]. Here we only make a few points: The entropy function can be taken as $\eta = \frac{1}{2}v^2 + u \ln u - u$ [4]. The partition $n = n_1 + n_2$ is $2 = 1 + 1$ if $a > 0$, and $2 = 2 + 0$ if $a = 0$. The partition $n = n_3 + n_4$ is $2 = 0 + 2$ if $\varepsilon > 0$ and $D > 0$, $2 = 1 + 1$ if $\varepsilon > 0$ and $D = 0$, or $\varepsilon = 0$ and $D > 0$, and $2 = 2 + 0$ if $\varepsilon = D = 0$. The diffeomorphism φ is the identity. The constant orthogonal matrix $P \in \mathbb{R}^{2\times 2}$ is the identity as well, except when $\varepsilon > 0$ and $D = 0$. In that case P is the permutation to interchange the two equations in (2.8). Condition (4), hence Assumption, 1.1, is satisfied if at least one of ε, D and a is positive. Therefore, we have seven cases to consider:

Case 1. $\varepsilon > 0$, $D > 0$ and $a > 0$;

Case 2. $\varepsilon = 0$, $D > 0$ and $a > 0$;

Case 3. $\varepsilon > 0$, $D = 0$ and $a > 0$;

Case 4. $\varepsilon = D = 0$ and $a > 0$;

Case 5. $\varepsilon > 0$, $D > 0$ and $a = 0$;

Case 6. $\varepsilon = 0$, $D > 0$ and $a = 0$;

Case 7. $\varepsilon > 0$, $D = 0$ and $a = 0$.

Here Case 4 fits the special case of hyperbolic balance laws (1.3), while Cases 5-7 fit the special case of hyperbolic-parabolic conservation laws (1.2). Since asymptotic behavior of solution is clear for these two special cases [5, 12], we focus on Cases 1-3. In these cases $r_2(w) = au(1-u)$, which is equivalent to $1-u$ since u is about one and $a > 0$. Applying Theorem 1.2, Theorem 1.3 and Corollary 1.4 to (2.8), (2.9), we have the following.

Theorem 2.1. *Let $a > 0$, $s \geq 2$ be an integer, and $(v_0, u_0 - 1) \in H^s(\mathbb{R})$. Then there exists a constant $\delta > 0$ such that if $\|(v_0, u_0 - 1)\|_s \leq \delta$, the Cauchy problem* (2.8), (2.9) *has a unique global solution. The solution has an energy estimate as follows.*

Case 1. If $\varepsilon > 0$ and $D > 0$ then

$$\sup_{t\geq 0} \|(v, u-1)\|_s^2(t) + \int_0^\infty (\|v_x\|_s^2 + \|u-1\|_{s+1}^2)(t)\, dt \leq C\|(v_0, u_o-1)\|_s^2. \tag{2.10}$$

Case 2. If $\varepsilon = 0$ and $D > 0$ then

$$\sup_{t\geq 0} \|(v, u-1)\|_s^2(t) + \int_0^\infty (\|v_x\|_{s-1}^2 + \|u-1\|_{s+1}^2)(t)\, dt \leq C\|(v_0, u_o-1)\|_s^2. \tag{2.11}$$

Case 3. If $\varepsilon > 0$ and $D = 0$ then

$$\sup_{t\geq 0} \|(v, u-1)\|_s^2(t) + \int_0^\infty (\|v_x\|_s^2 + \|u-1\|_s^2)(t)\, dt \leq C\|(v_0, u_o-1)\|_s^2. \tag{2.12}$$

Here in (2.10)-(2.12), *$C > 0$ is a constant.*

Theorem 2.2. *Let $a > 0$, $s \geq 4$ be an integer, and $(v_0, u_0 - 1) \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then there exists a constant $\delta > 0$ such that if $\delta_0 \equiv \|(v_0, u_0 - 1)\|_s + \|(v_0, u_0 - 1)\|_{L^1} \leq \delta$, the solution of (2.8), (2.9) given in Theorem 2.1 has the following L^p estimate with $p \geq 2$: For $t \geq 0$,*

$$\|D_x^l(v, u-1)\|_{L^p}(t) \leq C\delta_0(t+1)^{-\frac{1}{2}+\frac{1}{2p}-\frac{l}{2}}, \quad 0 \leq l \leq s - \frac{5}{2} + \frac{1}{p}, \tag{2.13}$$

$$\|D_x^l(u-1)\|_{L^p}(t) \leq C\delta_0(t+1)^{-1+\frac{1}{2p}-\frac{l}{2}}, \quad 0 \leq l \leq s - \frac{9}{2} + \frac{1}{p}, \tag{2.14}$$

where $C > 0$ is a constant.

To define the asymptotic solution we note from the integrals in (2.10)-(2.12) that $u - 1$ decays like v_x, hence faster than v. Taking leading terms in the second equation of (2.8) we have $v_x \approx a(1 - u)$ or

$$u - 1 \approx -\frac{1}{a}v_x. \tag{2.15}$$

Substituting (2.15) into the first equation of (2.8) we have

$$v_t - \varepsilon(v^2)_x \approx (\varepsilon + \frac{1}{a})v_{xx}. \tag{2.16}$$

Motivated by (2.16) we define a diffusion wave θ that is the self similar solution of

$$\theta_t - \varepsilon(\theta^2)_x = \alpha\theta_{xx}, \quad \alpha \equiv \varepsilon + \frac{1}{a}, \tag{2.17}$$

carrying the same mass as v:

$$\int_{\mathbb{R}} \theta(x,t)\,dx = \int_{\mathbb{R}} v(x,t)\,dx = \int_{\mathbb{R}} v_0(x)\,dx \equiv d_0. \tag{2.18}$$

Based on (2.15) and (2.16), we define the asymptotic solution:

$$(v, u) \approx (\theta, 1 - \frac{1}{a}\theta_x). \tag{2.19}$$

The asymptotic solution can be found explicitly. This is to solve (2.17), (2.18). If $\varepsilon = 0$, (2.17) is the heat equation, and θ is the heat kernel. If $\varepsilon > 0$, (2.17) is the Burgers equation, and can be solved by the Hopf-Cole transformation:

$$\theta = \frac{\alpha}{\varepsilon}(\ln\tilde{\theta})_x,$$

$$\tilde{\theta}_t = \alpha \tilde{\theta}_{xx}.$$

The formulas for θ are

$$\begin{aligned}
\theta(x,t) =& \frac{d_0}{\sqrt{4\pi\alpha(t+1)}} e^{-\frac{x^2}{4\alpha(t+1)}}, \quad \text{if } \varepsilon = 0, \\
\theta(x,t) =& -\frac{\sqrt{\alpha}}{2\varepsilon}(t+1)^{-\frac{1}{2}}(e^{-d_0\varepsilon/\alpha} - 1)e^{-\frac{x^2}{4\alpha(t+1)}} \\
& \times [\pi + (e^{-d_0\varepsilon/\alpha} - 1)\int_{\frac{x}{\sqrt{4\alpha(t+1)}}}^{\infty} e^{-y^2}\, dy]^{-1}, \quad \text{if } \varepsilon > 0.
\end{aligned} \tag{2.20}$$

The main result of this paper is the following theorem:

Theorem 2.3. *Let $a > 0$, and $(v_0, u_0 - 1) \in H^5(\mathbb{R}) \cap L^1(\mathbb{R})$. Let $|v_0(x)| \sim |x|^{-\alpha}$ as $|x| \to \infty$ with $\alpha > 2$. Then there exists a constant $\delta > 0$ such that if $\delta_1 \equiv \|(v_0, u_0 - 1)\|_5 + \|u_0 - 1\|_{L^1} + \sup_{\mathbb{R}}[(|x|+1)^\alpha |v_0(x)|] \le \delta$, the solution of* (2.8), (2.9) *given in Theorem 2.1 has the following property: For $t \ge 0$,*

$$\|D_x^l(v - \theta)\|(t) \le C\delta_1(t+1)^{-\frac{3}{4}-\frac{l}{2}+\gamma}, \quad l = 0, 1, \tag{2.21}$$

$$\|u - 1 + \frac{1}{a}\theta_x\|(t) \le C\delta_1(t+1)^{-\frac{5}{4}+\gamma}, \tag{2.22}$$

where $\gamma = 0$ if $\varepsilon = 0$ or $d_0 = 0$, and $\gamma > 0$ is arbitrarily chosen if $\varepsilon > 0$ and $d_0 \neq 0$, θ is defined in (2.20), *and $C > 0$ is a constant.*

The proof of Theorem 2.3 is given in next section. We comment that $\|D_x^l v\| \le C\delta_0(t+1)^{-\frac{1}{4}-\frac{l}{2}}$ for $l = 0, 1$, and $\|u - 1\| \le C\delta_0(t+1)^{-\frac{3}{4}}$from (2.13) and (2.14). Comparing with (2.21) and (2.22), it is clear that $(\theta, 1 - \frac{1}{a}\theta_x)$ is an asymptotic solution. Since the rates of $\|D_x^l\theta\|$ and $\|D_x^l(\frac{1}{a}\theta_x)\|$ are $(t+1)^{-\frac{1}{4}-\frac{l}{2}}$ and $(t+1)^{-\frac{3}{4}-\frac{l}{2}}$, respectively, see (3.23) below, the rates given in Theorem 2.2 are optimal for generic perturbations ($d_0 \neq 0$). Also, if $\varepsilon > 0$, i.e., the first equation in (2.8), which is the reduced equation, is nonlinear, the asymptotic solution is necessarily the solution of Burgers equation, and cannot be the solution of a linear equation. Thus results similar to Theorem 1.5 do not apply to one space dimension unless the system itself is linear in nature.

3. PROOF OF MAIN RESULT

In this section we prove our main result, Theorem 2.3. Throughout this section we use C and c for universal positive constants, which may vary line by line according to the context.

Recall from section 2 that (2.8) is in the form

$$w_t + f(w)_x = (B(w)w_x)_x + r(w), \tag{3.1}$$

satisfying Assumption 1.1 for $\varepsilon \geq 0$ and $D \geq 0$ as we have assumed $a > 0$ in Theorem 2.3. Now we linearize (2.8) around the constant equilibrium state $(0, 1)$ to write it as

$$\tilde{w}_t + A\tilde{w}_x = B\tilde{w}_{xx} + L\tilde{w} + R, \tag{3.2}$$

where

$$\begin{aligned} &\tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = \begin{pmatrix} v \\ u-1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \varepsilon & 0 \\ 0 & D \end{pmatrix}, \\ &L = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}, \quad R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} \varepsilon(v^2)_x \\ -u_x v - (u-1)v_x - a(u-1)^2 \end{pmatrix}. \end{aligned} \tag{3.3}$$

To study (3.2) we perform Fourier transform with respect to x:

$$\begin{aligned} \hat{\tilde{w}}(\xi, t) &= \int_{\mathbb{R}} \tilde{w}(x, t) e^{-ix\xi}\, dx, \\ \tilde{w}(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\tilde{w}}(\xi, t) e^{ix\xi}\, d\xi. \end{aligned} \tag{3.4}$$

This is gives us

$$\hat{\tilde{w}}_t = E(i\xi)\hat{\tilde{w}} + \hat{R}, \tag{3.5}$$

$$E(i\xi) = -i\xi A - \xi^2 B + L. \tag{3.6}$$

The solution of (3.5) is

$$\hat{\tilde{w}}(\xi, t) = e^{tE(i\xi)}\hat{\tilde{w}}(\xi, 0) + \int_0^t e^{(t-\tau)E(i\xi)}\hat{R}(\xi, \tau)\, d\tau. \tag{3.7}$$

We cite Lemma 3.2 in [11]:

Lemma 3.1. *Conditions (1), (2) and (4) of Assumption 1.1 imply*

$$|e^{tE(i\xi)}| \leq Ce^{-\frac{c\xi^2 t}{1+\xi^2}}, \quad \xi \in \mathbb{R}. \quad t \geq 0 \tag{3.8}$$

for some positive constants C and c.

For a more refined estimate we need the spectral decomposition of $E(i\xi)$. By straightforward calculation, the eigenvalues of $E(i\xi)$ are

$$\lambda_{1,2}(i\xi) = -\frac{1}{2}[a+(\varepsilon+D)\xi^2] \pm \sqrt{\frac{1}{4}[a+(\varepsilon+D)\xi^2]^2 - \xi^2[(\varepsilon a+1)+D\varepsilon\xi^2]}, \tag{3.9}$$

and the corresponding eigenprojections are

$$P_{1,2}(i\xi) = \frac{1}{-\xi^2+(\lambda_{1,2}+\varepsilon\xi^2)^2}\begin{pmatrix} -\xi^2 & -i\xi(\lambda_{1,2}+\varepsilon\xi^2) \\ -i\xi(\lambda_{1,2}+\varepsilon\xi^2) & (\lambda_{1,2}+\varepsilon\xi^2)^2 \end{pmatrix}. \tag{3.10}$$

Therefore,

$$e^{tE(i\xi)} = e^{\lambda_1(i\xi)t}P_1(i\xi) + e^{\lambda_2(i\xi)t}P_2(i\xi), \tag{3.11}$$

with $\lambda_{1,2}$ and $P_{1,2}$ given in (3.9) and (3.10).

The leading term in $e^{tE(i\xi)}$ comes from small ξ. Thus we consider Taylor expansions for $|\xi| \ll 1$:

$$\begin{aligned} &\lambda_1(i\xi) = -\alpha\xi^2 + O(\xi^4), \qquad \lambda_2(i\xi) = -a + O(\xi^2), \\ &P_1(i\xi) = \begin{pmatrix} 1+O(\xi^2) & -\frac{i\xi}{a}+O(\xi^3) \\ -\frac{i\xi}{a}+O(\xi^3) & O(\xi^2) \end{pmatrix}, \qquad P_2(i\xi) = \begin{pmatrix} O(\xi^2) & O(\xi) \\ O(\xi) & 1+O(\xi^2) \end{pmatrix}. \end{aligned} \tag{3.12}$$

Lemma 3.2. *Let $\eta > 0$ be small and $l \geq 0$ be an integer. Let $(e^{tE(i\xi)})_1$ and $(e^{tE(i\xi)})_2$ be the first and second rows of $e^{tE(i\xi)}$, respectively. Then*

$$\int_{|\xi|\leq\eta} |(i\xi)^l[(e^{tE(i\xi)})_1 \begin{pmatrix} \hat{h}(\xi) \\ 0 \end{pmatrix} - e^{-\alpha\xi^2 t}\hat{h}(\xi)]|^2\, d\xi$$

$$=O(1)(t+1)^{-\frac{5}{2}}\|D_x^l h\|_{L^1}^2 + O(1)e^{-at}\|D_x^l h\|^2 \tag{3.13}$$

$$=O(1)(t+1)^{-l-\frac{5}{2}}\|h\|_{L^1}^2 + O(1)e^{-at}\|D_x^l h\|^2, \tag{3.14}$$

$$\int_{|\xi|\leq\eta} |(i\xi)^l(e^{tE(i\xi)})_1 \begin{pmatrix} 0 \\ \hat{h}(\xi) \end{pmatrix} |^2\, d\xi$$

$$=O(1)(t+1)^{-l-\frac{3}{2}}\|h\|_{L^1}^2 + O(1)e^{-at}\|h\|^2, \tag{3.15}$$

$$\int_{|\xi|\leq\eta} |(e^{tE(i\xi)})_2 \begin{pmatrix} \hat{h}(\xi) \\ 0 \end{pmatrix} + \frac{1}{a}e^{-\alpha\xi^2 t}i\xi\hat{h}(\xi)|^2\, d\xi$$

$$=O(1)(t+1)^{-\frac{5}{2}}\|h_x\|_{L^1}^2 + O(1)e^{-at}\|h_x\|^2 \tag{3.16}$$

$$=O(1)(t+1)^{-\frac{7}{2}}\|h\|_{L^1}^2 + O(1)e^{-at}\|\ h_x\|^2, \tag{3.17}$$

$$\int_{|\xi|\le\eta} |(e^{tE(i\xi)})_2 \begin{pmatrix} 0 \\ \hat{h}(\xi) \end{pmatrix} |^2\, d\xi$$
$$= O(1)(t+1)^{-\frac{5}{2}}\|h\|_{L^1}^2 + O(1)e^{-at}\|h\|^2. \tag{3.18}$$

Proof. For small $\eta > 0$ the Taylor expansions in (3.12) hold in $[-\eta, \eta]$. With (3.11), we have

$$(e^{tE(i\xi)})_1 \begin{pmatrix} \hat{h}(\xi) \\ 0 \end{pmatrix} = e^{\lambda_1(i\xi)t}[1 + O(\xi^2)]\hat{h}(\xi) + e^{\lambda_2(i\xi)t}O(\xi^2)\hat{h}(\xi)$$
$$= \{e^{-\alpha\xi^2 t} + e^{-\alpha\xi^2 t}[e^{O(\xi^4)t} - 1] + e^{-\alpha\xi^2 t + O(\xi^4)t}O(\xi^2) + e^{-at+O(\xi^2)t}O(\xi^2)\}\hat{h}(\xi)$$
$$= [e^{-\alpha\xi^2 t} + e^{-\alpha\xi^2 t+O(\xi^4)t}O(\xi^4)t + e^{-\alpha\xi^2 t+O(\xi^4)t}O(\xi^2) + e^{-at+O(\xi^2)t}O(\xi^2)]\hat{h}(\xi).$$

Therefore,

$$\int_{|\xi|\le\eta} |(i\xi)^l[(e^{tE(i\xi)})_1 \begin{pmatrix} \hat{h}(\xi) \\ 0 \end{pmatrix} - e^{-\alpha\xi^2 t}\hat{h}(\xi)]|^2\, d\xi$$
$$\le C\int_{|\xi|\le\eta} \xi^{2l}[e^{-\alpha\xi^2 t/2}\xi^2 + e^{-at/2}\xi^2]^2|\hat{h}(\xi)|^2\, d\xi$$
$$\le C\int_{|\xi|\le\eta} e^{-\alpha\xi^2 t}\xi^{2l+4}\, d\xi\|\hat{h}\|_{L^\infty}^2 + Ce^{-at}\int_{|\xi|\le\eta} |(i\xi)^l\hat{h}(\xi)|^2\, d\xi$$
$$\le C(t+1)^{-l-\frac{5}{2}}\|h\|_{L^1}^2 + Ce^{-at}\|D_x^l h\|^2,$$

where we have applied Plancherel theorem. It is clear that the first term on the right-hand side can be replaced by

$$C\int_{|\xi|\le\eta} e^{-\alpha\xi^2 t}\xi^4\, d\xi\|(i\xi)^l\hat{h}\|_{L^\infty}^2 \le C(t+1)^{-\frac{5}{2}}\|D_x^l h\|_{L^1}^2.$$

This gives us (3.13) and (3.14). Equations (3.15)-(3.18) can be verified similarly. □

For $t \ge 0$ let

$$N(t) = \sup_{0\le\tau\le t}[(\tau+1)^{\frac{3}{4}-\gamma}\|v-\theta\|(\tau) + (\tau+1)^{\frac{5}{4}-\gamma}\|v_x-\theta_x\|(\tau)], \tag{3.19}$$

where $\gamma \ge 0$ is defined in Theorem 2.3. Equation (3.19) gives us

$$\begin{aligned} \|v-\theta\|(t) &\le N(t)(t+1)^{-\frac{3}{4}+\gamma}, \\ \|v_x-\theta_x\|(t) &\le N(t)(t+1)^{-\frac{5}{4}+\gamma}, \end{aligned} \qquad t \ge 0. \tag{3.20}$$

From (2.13) and (2.14), noting $s \geq 5$, we have

$$\begin{aligned} &\|D_x^l v\|(t) \leq C\delta_1(t+1)^{-\frac{1}{4}-\frac{l}{2}}, \quad 0 \leq l \leq 3, \\ &\|D_x^l (u-1)\|(t) \leq C\delta_1(t+1)^{-\frac{3}{4}-\frac{l}{2}}, \quad l = 0,1, \\ &\|D_x^2 u\|(t) \leq C\delta_1(t+1)^{-\frac{5}{4}}. \end{aligned} \tag{3.21}$$

Here we have applied the fact that $\delta_0 \leq C\delta_1$. Equations (3.20) and (3.21), together with Sobolev inequality, imply

$$\begin{aligned} &\|v\|_{L^\infty}(t) \leq C\|v\|^{\frac{1}{2}}(t)\|v_x\|^{\frac{1}{2}}(t) \leq C\delta_1(t+1)^{-\frac{1}{2}}, \\ &\|v_x\|_{L^\infty}(t) \leq C\delta_1(t+1)^{-1}, \qquad \|u-1\|_{L^\infty}(t) \leq C\delta_1(t+1)^{-1}. \end{aligned} \tag{3.22}$$

For θ given in (2.20), by straightforward calculation, we have

$$|D_x^l \theta(x,t)| \leq C\delta_1(t+1)^{-\frac{1}{2}-\frac{l}{2}} e^{-\frac{x^2}{C(t+1)}}, \quad 0 \leq l \leq 2 \tag{3.23}$$

for $\varepsilon \geq 0$, where we have used the fact that $|d_0| \leq C\delta_1$.

Let Θ be the asymptotic solution of the perturbation:

$$\Theta = \begin{pmatrix} \theta \\ -\frac{1}{a}\theta_x \end{pmatrix} \equiv \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}. \tag{3.24}$$

Introduce the following notations:

$$q_1 = \varepsilon v^2, \quad q_2 = \varepsilon\theta^2, \quad Q = \begin{pmatrix} q_{2x} \\ -\frac{1}{a}q_{2xx} \end{pmatrix} \equiv \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}. \tag{3.25}$$

By (2.17) we write $\hat{\Theta}$ as

$$\hat{\Theta}(\xi,t) = e^{-\alpha\xi^2 t}\hat{\Theta}(\xi,0) + \int_0^t e^{-\alpha\xi^2(t-\tau)}\hat{Q}(\xi,\tau)\,d\tau. \tag{3.26}$$

Combining (3.7) and (3.26) gives us

$$\begin{aligned} (\hat{\tilde{w}} - \hat{\Theta})(\xi,t) = {} & e^{tE(i\xi)}\hat{\tilde{w}}(\xi,0) - e^{-\alpha\xi^2 t}\hat{\Theta}(\xi,0) \\ & + \int_0^t [e^{(t-\tau)E(i\xi)}\hat{R}(\xi,\tau) - e^{-\alpha\xi^2(t-\tau)}\hat{Q}(\xi,\tau)]\,d\tau. \end{aligned} \tag{3.27}$$

Lemma 3.3. *With the same notations as in Lemma 3.2 for the rows of $e^{tE(i\xi)}$, we have*

$$\|(i\xi)^l[(e^{tE(i\xi)})_1\hat{\tilde{w}}(\xi,0) - e^{-\alpha\xi^2 t}\hat{\Theta}_1(\xi,0)]\|^2 \leq C\delta_1^2(t+1)^{-\frac{3}{2}-l}, \quad l = 0,1, \tag{3.28}$$

$$\|[(e^{tE(i\xi)})_2\hat{\tilde{w}}(\xi,0) - e^{-\alpha\xi^2 t}\hat{\Theta}_2(\xi,0)\|^2 \leq C\delta_1^2(t+1)^{-\frac{5}{2}}. \tag{3.29}$$

Proof. From (2.18) we may define

$$V_0(x) = \int_{-\infty}^{x} [v_0(y) - \theta(y,0)]\, dy = -\int_{x}^{\infty} [v_0(y) - \theta(y,0)]\, dy. \tag{3.30}$$

This implies

$$V_0'(x) = v_0(x) - \theta(x,0). \tag{3.31}$$

From the hypotheses of Theorem 2.3,

$$|v_0(x)| \le \delta_1(|x|+1)^{-\alpha}, \qquad \alpha > 2. \tag{3.32}$$

Therefore, for $x \le 0$,

$$\begin{aligned} |V_0(x)| = |\int_{-\infty}^{x} [v_0(y) - \theta(y,0)]\, dy| &\le \int_{-\infty}^{x} |v_0(y)|\, dy + \int_{-\infty}^{x} |\theta(y,0)|\, dy \\ \le \delta_1 \int_{-\infty}^{x} (-y+1)^{-\alpha}\, dy &+ C\delta_1 \int_{-\infty}^{x} e^{-\frac{y^2}{C}}\, dy \le C\delta_1(|x|+1)^{-\alpha+1}, \end{aligned}$$

where we have applied (3.23). Similarly, for $x \ge 0$,

$$\begin{aligned} |V_0(x)| = |-\int_{x}^{\infty} [v_0(y) - \theta(y,0)]\, dy| \le & \int_{x}^{\infty} |v_0(y)|\, dy + \int_{x}^{\infty} |\theta(y,0)|\, dy \\ \le & C\delta_1(|x|+1)^{-\alpha+1}. \end{aligned}$$

That is,

$$|V_0(x)| \le C\delta_1(|x|+1)^{-\alpha+1}, \qquad x \in \mathbb{R}.$$

With (3.32), we have

$$\|V_0\|_{L^1} \le C\delta_1, \qquad \|v_0\|_{L^1} \le C\delta_1. \tag{3.33}$$

Now we prove (3.28). By definition,

$$\|(i\xi)^l[(e^{tE(i\xi)})_1 \hat{\tilde{w}}(\xi,0) - e^{-\alpha\xi^2 t}\hat{\Theta}_1(\xi,0)]\|^2 = I_1 + I_2, \tag{3.34}$$

where for a small constant $\eta > 0$,

$$\begin{aligned} I_1 &= \int_{|\xi|\le\eta} |(i\xi)^l[(e^{tE(i\xi)})_1 \hat{\tilde{w}}(\xi,0) - e^{-\alpha\xi^2 t}\hat{\Theta}_1(\xi,0)]|^2\, d\xi, \\ I_2 &= \int_{|\xi|\ge\eta} |(i\xi)^l[(e^{tE(i\xi)})_1 \hat{\tilde{w}}(\xi,0) - e^{-\alpha\xi^2 t}\hat{\Theta}_1(\xi,0)]|^2\, d\xi. \end{aligned}$$

For I_1 we apply (3.3) and (3.24) to write

$$\begin{aligned}
&(i\xi)^l[(e^{tE(i\xi)})_1\hat{\tilde{w}}(\xi,0)-e^{-\alpha\xi^2t}\hat{\Theta}_1(\xi,0)]\\
=&(i\xi)^le^{-\alpha\xi^2t}[\hat{v}_0(\xi)-\hat{\theta}(\xi,0)]+(i\xi)^l[(e^{tE(i\xi)})_1\begin{pmatrix}\hat{v}_0(\xi)\\0\end{pmatrix}-e^{-\alpha\xi^2t}\hat{v}_0(\xi)]\\
&+(i\xi)^l(e^{tE(i\xi)})_1\begin{pmatrix}0\\\hat{\tilde{w}}_2(\xi,0)\end{pmatrix}.
\end{aligned}$$

From (3.31), (3.14), (3.15), (3.3) and (3.33), we have

$$\begin{aligned}
I_1\le&C\int_{|\xi|\le\eta}\xi^{2l+2}e^{-2\alpha\xi^2t}|\hat{V}_0(\xi)|^2\,d\xi+C[(t+1)^{-l-\frac{5}{2}}\|v_0\|_{L^1}^2+e^{-at}\|D_x^lv_0\|^2]\\
&+C[(t+1)^{-l-\frac{3}{2}}\|\tilde{w}_2\|_{L^1}^2(0)+e^{-at}\|\tilde{w}_2\|^2(0)]\\
\le&C(t+1)^{-l-\frac{3}{2}}(\|V_0\|_{L^1}^2+\|v_0\|_{L^1}^2+\|v_0\|_l^2+\|u_0-1\|_{L^1}^2+\|u_0-1\|^2)\\
\le&C\delta_1^2(t+1)^{-l-\frac{3}{2}},\qquad l=0,1.
\end{aligned}\tag{3.35}$$

For I_2 we apply Lemma 3.1, (3.24), (3.3) and (3.23) to have

$$\begin{aligned}
I_2\le&C\int_{|\xi|\ge\eta}\xi^{2l}e^{-\frac{2c\xi^2t}{1+\xi^2}}|\hat{\tilde{w}}(\xi,0)|^2\,d\xi+C\int_{|\xi|\ge\eta}\xi^{2l}e^{-2\alpha\xi^2t}|\hat{\theta}(\xi,0)|^2\,d\xi\\
\le&Ce^{-\tilde{c}t}[\|D_x^l\tilde{w}\|^2(0)+\|D_x^l\theta\|^2(0)]\le C\delta_1^2e^{-\tilde{c}t},\quad l=0,1,
\end{aligned}\tag{3.36}$$

where $\tilde{c}>0$ is a constant. Combining (3.34)-(3.36) we arrive at (3.28). Equation (3.29) can be proved in a similar way, using (3.17) and (3.18). $\square$

Lemma 3.4. *With the same notations as in Lemma 3.2 for the rows of $e^{tE(i\xi)}$, for $l=0,1$ we have*

$$\begin{aligned}
&\|(i\xi)^l[(e^{(t-\tau)E(i\xi)})_1\hat{R}(\xi,\tau)-e^{-\alpha\xi^2(t-\tau)}\hat{Q}_1(\xi,\tau)]\|^2\\
=&O(1)[\varepsilon^2\delta_1^2N^2(\tau)(t-\tau+1)^{-\frac{3}{2}}(\tau+1)^{-2-l+2\gamma}+\varepsilon^2\delta_1^4(t-\tau+1)^{-\frac{5}{2}}(\tau+1)^{-2-l}]\\
&+O(1)\delta_1^4(t-\tau+1)^{-\frac{3}{2}-l}(\tau+1)^{-3}
\end{aligned}\tag{3.37}$$

$$\begin{aligned}
=&O(1)[\varepsilon^2\delta_1^2N^2(\tau)(t-\tau+1)^{-\frac{3}{2}-l}(\tau+1)^{-2+2\gamma}+\varepsilon^2\delta_1^4(t-\tau+1)^{-\frac{5}{2}-l}(\tau+1)^{-2}]\\
&+O(1)\delta_1^4(t-\tau+1)^{-\frac{3}{2}-l}(\tau+1)^{-3},
\end{aligned}\tag{3.38}$$

$$
\begin{aligned}
&\|(e^{(t-\tau)E(i\xi)})_2\hat{R}(\xi,\tau)-e^{-\alpha\xi^2(t-\tau)}\hat{Q}_2(\xi,\tau)\|^2\\
=&O(1)[\varepsilon^2\delta_1^2N^2(\tau)(t-\tau+1)^{-\frac{3}{2}}(\tau+1)^{-3+2\gamma}+\varepsilon^2\delta_1^4(t-\tau+1)^{-\frac{5}{2}}(\tau+1)^{-3}\\
&+\varepsilon^2\delta_1^4e^{-\tilde{c}(t-\tau)}(\tau+1)^{-\frac{5}{2}}]+O(1)\delta_1^4(t-\tau+1)^{-\frac{5}{2}}(\tau+1)^{-3} \qquad (3.39)\\
=&O(1)[\varepsilon^2\delta_1^2N^2(\tau)(t-\tau+1)^{-\frac{5}{2}}(\tau+1)^{-2+2\gamma}+\varepsilon^2\delta_1^4(t-\tau+1)^{-\frac{5}{2}}(\tau+1)^{-3}\\
&+\varepsilon^2\delta_1^4e^{-\tilde{c}(t-\tau)}(\tau+1)^{-\frac{5}{2}}]+O(1)\delta_1^4(t-\tau+1)^{-\frac{5}{2}}(\tau+1)^{-3}, \qquad (3.40)
\end{aligned}
$$

where $\tilde{c}>0$ is a constant, and if $d_0=0$, i.e., $\theta=0$, the terms with $N^2(\tau)$ in (3.37), (3.38) *and* (3.40) *are replaced by $\varepsilon^2N^4(\tau)(t-\tau+1)^{-\frac{3}{2}}(\tau+1)^{-3-l}$, $\varepsilon^2N^4(\tau)(t-\tau+1)^{-\frac{3}{2}-l}(\tau+1)^{-3}$ and $\varepsilon^2N^4(\tau)(t-\tau+1)^{-\frac{5}{2}}(\tau+1)^{-3}$, respectively.*

Proof. By definition,

$$
\|(i\xi)^l[(e^{(t-\tau)E(i\xi)})_1\hat{R}(\xi,\tau)-e^{-\alpha\xi^2(t-\tau)}\hat{Q}_1(\xi,\tau)]\|^2=I_1+I_2, \qquad (3.41)
$$

where for a small constant $\eta>0$,

$$
\begin{aligned}
I_1&=\int_{|\xi|\le\eta}|(i\xi)^l[(e^{(t-\tau)E(i\xi)})_1\hat{R}(\xi,\tau)-e^{-\alpha\xi^2(t-\tau)}\hat{Q}_1(\xi,\tau)]|^2\,d\xi,\\
I_2&=\int_{|\xi|\ge\eta}|(i\xi)^l[(e^{(t-\tau)E(i\xi)})_1\hat{R}(\xi,\tau)-e^{-\alpha\xi^2(t-\tau)}\hat{Q}_1(\xi,\tau)]|^2\,d\xi.
\end{aligned}
$$

For I_1 from (3.3) and (3.25) we have

$$
\begin{aligned}
&(i\xi)^l[(e^{(t-\tau)E(i\xi)})_1\hat{R}(\xi,\tau)-e^{-\alpha\xi^2(t-\tau)}\hat{Q}_1(\xi,\tau)]\\
=&(i\xi)^le^{-\alpha\xi^2(t-\tau)}[i\xi\hat{q}_1(\xi,\tau)-i\xi\hat{q}_2(\xi,\tau)]+(i\xi)^l[(e^{(t-\tau)E(i\xi)})_1\begin{pmatrix}i\xi\hat{q}_1(\xi,\tau)\\0\end{pmatrix}\\
&-e^{-\alpha\xi^2(t-\tau)}i\xi\hat{q}_1(\xi,\tau)]+(i\xi)^l(e^{(t-\tau)E(i\xi)})_1\begin{pmatrix}0\\\hat{R}_2(\xi,\tau)\end{pmatrix}.
\end{aligned}
$$

Applying (3.14) and (3.15) gives us

$$
\begin{aligned}
I_1\le&C\int_{|\xi|\le\eta}\xi^{2l+2}e^{-2\alpha\xi^2(t-\tau)}|\hat{q}_1-\hat{q}_2|^2(\xi,\tau)\,d\xi+C[(t-\tau+1)^{-l-\frac{5}{2}}\|D_xq_1\|_{L^1}^2(\tau)\\
&+e^{-a(t-\tau)}\|D_x^{l+1}q_1\|^2(\tau)+(t-\tau+1)^{-l-\frac{3}{2}}\|R_2\|_{L^1}^2(\tau)+e^{-a(t-\tau)}\|R_2\|^2(\tau)]. \qquad (3.42)
\end{aligned}
$$

From (3.25), (3.3) and (3.20)-(3.23) we have

$$\begin{aligned}
\|q_1-q_2\|_{L^1}(\tau) &= \varepsilon\|(v+\theta)(v-\theta)\|_{L^1}(\tau) \le \varepsilon(\|v\|+\|\theta\|)(\tau)\|v-\theta\|(\tau)\\
&\le C\varepsilon\delta_1 N(\tau)(\tau+1)^{-1+\gamma},\\
\|D_x q_1\|_{L^1}(\tau) &= 2\varepsilon\|vv_x\|_{L^1}(\tau)\le 2\varepsilon\|v\|(\tau)\|v_x\|(\tau)\le C\varepsilon\delta_1^2(\tau+1)^{-1},\\
\|R_2\|_{L^1}(\tau) &\le (\|u_x\|\|v\|+\|u-1\|\|v_x\|+a\|u-1\|^2)(\tau)\le C\delta_1^2(\tau+1)^{-\frac{3}{2}},\\
\|D_x q_1\|(\tau) &\le 2\varepsilon\|v\|_{L^\infty}(\tau)\|v_x\|(\tau)\le C\varepsilon\delta_1^2(\tau+1)^{-\frac{5}{4}},\\
\|D_x^2 q_1\|(\tau) &\le 2\varepsilon(\|v_x\|_{L^\infty}\|v_x\|+\|v\|_{L^\infty}\|v_{xx}\|)(\tau)\le C\varepsilon\delta_1^2(\tau+1)^{-\frac{7}{4}},\\
\|R_2\|(\tau) &\le (\|v\|_{L^\infty}\|u_x\|+\|u-1\|_{L^\infty}\|v_x\|+\|u-1\|_{L^\infty}\|u-1\|)(\tau)\le C\delta_1^2(\tau+1)^{-\frac{7}{4}}.
\end{aligned}\tag{3.43}$$

Substituting (3.43) into (3.42) gives us

$$\begin{aligned}
I_1 \le & C(t-\tau+1)^{-l-\frac{3}{2}}\|q_1-q_2\|^2_{L^1}(\tau)+C\varepsilon^2\delta_1^4(t-\tau+1)^{-l-\frac{5}{2}}(\tau+1)^{-2}\\
&+C\delta_1^4(t-\tau+1)^{-l-\frac{3}{2}}(\tau+1)^{-3}\\
\le & C\varepsilon^2\delta_1^2[N^2(\tau)(t-\tau+1)^{-l-\frac{3}{2}}(\tau+1)^{-2+2\gamma}+\delta_1^2(t-\tau+1)^{-l-\frac{5}{2}}(\tau+1)^{-2}]\\
&+C\delta_1^4(t-\tau+1)^{-l-\frac{3}{2}}(\tau+1)^{-3}, \qquad l=0,1.
\end{aligned}\tag{3.44}$$

If we apply (3.13) and (3.15) instead, (3.42) becomes

$$\begin{aligned}
I_1 \le & C\int_{|\xi|\le\eta}\xi^2 e^{-2\alpha\xi^2(t-\tau)}|(i\xi)^l(\hat{q}_1-\hat{q}_2)|^2(\xi,\tau)\,d\xi+C[(t-\tau+1)^{-\frac{5}{2}}\|D_x^{l+1}q_1\|^2_{L^1}(\tau)\\
&+e^{-a(t-\tau)}\|D_x^{l+1}q_1\|^2(\tau)+(t-\tau+1)^{-l-\frac{3}{2}}\|R_2\|^2_{L^1}(\tau)+e^{-a(t-\tau)}\|R_2\|^2(\tau)].
\end{aligned}\tag{3.45}$$

Since (3.20)-(3.23) imply

$$\begin{aligned}
\|D_x(q_1-q_2)\|_{L^1}(\tau) &\le \varepsilon[(\|v_x\|+\|\theta_x\|)\|v-\theta\|+(\|v\|+\|\theta\|)\|v_x-\theta_x\|](\tau)\\
&\le C\varepsilon\delta_1 N(\tau)(\tau+1)^{-\frac{3}{2}+\gamma},\\
\|D_x^2 q_1\|_{L^1}(\tau) &\le 2\varepsilon(\|v_x\|^2+\|v\|\|v_{xx}\|)(\tau)\le C\varepsilon\delta_1^2(\tau+1)^{-\frac{3}{2}},
\end{aligned}$$

we have

$$\begin{aligned}
I_1 \le & C\varepsilon^2\delta_1^2[N^2(\tau)(t-\tau+1)^{-\frac{3}{2}}(\tau+1)^{-2-l+2\gamma}+\delta_1^2(t-\tau+1)^{-\frac{5}{2}}(\tau+1)^{-2-l}]\\
&+C\delta_1^4(t-\tau+1)^{-l-\frac{3}{2}}(\tau+1)^{-3}, \qquad l=0,1.
\end{aligned}\tag{3.46}$$

For I_2 we apply Lemma 3.1 to have

$$
\begin{aligned}
I_2 &\le Ce^{-\tilde{c}(t-\tau)} \int_{|\xi|\ge\eta} [|(i\xi)^l \hat{R}(\xi,\tau)|^2 + |(i\xi)^l \hat{Q}_1(\xi,\tau)|^2]\, d\xi \\
&\le Ce^{-\tilde{c}(t-\tau)}(\|D_x^l R\|^2 + \|D_x^l Q_1\|^2)(\tau) \\
&= Ce^{-\tilde{c}(t-\tau)}(\|D_x^{l+1} q_1\|^2 + \|D_x^l R_2\|^2 + \|D_x^{l+1} q_2\|^2),
\end{aligned}
$$

where $\tilde{c} > 0$ is a constant. From (3.3) and (3.20)-(3.23),

$$
\begin{aligned}
\|D_x R_2\|(\tau) &\le C[\|v\|_{L^\infty}\|u_{xx}\| + \|v_x\|_{L^\infty}\|u_x\| + \|u-1\|_{L^\infty}(\|v_{xx}\| + \|u_x\|)](\tau) \\
&\le C\delta_1^2(\tau+1)^{-\frac{7}{4}}, \\
\|D_x^{l+1} q_2\|(\tau) &\le C\varepsilon\delta_1^2(\tau+1)^{-\frac{5}{4}-\frac{l}{2}}, \quad l = 0, 1.
\end{aligned}
$$

Therefore, with (3.43) we have

$$
I_2 \le C\delta_1^4 e^{-\tilde{c}(t-\tau)}[\varepsilon^2(\tau+1)^{-\frac{5}{2}-\frac{l}{2}} + (\tau+1)^{-\frac{7}{2}}], \quad l = 0, 1. \tag{3.47}
$$

Combining (3.41) with (3.46) and (3.47), or with (3.44) and (3.47), we obtain (3.37) and (3.38), respectively.

We note that in (3.38), the term with $N^2(\tau)$ is from the first term of (3.42). If $d_0 = 0$, i.e., v_0 has zero mass, then $\theta = 0$. With $\gamma = 0$, (3.20) gives

$$
\|v\|(t) \le N(t)(t+1)^{-\frac{3}{4}}. \tag{3.48}
$$

Therefore, that term is replaced by $C\varepsilon^2 N^4(\tau)(t-\tau+1)^{-\frac{3}{2}-l}(\tau+1)^{-3}$.

Equations (3.39) and (3.40) can be proved in a similar way, applying (3.16) and (3.18) □

From (3.27) and by Plancherel theorem,

$$
\begin{aligned}
\|\tilde{w} - \Theta\|(t) = \|\hat{\tilde{w}} - \hat{\Theta}\|(t) \le & \|e^{tE(i\xi)}\hat{\tilde{w}}(\xi,0) -^{-\alpha\xi^2 t}\hat{\Theta}(\xi,0)\| \\
& + \int_0^t \|e^{(t-\tau)E(i\xi)}\hat{R}(\xi,\tau) - e^{-\alpha\xi^2(t-\tau)}\hat{Q}(\xi,\tau)\|\, d\tau.
\end{aligned}
$$

Noting (3.3) and (3.24), and applying Lemmas 3.3 and 3.4, we have

$$
\begin{aligned}
\|v-\theta\|(t) \le & C\delta_1(t+1)^{-\frac{3}{4}} + C\varepsilon\delta_1 N(t)\int_0^t (t-\tau+1)^{-\frac{3}{4}}(\tau+1)^{-1+\gamma}\, d\tau \\
& + C\varepsilon\delta_1^2\int_0^t a(t-\tau+1)^{-\frac{5}{4}}(\tau+1)^{-1}\, d\tau + C\delta_1^2\int_0^t (t-\tau+1)^{-\frac{3}{4}}(\tau+1)^{-\frac{3}{2}}\, d\tau.
\end{aligned}
$$

In the case of $d_0 = 0$, the second term on the right-hand side is replaced by

$$C\varepsilon N^2(t)\int_0^t (t-\tau+1)^{-\frac{3}{4}}(\tau+1)^{-\frac{3}{2}}\,d\tau$$

according to Lemma 3.4. Therefore,

$$\|v-\theta\|(t) \le \begin{cases} C\delta_1(t+1)^{-\frac{3}{4}}, & \text{if } \varepsilon = 0, \\ C[\delta_1(t+1)^{-\frac{3}{4}} + \varepsilon\delta_1 N(t)(t+1)^{-\frac{3}{4}+\gamma}], & \text{if } \varepsilon > 0 \text{ and } d_0 \neq 0, \\ C[\delta_1(t+1)^{-\frac{3}{4}} + \varepsilon N^2(t)(t+1)^{-\frac{3}{4}}], & \text{if } \varepsilon > 0 \text{ and } d_0 = 0. \end{cases} \tag{3.49}$$

Similarly, applying (3.38) when integrating on $[0, t/2]$, and (3.37) on $[t/2, t]$, we have

$$\|v_x-\theta_x\|(t) \le \begin{cases} C\delta_1(t+1)^{-\frac{5}{4}}, & \text{if } \varepsilon = 0, \\ C[\delta_1(t+1)^{-\frac{5}{4}} + \varepsilon\delta_1 N(t)(t+1)^{-\frac{5}{4}+\gamma}], & \text{if } \varepsilon > 0 \text{ and } d_0 \neq 0, \\ C[\delta_1(t+1)^{-\frac{5}{4}} + \varepsilon N^2(t)(t+1)^{-\frac{5}{4}}], & \text{if } \varepsilon > 0 \text{ and } d_0 = 0. \end{cases} \tag{3.50}$$

Applying (3.29), (3.39) and (3.40) gives us

$$\|u-1+\frac{1}{a}\theta_x\|(t) \le \begin{cases} C\delta_1(t+1)^{-\frac{5}{4}}, & \text{if } \varepsilon = 0, \\ C[\delta_1(t+1)^{-\frac{5}{4}} + \varepsilon\delta_1 N(t)(t+1)^{-\frac{5}{4}+\gamma}], & \text{if } \varepsilon > 0 \text{ and } d_0 \neq 0, \\ C[\delta_1(t+1)^{-\frac{5}{4}} + \varepsilon N^2(t)(t+1)^{-\frac{5}{4}}], & \text{if } \varepsilon > 0 \text{ and } d_0 = 0. \end{cases} \tag{3.51}$$

Combining (3.49)-(3.51), Theorem 2.3 is proved for the case $\varepsilon = 0$. For $\varepsilon > 0$ and $d_0 \neq 0$, (3.19), (3.49) and (3.50) imply

$$N(t) \le C\delta_1 + C\varepsilon\delta_1 N(t)$$

or

$$(1 - C\varepsilon\delta_1)N(t) \le C\delta_1.$$

Therefore, if δ_1 is small, we have

$$N(t) \le C\delta_1, \tag{3.52}$$

which implies (2.21). Substituting (3.52) into (3.51) we also have (2.22). For $\varepsilon > 0$ and $d_0 = 0$, similarly we have

$$N(t) \le C\delta_1 + C\varepsilon N^2(t)$$

or

$$[1 - C\varepsilon N(t)]N(t) \le C\delta_1.$$

If $N(t)$ is small, we have (3.52). By a standard continuity argument, (3.52) holds if δ_1 is small. This implies (2.21) and (2.22) as well.

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