Communications in Applied Analysis, **23**, No. 1 (2019), 179-207 ISSN: 1083-2564

VARIABLE ORDER DIFFERENTIAL EQUATIONS AND APPLICATIONS

A. ZACHARIAS[1], H. KHANAL[2], AND A. LUDU[3]

[1,2,3]Department of Mathematics
Embry-Riddle Aeronautical University
Daytona Beach, FL 32114, USA

ABSTRACT: We study the numerical solutions of a time dependent order differential equation by collocation method. Two variable order of differentiation with respect to time are used for two different terms, in order to manage the dynamics, loss and dispersion in the model equation. The numerical procedure is based on a Taylor expansion of the solution and identification of the series coefficients. In this way the critical open problem of the initial conditions, when the order of the differential equation varies from one to two, is solved in an efficient way. The model is applied to two physical situations, namely an electric circuit and acoustic waves in a loss-fluid. The results are in agreement with the expected behavior of such systems.

AMS Subject Classification: 34A08, 45G10, 65D30

Received: August 19, 2018 ; **Accepted:** December 6, 2018 ;
Published: December 15, 2018 **doi:** 10.12732/caa.v23i1.12
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION

In mathematics, a dynamical system is a system in which a function describes the time dependence of a point (the dynamical state) in a differentiable manifold. This function represents the time evolution of the dynamical system, and it describes what future states follow from the current state. It is a general consent in mathematical physics that this function fulfills two basic constraints: it obeys a certain system of differential equations involving time derivatives, and it must be deterministic (even stochastic dynamical systems are described like this). The study of dynamical systems is useful in a large class of applications in mathematics, physics, biology, chemistry, engineering, economics, and medicine. The differential equations describing the dynamical system are in general defined in time and space $\mathbf{R} \times \mathbf{R}^3$ and are classified by the highest order of the derivatives involved.

In these differential equations, the orders of the time derivative control the structure of the solutions, and in the end control the dynamical behavior of the system in time [1]. For example, the parity of the order of differentiation with respect to time determines if the evolution is reversible or irreversible, or the value of the highest order of time differentiation determines to what extent the system is history dependent. Different orders of time derivatives in the differential equation generate different types of solutions with most dissimilar behavior. This is the reason for choosing certain order of time derivative when modeling different types of processes. In general, first order is responsible for time evolution, the drift, and for the diffusion, the second order is responsible for wave propagation and oscillations, third order is responsible for dispersion, and so on. Consequently, controlling the orders for the time derivatives can generate drastic changes in the behavior of solutions.

In the real world, complex systems may change drastically their behavior even during their evolution in time. For example, a system close to a phase transition can spontaneously switch behavior from deterministic to stochastic, or in fluid dynamics can transit from laminar to turbulent regimes. A simple physical example is provided by the dynamics of a solid projectile with high velocity impact on water: in the first part of fast motion under water the flow around the projectile is turbulent, cavitation occurs, etc. When the speed decreases the flow becomes laminar with Rayleigh drag proportional to the square of the speed, and when the velocity is very small, and the projectile just sinks, the surrounding fluid enters in a creep flow with Stokes drag proportional to the speed. Since the velocity is the time derivative of the position, it results that this projectile motion under water is initially controlled by second order time derivative, and later on by first order time derivative. Other examples can be mentioned from behavior models, social systems models, population growth models, smart materials dynamics, etc. where the order of differentiation in time controls the memory of the process.

The traditional way to model such drastic changes in the dynamical laws of a system, during its evolution, is to provide time dependent variable coefficients to the time derivatives. Especially when these coefficient functions are nonlinear, such modulation of the behavior influences the temporal dependence of the solutions. However, this type of control does not actually change the nature and geometric structure of the differential equation. In addition, there are situations when the transition of the system from one type of behavior to another type of behavior happens in the same external conditions, with the same material properties, and the only cause of dynamical transition are the dynamical parameters themselves (like position, velocity, pressure, density, field intensity, etc.). Moreover, this approach does not provide a homotopy equivalence between different models. All theorems of existence and uniqueness are valid only within open intervals for the coefficients values, because the cancellation of a derivative may introduce singularities. For example, one cannot use anymore theorems of existence and uniqueness valid for a second order ODE, if the second derivative approaches zero.

In our paper we present another approach to this model for breaking of dynamical continuity. Instead of enhancing or suppressing the importance of various orders of time derivatives in the differential equation by adjusting the coefficients in front of them, we rather adjust the orders of differentiation. In this way we keep all the terms in the ODE but we can merge different orders of differentiation homotopically. The tool for this procedure is very new and still in its stage of infancy, and it is the Variable Order of Differentiation Equation method (VODE). This procedure is also known under the name of dynamical order of differentiation equations (DODE) [2, 3].

Our different approach on time dependent order differential equation can provide an alternative for modeling systems experiencing changes in their dynamical laws. That may include population growth rates in variable environments [4], memory dependent diffusion [5], stochastic processes and multiplex networks described by higher-order Markovian processes[6] coupling of nonlinear variable boundary conditions with nonlinear waves [7] boundary area and speed of action in self-replicating clusters [8]. Continuous order of differentiation was considered for differential equations systems through the fractional calculus [9], while the research for systems with time-dependent or variable-dependent order of differentiation is still in its initial stage [10, 11, 2, 12].

Basically, in this new type of differential equation all orders of time differentiation are variable, and are given by functions of time. In this way, the system can totally change its dynamics, if some of these orders of differentiation approach different integer values, including zero. The method has the advantage of mapping the ODE in various other types of ODE of different orders without artificial coefficients management. Moreover, this procedure provides indeed a deep change in the dynamics of the system, because smooth change of orders of derivatives between their traditional integer values involves a smooth change the geometry of the differential equation as projected in its jet space [2]. Simply explaining, if the jet bundle is understood as a vector space such that each independent and dependent variable, as well as all their possible derivatives up to a maximal order n, are linear independent coordinates of this space, then the differential equation is represented by a hypersurface in this space, and the solutions are sub-manifolds lying in this hypersurface. The traditional way of handling changes in the dynamical regime by time variable coefficient functions in front of the derivatives is equivalent with shrinking such hypersurfaces to some of their degenerate limits. On the contrary, the VODE procedure deforms the hypersurface by submerging it in higher dimensional spaces.

When the order of differentiation changes with time, the ODE becomes an intermediate fractional derivative ordinary differential equation, also known as FODE. This is actually an integral equation. During this process, not only the type of dynamics described by the VODE is changing, but the space of solutions, i.e. number of independent solution, is changing continuously. This feature makes this procedure more interesting, and also more challenging because mapping between linear spaces of different dimensions may create difficulties in choosing the appropriate initial conditions.

We apply the VODE procedure on a linear second order differential equation. Obviously, VODE procedure can be applied equally to nonlinear differential equations, but here we limit ourselves to linear equations just to make sure we understand the effects of the VODE procedure on very well known solutions and solution spaces, and to take profit of the simplicity of linear spaces of solutions. Even if the most interesting systems, and in particular the complex systems, are more complete as described by nonlinear differential equations, the study of the subsequent linear models is still very important. On one hand, a reasonable approximation of the behavior of a nonlinear system is the linear and local one, and on another hand the nonlinear solutions are usually expanded in terms of the linear solutions. Any approach of a nonlinear problem involves at some point the expansion of the nonlinear solution in terms of the corresponding linear modes of the linearized approximation of the differential equation. Many features of the nonlinear system can be thus obtained first hand from the study of their linearized versions. For example, systems like critical processes and phase transitions described by the Ginzburg-Landau equation, bifurcations, solitons described by nonlinear Schr̆odinger equations, fluid systems described by Boussinesq equation, bosons and superfluid systems described by Gross-Pitaevskii equation, Ricci flows in geometry and gravitation, and many others are solved only based on the spectrum of eigenvalues and eigen spaces of solutions obtained from their linearization [3, 4, 6, 10, 11, 13, 14, 15].

We direct our applications on linear differential equations of order two which describes time dependent physical models. Such differential systems are usually build by generalizations of classical mechanics, thermodynamics or electromagnetism, where the equations of evolution are inspired by Euler-Lagrange equations of a least action principle. This explains the occurrence of, and limitation only to the 2^{nd} order of the time derivative. The 1^{st} order time derivative usually stays in the equation (is relevant) if conservation and transfer laws are involved. Conservation laws arise usually from the Stokes-Cartan theorem, namely involving integration of a form and its differential, hence just the 1^{st} order of differentiation involvement. These equations contain in addition linear differential operators in the space coordinates. We study one dimensional, linear, constant coefficient homogeneous dynamical system by using the new type of differential equation with variable order of differentiation (VODE or DODE) which we introduced in [2].

The paper is organized as follows: after the introduction we present in section **2** several physical motivations for the use of the VODE models. In section **3** we introduced the time variable order derivative and its properties, establish the model differential equation and briefly elaborate on the existence and uniqueness of its solutions. In section **4** we present the numerical approach for solving the initial value problem for linear non-homogeneous VODE of order between one and two and some test examples are presented. In section **5** we apply the procedure to two different physical systems. The paper is closed by Conclusions and References.

2. PHYSICAL MOTIVATION

A dynamical system governed by a second order differential equation in time exhibits totally different behavior if the first order derivative exists, or if it is removed. We give here a typical example, the damped linear and homogeneous wave equation in $1+1$ dimension. In one space dimension its form is

$$a\frac{\partial^2 u}{\partial x^2} - b\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + du = 0, \tag{2.1}$$

for $u(t,x)$ and $a>0, b>0, c, d$ functions of x, t in general, or in simpler cases just constants. If the coefficient $c = 0$ we have a model for the ideal wave equation whose solution propagate sin space with finite velocity, fulfilling thus the Huygens causality principle. Namely, if we have a localized function as initial condition (like compact supported initial condition) and if we study the evolution in the whole space, then points at very large distance from the source initially at rest, are disturbed only when the front of the wave arrives at their location, in a certain finite time. This propagation of the solution is a fundamental property coming from the causal nature of the Green function for the hyperbolic differential equations. However, if we have $b = 0$ the resulting equation is the heat equation which has a totally different behavior of solutions. For any given initial ($t = 0$) and boundary conditions, the solution propagates with infinite velocity in the whole space. In other words, for any $t > 0, x$ we have $u(x,t) \neq u(x,0)$, so the information propagates with infinite speed and the solution is non-causal. In other words, no matter how small the coefficient c is, the nature of the solution changes dramatically and abruptly when $c \to 0$, which situation is not covered by the Cauchy-Kovalevskaya theorem of continuous dependence of the solution with the parameters. Only the damped equation with $c \neq 0$ is causal, while its reduction to ideal wave equation is not. This situation emphasizes a defect of the damped wave equation model. It is obviously that this problem cannot be solved by an appropriate choice of the coefficients of the equation.

ELECTRIC CIRCUIT

We give here another example experiencing the same difficulty, inspired by electromagnetism, namely the telegrapher's equation. This equation is a linear differential equations that describe the voltage or current on an electrical transmission line with distance and time. The model applies to transmission lines of all frequencies (historically the telegraph wires, and at present the radio frequency conductors, telephone lines, power lines, and even under the ocean direct currents power cables). The model result from Maxwell's equations by assuming that the electric lines are composed of an infinite series of two-port elementary components, each representing an infinitesimally short segment of the transmission line with distributed resistance R of the conductors, distributed inductance L, distributed capacitance C, and distributed conductance G of the dielectric material separating the two lines. The simplest linear model for telegrapher's equation has the form:

$$\frac{\partial^2 V}{\partial x^2} - LC\frac{\partial^2 V}{\partial t^2} - (RC + GL)\frac{\partial V}{\partial t} + GRV = 0, \tag{2.2}$$

where $V(x,t)$ is the electric potential difference across the wires (the voltage). It easy to check that no physical model for the coefficients L, C, R, G all greater than zero, can accommodate a smooth transition of the solution from causal to non-causal. That is because if we want to remove the first order time derivative term we need all coefficients zero, which is impossible. We can remove the second order time derivative by asking $C = 0$ or $L = 0$, and hence reduce the equation to a first order in time. However, there is no way to smoothly map such a first order in time equation into a pure second order it time equation, without the first order derivative:

$$\frac{\partial^2 V}{\partial x^2} - GL\frac{\partial V}{\partial t} + GRV \not\to \frac{\partial^2 V}{\partial x^2} - LC\frac{\partial^2 V}{\partial t^2} + GRV, \tag{2.3}$$

where $C \to 0$, for example.

FLUID DYNAMICS

Our last example is brought by the Navier-Stokes equations in fluid dynamics

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}\right)=-\nabla P+\rho\vec{f}+\mu\triangle\vec{v}, \tag{2.4}$$

where $\vec{v}$ is the fluid velocity field, ρ is fluid density, P is the fluid pressure, μ is dynamic viscosity coefficient, and $\vec{f}$ is the volume density of forces acting on the fluid from exterior, all being functions of point $\vec{r}$ and time t. In order to model waves in the fluid, we need the equation of state relationship between density and pressure, which, in almost all fluid models, is given by

$$\frac{\partial P}{\partial t}=\frac{K}{\rho}\frac{\partial\rho}{\partial t}, \tag{2.5}$$

where $K>0$ is the bulk modulus constant for the fluid, and it is related to the speed of propagation of sound waves in the fluid c_s by

$$c_s=\sqrt{\frac{K}{\rho}}. \tag{2.6}$$

We also need to implement mass conservation law through the equation of continuity

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0, \tag{2.7}$$

From Eq. (2.4-2.7) we obtain the system

$$\frac{1}{c_s^2}\frac{\partial P}{\partial t}+\nabla\cdot\vec{v}=0,$$

$$\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot(\nabla\cdot\vec{v})=-\nabla P+\vec{f}+\frac{\mu}{\rho}\triangle\vec{v}. \tag{2.8}$$

If we drop the nonlinear terms, as we mentioned above, and differentiate with respect to time, we have

$$\frac{\partial^2\vec{v}}{\partial t^2}+\nabla\frac{\partial P}{\partial t}-\frac{\mu}{\rho}\triangle\left(\frac{\partial\vec{v}}{\partial t}\right)=0,$$

$$\frac{\partial^2\vec{v}}{\partial t^2}-\nabla(c_s^2\nabla\cdot\vec{v})-\left(\frac{\mu}{\rho}\right)^2\triangle^2\vec{v}+\triangle\nabla P=0. \tag{2.9}$$

By using the identity $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \triangle \vec{v}$ we have, in the simplified case of irrotational flow ($\nabla \times \vec{v} = 0$) the equation

$$\frac{\partial^2 \vec{v}}{\partial t^2} - \left(\frac{\mu}{\rho}\right) \triangle \frac{\partial \vec{v}}{\partial t} - \left(\frac{\mu}{\rho}\right)^2 \triangle^2 \vec{v} + \triangle \left[\frac{\mu}{\rho} \triangle \vec{v} - \vec{v} \cdot (\nabla \cdot \vec{v})\right] = 0. \tag{2.10}$$

The first two terms of the equation above represent the damped wave equation part of the Navier-Stokes equation. In this case the dynamic viscosity coefficient μ plays the role of the transition parameter from causal to non-causal ($\mu = 0$) solutions.

The mathematical difficulty of smoothly switching the second order differential equation model between loss-free, ideal case and damped, frictional, viscous cases, which involves the transition between causal and non-causal classes of solution, includes variable coefficients, which complicates the structure of the solution. Moreover, trying to tune the behavior of the linear differential equation by adjusting the variable coefficients is not expected to bring qualitative changes in the solutions because the geometry and topology of the linear ordinary differential equation is not changed by changing its coefficients or the source term because the structure of the jet space for the linear ordinary differential equation is invariant to the functional dependence on coefficients, or on the nonhomogenous term. On the contrary, it is known that the order of differentiation changes the nature of the physical laws behind the model. The drag upon a submerged object changes from inertia-less creep-flow (when force is proportional to the velocity) to Rayleigh drag (force is proportional to acceleration). In this example, the force term must change from a first order to the second order time derivative when one smoothly accelerate a submerged object.

POPULATION DYNAMICS

The present models for demographic growth include three factors: exponential growth of the species under steady favorable conditions, limits imposed by Earth's carrying capacity, and impact of technological change on the two other factors [4]. The traditional model consists of three differential equations for population, technological level, and carrying capacity. Solution of this system was approximated by a tamed-quasi-hyperbolic function that gives quasi-hyperbolic growth at early stages of the process after which population reaches a plateau [16]. The real data, shows an accelerated rate of growth from power law, to exponential, to hyperbolic with singularity. At present, there are two main problems with population growth models. On one hand, while exponential dynamics is easy to be modeled with linear differential equations, the hyperbolic laws (of the form $P(t) \sim 1/(t - t_\infty)$) need nonlinear differential equations. On the other hand, there is no differential equation whose solution can change in time from exponential to hyperbolic.

A very recent model, based on fractional derivatives, [17], emphasizes a better fit to the general laws of predator-prey biological population dynamics, yet it doesn't deliver a sufficiently explanation for the human population dynamics. We believe that the only way to combine in the same solution a smooth transition between exponential and hyperbolic growth is by using time dependent order derivatives. The human civilization is an informational system that generates knowledge needed for its survival, and knowledge can be modeled as a set of algorithms for processing information. Solving of problems or formulation of new problems are accompanied by the introduction of new algorithms. This time variable dynamical law could be modeled by variable order differential equations of VODE type. In addition to this feature, the VODE models can generate exponential behavior of solutions when the order of differentiation approaches an integer number, yet it can generate at the same time hyperbolic dynamics through the singular kernel of the integral equation.

3. DIFFERENTIAL EQUATIONS WITH VARIABLE ORDER OF DIFFERENTIATION

We introduce the variable order differential equation (VODE) through the theory of fractional differential operators [9, 14, 18]. Such constructions as the fractional derivatives and fractional integrals have applications in viscoelasticity, feedback amplifiers, electrical circuits, electro-analytical chemistry, fractional multipoles, neuron modeling and related areas in physics, chemistry, and biological sciences [14, 18]. There are several ways to introduce the fractional differential: Riemann-Liouville, Caputo, Jumariè, Erdèly-Kober, Baleanu-Atangana, Caputo-Almeida, etc. [19, 20, 21] and more recently [6, 7, 10, 11], each generating well-defined operators with convenient properties. Based on previous literature we introduce in the following an integration operator of time-dependent order $\alpha : [t_0, \infty) \to (0,1), t_0 > 0$ by

$$ {}_{t_0}I_t^{\alpha(t)}\ x(t) = \frac{1}{\Gamma(\alpha(t))} \int_{t_0}^{t} (t-s)^{\alpha(t)-1} x(s) ds, \tag{3.1} $$

where Γ is the Gamma function, and the derivative of variable order by

$$ {}_{t_0}D_t^{\alpha(t)} x(t) = \frac{d}{dt}\Big({}_{t_0}I_t^{\alpha(t)} \Big) x(t), \tag{3.2} $$

for $\alpha(t) \in (0,1)$. When α is a constant this definition generates the Riemann-Liouville fractional derivative [9]. Since definition Eq. (3.2) can be generalized for higher values of $\alpha(t)$ by

$$ {}_{t_0}D_t^{\alpha(t)} x(t) = \frac{d^m}{dt^m}\Big({}_{t_0}I_t^{m-\alpha(t)} \Big) x(t), \tag{3.3} $$

with $\alpha(t) \in (m-1, m)$, m positive integer, we can use the following expression of the time-dependent order Riemann-Liouville derivative of order $\alpha(t) : \mathbb{R}_+ \to [m-1, m)$

$$ {}_{t_0}D_t^{\alpha(t)}\ x(t) = \frac{d^m}{dt^m} \frac{1}{\Gamma(m-\alpha(t))} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{\alpha(t)-m+1}} ds, \tag{3.4} $$

for any positive integer m. In the following calculations the independent variable is t, and the initial moment $t_0 > 0$. We maintain t_0 constant at a very small positive value, and for this reason we skip the t_0 subscript from the equations and definitions, unless it becomes relevant. For more detailed properties of fractional derivatives one can use for example [9, 20].

When $\alpha(t) = \alpha_0 = \text{const}$ and $x \in C^m(I)$, by applying consecutive integration by parts in Eq.(3.4) we obtain the expression [9, 22]

$$ D^{\alpha_0}\ x(t) = \sum_{k=0}^{m-1} \frac{x^{(k)}(t_0)(t-t_0)^{k-\alpha_0}}{\Gamma(k-\alpha_0+1)} + \frac{1}{\Gamma(m-\alpha_0)} \int_{t_0}^{t} (t-s)^{m-\alpha_0-1} x^{(m)}(s) ds. \tag{3.5} $$

The integral in the right hand side term represents the Caputo fractional derivative of $x(t)$, [5, 9, 22], and it can be obtained directly from the definition Eq. (3.3) by an inverted sequence of operators

$$ DC^{\alpha_0} x(t) = I^{m-\alpha_0} \frac{d^m x(t)}{dt^m}. \tag{3.6} $$

From Eq. (3.5) for constant α_0 one can verify that the fractional derivative converges uniformly towards the integer order derivative when α_0 approaches its domain limits. For example if $m = 1, \alpha_0 \in (0, 1)$ we have

$$\lim_{\alpha_0->0^+} D^{\alpha_0}x(t) = \lim_{\alpha_0->0^+} \Big[\frac{x(t_0)(t-t_0)^{-\alpha_0}}{\Gamma(1-\alpha_0)} + \frac{1}{\Gamma(1-\alpha_0)}\int_{t_0}^{t}\frac{x'(s)}{(t-s)^{\alpha_0}}ds\Big] = x(t),$$

as well as

$$\begin{aligned}&\lim_{\alpha_0->1^-} D^{\alpha_0}x(t)\\ &= \lim_{\alpha_0->1^-}\Big[\frac{x(t_0)(t-t_0)^{-\alpha_0}}{\Gamma(1-\alpha_0)} + \frac{x'(t_0)(t-t_0)^{1-\alpha_0}}{\Gamma(2-\alpha_0)}\\ &+ \frac{1}{(1-\alpha_0)\Gamma(1-\alpha_0)}\int_{t_0}^{t}\frac{x''(s)}{(t-s)^{\alpha_0-1}}ds\Big] = x'(t).\end{aligned}$$

However, Eq. (3.5) is not valid anymore when $\alpha(t)$ is not constant. We need to substitute it with

$$\begin{aligned}&D^{\alpha(t)}x(t)\\ &= \frac{1}{(\alpha(t)-1)\Gamma(1-\alpha(t))}\Big[\frac{x(t_0)(t-t_0)^{-\alpha(t)}}{\Gamma(1-\alpha(t))} - DC^{\alpha(t)}x(t)\Big]\\ &\times \frac{\alpha'(t)}{\Gamma^2(2-\alpha(t))}\Big[\Gamma(2-\alpha(t))DC^{\alpha(t)-1}x(t)\Big(\frac{1}{\alpha(t)-1} - \frac{\Gamma'(1-\alpha(t))}{\Gamma(1-\alpha(t))}\Big)\\ &- L_1 + x(t_0)(t-t_0)^{1-\alpha(t)}\Big(\frac{1}{\alpha(t)-1} + \ln(t-t_0) - \frac{\Gamma'(1-\alpha(t))}{\Gamma(1-\alpha(t))}\Big)\Big],\end{aligned} \tag{3.7}$$

where

$$L_1 = \frac{1}{\Gamma(2-\alpha(t))}\int_{t_0}^{t}\frac{x'(s)\ln(t-s)}{(t-s)^{\alpha(t)-1}}ds$$

is a convolution with singular kernel [10, 11, 12].

In order to implement initial conditions (IC) in the VODE one should take some cautions. The physical meaning of IC is not fully understood in this case [9, 21]. Present approaches incorporate the derivatives of the initial data directly in the differential equation [5, 20, 21]

$$\begin{aligned}&D^{\alpha(t)}\Big(x(t) - T_{m-1}[x]\Big) = f(t, x(t)),\\ &x^{(k)}(t_0) = x_k, \quad k = 0, 1, \ldots, m-1,\end{aligned} \tag{3.8}$$

for given initial data $\{x_k \in \mathbb{R}\}_{k=0,\ldots,m-1}$, $t_0 > 0$, with $T_{m-1}[x](t)$ the Taylor polynomial of order $m-1$ for $x(t)$, the source term f a continuous function

$f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, and the variable order of differentiation also a continuous function $\alpha : \mathbb{R}^+ \to (m-1, m)$.

In our paper we study the case $m = 1$ with variable order of differentiation in the range $\alpha \in (1, 2)$. According to the definition of fractional derivatives Eq. (3.4) and Eq. (3.8) we have

$$\begin{aligned} &D^{\alpha(t)}\Big(x(t) - x(t_0) - (t-t_0)x'(t_0)\Big) \\ &= \frac{d^2}{dt^2}\frac{1}{\Gamma(2-\alpha(t))}\int_{t_0}^{t}\frac{x(s) - x_0 - sx_1}{(t-s)^{\alpha(t)-1}}ds \\ &= f(t, x(t)). \end{aligned} \tag{3.9}$$

Following the same procedure introduced in [2] we map the VODE initial value problem from Eq. (3.8) into a Volterra integral equation of second kind with singular integrable kernel, $k(t,\tau) = (t-\tau)^{\alpha(t)-2}$ for $\alpha(t) \in (1, 2)$

$$x(t) = x_0 + (t-t_0)x_1 + \frac{1}{\Gamma(\alpha(t)-1)}\int_{t_0}^{t}\frac{\int_{t_0}^{\tau} f(s, x(s))ds}{(t-\tau)^{2-\alpha(t)}}d\tau. \tag{3.10}$$

Any solution of Eq. (3.10) is a solution of the initial value VODE problem Eq. (3.8) with $m = 2$, being represented by a continuous function $x(t)$, $t \geq t_0$. In order to compare solutions for Eq. (3.8) with the limiting traditional situations where $\alpha \in \{1, 2\}$ is integer we notice that $\lim_{\alpha\to 2^-} D^{\alpha(t)}[x(t)-x_0-(t-t_0)x_1] = x''(t) = f(t, x(t))$. This result represents the $\alpha \to 2^-$ limiting solution for the ODE Eq. (3.8) and initial problem $x(t_0) = x_0, x'(t_0) = x_1, f(t_0, x_0) = 0$.

For the lower limit $\alpha \to 1^+$ we use the integral equation version Eq. (3.10):

$$x'(t) - x_1 \to \lim_{\alpha\to 1^+}\frac{1}{\Gamma(\alpha(t)-1)}\int_{t_0}^{t}\frac{f(\tau, x(\tau))d\tau}{(t-\tau)^{2-\alpha(t)}} = 0,$$

because $f(t, x)$ is continuous on any compact $[0, t]$, and hence upper bounded, and because the corresponding kernel is integrable.

The existence and uniqueness of the solutions in our problem is covered by our previous results [2, 3] where we proved that the initial value VODE problem given in Eq. (3.8) has a unique solution defined on $\mathbb{R}_+$ if $\alpha(t) : \mathbb{R}_+ \to (1, 2)$ is continuous, if it exists

$$p \in \big(2, \min_{t\geq 0}\big\{\frac{1}{\alpha(t)-1}, \frac{1}{2-\alpha(t)}\big\}\big)$$

such that

$$\sup_{t\geq 0} \frac{\Gamma(1+p(\alpha(t)-2))}{\Gamma(\alpha(t)-1)} \leq +\infty,$$

and also if $f(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|f(t,x) - f(t,y)| \leq G(t)|x-y|,$$

for all $t \geq t_0 > 0, x, y \in \mathbb{R}$ for some continuous function $G(t) : \mathbb{R}_+ \to \mathbb{R}_+$.

4. THE MODEL EQUATION AND NUMERICAL APPROXIMATION

The model equation proposed has the form

$$A\frac{d^{\alpha(t)}x}{dt^{\alpha(t)}} + B\frac{d^{\beta(t)}x}{dt^{\beta(t)}} + Cx = f(t,x),\ x(0) = x_0 \tag{4.1}$$

for the unknown $x(t)$, with A, B, C constant coefficients, and f an arbitrary function so far, and $t \in [0,1]$. We have the time dependent fractional derivative

$$\frac{d^{\alpha(t)}x}{dt^{\alpha(t)}} = D^{\alpha(t)}[x]. \tag{4.2}$$

The smooth functions $\alpha(t) : (0,1) \to (1,2)$ and $\beta(t) : (0,1) \to (0,1)$ describe the variable order of differentiation. In this way our model equation can smoothly map into any of the following cases

$$A\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Cx = f(t,x), \tag{4.3}$$

or

$$(A+B)\frac{dx}{dt} + Cx = f(t,x), \tag{4.4}$$

or

$$A\frac{d^2x}{dt^2} + (B+C)x = f(t,x), \quad \text{or} \quad (A+B)\frac{d^2x}{dt^2} + Cx = f(t,x), \tag{4.5}$$

and anything in between.

NUMERICAL APPROACH I

We consider the following definition for the Riemann-Liouville derivative of variable order $1 < \alpha(t) < 2$

$$D^{\alpha(t)}[x(t)] = \frac{d^2}{dt^2}\frac{1}{\Gamma(2-\alpha(t))}\int_0^t \frac{x(s)}{(t-s)^{\alpha(t)-1}}ds. \tag{4.6}$$

The variable order derivative defined by Eq. (4.6) can be evaluated for the polynomials $x(t) = t^n$ as

$$D^{\alpha(t)}[t^n] = n!\,\frac{\mathcal{G}_n^{\alpha}(t)\,t^{n+2-\alpha(t)}}{\Gamma(2-\alpha(t))} \tag{4.7}$$

where

$$\mathcal{G}_n^{\alpha}(t) = \left\{\left(d\ln\Gamma(n+3-\alpha(t)) - t^{n+2-\alpha(t)}\right)^2 - d^2\ln\Gamma(n+3-\alpha(t)) \right.$$
$$\left. -t^{2(\alpha(t)-n-2)}\right\}(\alpha'(t))^2 + \left(d\ln\Gamma(n+3-\alpha(t)) - t^{n+2-\alpha(t)}\right)\alpha''(t) \tag{4.8}$$

Using Eq. (4.7), we can solve the VODE Eq. (4.1) numerically using the collocation method.

Assume that the solution of the VODE Eq. (4.1) can be expressed as

$$x(t) = \sum_{j=0}^{n} c_j t^j. \tag{4.9}$$

The values of the coefficients c_js can be obtained by collocating $n+1$ points t_i in $[0,1]$. The coefficient c_0 is obtained from the initial condition $c_0 = x(0) = x_0$, so we need to solve the following system of n equations for $c = (c_1, c_2, \cdots, c_n)^T$.

$$\sum_{j=1}^{n} c_j\left(AD^{\alpha} + BD^{\beta} + C\right)[t_i^j]$$
$$= f(t_i, x_0 + \sum_{j=1}^{n} c_j t_i^j) - \left(AD^{\alpha} + BD^{\beta} + C\right)[x_0] \tag{4.10}$$

If the function $f(t,x)$ in the ODE (4.1) is linear in x, Eq. (4.10) becomes a system of n linear equations.

NUMERICAL APPROACH II

Using the Taylor series approximation for $x(s)$

$$x(s) = x(t) + (s-t)x'(t) + \frac{(s-t)^2}{2!}x''(t) + \cdots + \frac{(s-t)^n}{n!}x^{(n)}(t) \quad (4.11)$$

the Riemann-Liouville derivative defined by Eq. (4.6) can be approximated by

$$D^{\alpha(t)}[x(t)] = \frac{d^2}{dt^2}\sum_{k=0}^{n}\frac{(-1)^k}{k!}\left(\frac{t^{k+2-\alpha(t)}}{(k+2-\alpha(t))\Gamma(2-\alpha(t))}\right)x^{(k)}(t). \quad (4.12)$$

Using Eq. (4.12) in Eq. (4.1), we obtain the following ODE of integer order $n+2$ as

$$\sum_{k=0}^{n}\left(\mathcal{C}_k x^{(k+2)}(t) + 2\mathcal{C}'_k x^{(k+1)}(t) + \mathcal{C}''_k x^{(k)}(t)\right) + Cx(t) = f(t, x(t)) \quad (4.13)$$

The coefficients $\mathcal{C}_k$ and their derivatives are computed as follows.

$$\mathcal{C}_k = \frac{(-1)^k}{k!}\left(\underbrace{\frac{At^{k+2-\alpha(t)}}{(k+2-\alpha(t))\Gamma(2-\alpha(t))}}_{:=\mathcal{A}_k} + \underbrace{\frac{Bt^{k+2-\beta(t)}}{(k+2-\beta(t))\Gamma(2-\beta(t))}}_{:=\mathcal{B}_k}\right) \quad (4.14)$$

$$\mathcal{A}'_k = \mathcal{A}_k\left\{\frac{k+2-\alpha(t)}{t} + \alpha'(t)\left(\frac{1}{k+2-\alpha(t)} + d\ln\Gamma(2-\alpha(t)) - \ln t\right)\right\} \quad (4.15)$$

$$\begin{aligned}\mathcal{A}''_k &= \mathcal{A}_k\Bigg[\left\{\frac{k+2-\alpha(t)}{t} + \alpha'(t)\left(\frac{1}{k+2-\alpha(t)} + d\ln\Gamma(2-\alpha(t)) - \ln t\right)\right\}^2 \\ &-\frac{k+2-\alpha(t)}{t^2} - \frac{2\alpha'(t)}{t} + (\alpha'(t))^2\left(d^2\ln\Gamma(2-\alpha(t)) - \frac{1}{(k+2-\alpha(t))^2}\right) \\ &+ \alpha''(t)\left(\frac{1}{k+2-\alpha(t)} + d\ln\Gamma(2-\alpha(t)) - \ln t\right)\Bigg] \end{aligned} \quad (4.16)$$

Similar expressions for $\mathcal{B}'_k$ and $\mathcal{B}''_k$ are obtained by replacing A and $\alpha(t)$ by B and $\beta(t)$ respectively in Eq. (4.16).

Let

$$\begin{aligned} a_0 &= \mathcal{C}''_0(t) + C \\ a_1 &= 2\mathcal{C}'_0(t) + \mathcal{C}''_1(t) \\ a_i &= \mathcal{C}_{i-2}(t) + 2\mathcal{C}'_{i-1}(t) + \mathcal{C}''_i(t), \quad (i = 2, \cdots, n) \\ a_{n+1} &= \mathcal{C}_{n-1}(t) + 2\mathcal{C}'_n(t) \\ a_{n+2} &= \mathcal{C}_n(t) \end{aligned} \tag{4.17}$$

Using the coefficients $a_k(t)$ from Eq. (4.17), the approximate integer order differential equation IODE Eq. (4.13) can be expressed in the following simplified form.

$$\sum_{k=0}^{n+2} a_k\, x^{(k)}(t) = f(t,x) \tag{4.18}$$

Letting $y = (x, x', x'', \cdots, x^{(n+1)})^T$, the IODE Eq. (4.13) can be expressed as a system of $n+2$ first order equations.

$$y'_i = y_{i+1}\ (i = 0, 1, 2, \cdots, n+1), \quad y'_{n+1} = \frac{1}{a_{n+2}}\left(f(t, y_0) - \sum_{k=0}^{n+1} a_k\, y_k\right) \tag{4.19}$$

The IODE Eq. (4.19) can be solved numerically employing different ODE integration techniques. In general, the $n+2$ order differential equation requires $n+2$ appropriate initial conditions

$$x(t_0) = x_0,\ x'(t_0) = x_1,\ \cdots, x^{(n)}(t_0) = x_n, x^{(n+1)}(t_0) = x_{n+1}. \tag{4.20}$$

For a given initial value problem of the VODE Eq. (4.1) with one initial condition, we can approximate the required $n+2$ initial conditions for the IODE using the collocation method (with few points) and solve the system of first oder differential equation Eq. (4.19) employing better numerical methods.

SOME TEST EXAMPLES

The numerical procedures described above are implemented in Python employing the NumPy and SciPy library for special functions, the linear system solver and the ODE integrators.

First, we tested the collocation method to the initial value problem of VODE Eq. (4.1) and the corresponding approximate integer order differential equation (5th order IODE) Eq. (4.18) and Eq. (4.19) with the known exact solution $x(t) = t^4+t^3+t^2+t+1$ by setting the right hand side function $f(t,x)$ accordingly. For example, for Eq. (4.1), we use

$$f(t,x) = \sum_{k=0}^{4} \left(A\frac{\mathcal{G}_k^{\alpha}(t)\, t^{k+2-\alpha(t)}}{\Gamma(2-\alpha(t))} + B\frac{\mathcal{G}_k^{\beta}(t)\, t^{k+2-\beta(t)}}{\Gamma(2-\beta(t))} \right) + C(t^4+t^3+t^2+t+1) \quad (4.21)$$

where $\mathcal{G}_k^{\alpha}$, $\mathcal{G}_k^{\beta}$ are as defined in Eq. (4.8).

We set $\alpha(t) = 2 - t^3$, $\beta(t) = 1 + t^3$, $A = 1$, $B = 1$ and $C = 100$ to make sure that each of the expressions in the equations is used in the computation. The solutions (shown in figure 1) from all the methods are identical as expected. Since the variable coefficients in the $a_k(t)$ has singularity at $t = 0$, the ODE solver could not start from $t = 0$. The collocation method was easy to implement as it does not require to evaluate these expressions at $t = 0$.

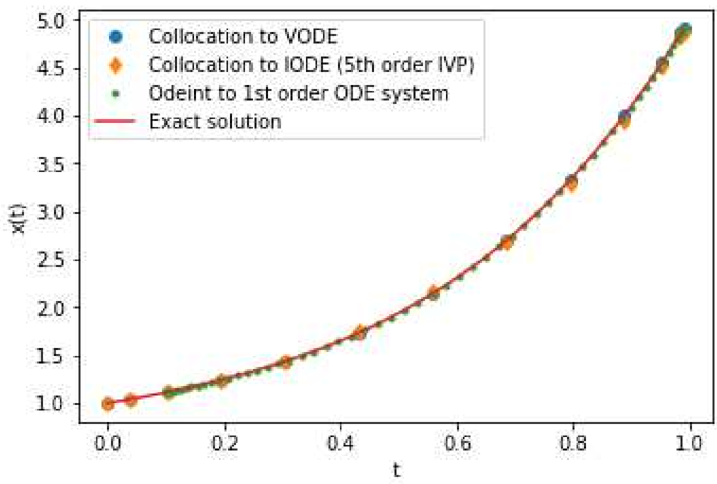

Figure 1: Tests on the variable order differential equation (4.10) and the integer order differential equation (4.13), (4.18) with the known (identical) solution.

Next, we compare the numerical solutions of VODE Eq. (4.10) and its approximation equation IODE Eq. (4.13) with the same function f given in

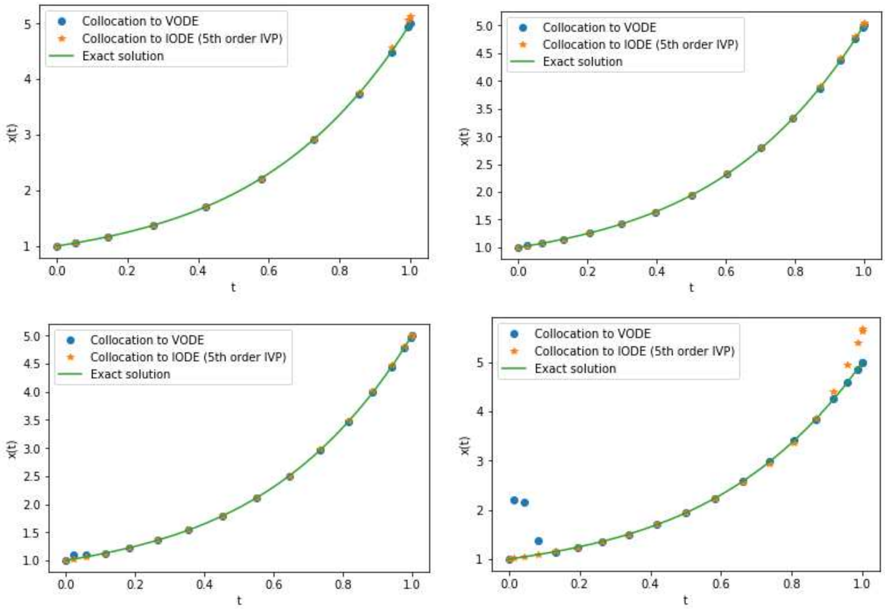

Figure 2: Comparison of numerical solutions to VODE (4.1) and the approximate IODE (4.13) with the exact solution.

Eq. (4.21) for both the equations. The numerical solutions using 10, 15, 16 and 19 collocation points are displayed in figure 2 and the relative errors in L_2-norm are plotted in figure 3.

The numerical results show that $n = 15$ has the least error, and for larger values of n numerical solutions to the VODE equation seems more stable at the right end ($t \approx 1$) whereas the IODE has better stability at the left end ($t = 0$). It should be noted that the VODE equation has singularity at both the end points and we are using the third order Taylor polynomial in the IODE implementation.

The solutions for the VODE Eq. (4.10) and the approximate fifth order ODE Eq. (4.13) for $f(t, x) = \pm C \sin(At + B)$ with different values of A, B, and C are presented in figures 4-7. The numerical results show that IODE Eq. (4.13) mimics the qualitative behavior of the VODE Eq. (4.10) in general. For the initial time, it is very accurate in almost all simulations. We also observed that adjusting the values of C in the model Eq. (4.10), we can make the two solutions identical. As the IODE Eq. (4.13) is an ordinary differential equation of order $n + 2$, we can not expect its unique solution without imposing all the

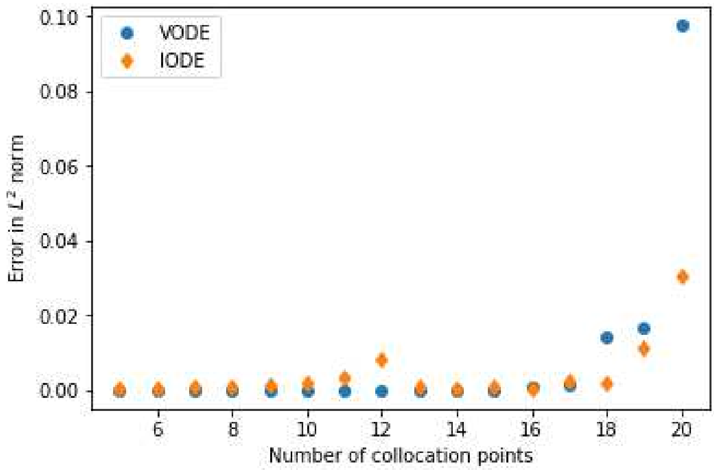

Figure 3: Error in approximation (in L_2-norm) for the VODE (4.1) and the IODE (4.13).

required initial conditions.

In general, we expect only one initial condition in applications with the variable order differential equation models. Creating extra initial conditions may not be physically meaningful in the models. As our experiment suggests, collocation method for IODE has stable solution at the initial time, and we may use this method to create artificial initial conditions to solve the complete ODE system of IODE.

In all the numerical results presented in the next section, the application problems in the form of VODE Eq. (4.1) and their approximate IODE Eq. (4.13) are collocated using 12 Chebyshev nodes in the interval $[0, 1]$ and the solutions are interpolated using the cubic splines.

5. VODE MODELING AND APPLICATIONS

In the following we present three examples of application of the VODE generic model given by Eq. (3.8) on physical phenomena. We have chosen examples from electricity and acoustic waves in compressible fluids. In order to emphasize and isolate the role of the variable order of differentiation we present a simplified one-dimensional version of the VODE model, described by Eq. (4.1).

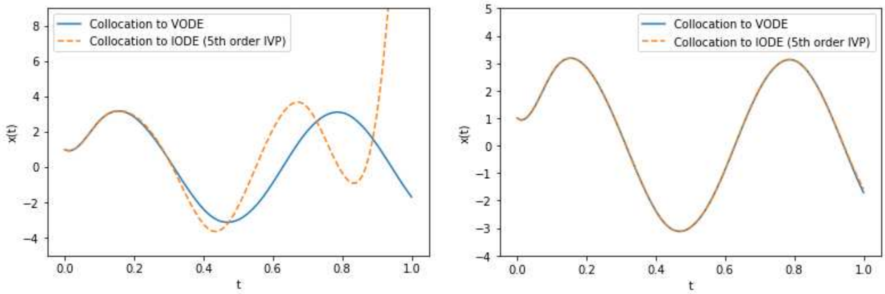

Figure 4: Comparison between the solutions of the VODE Eq. (4.1) and the approximate fifth order ODE Eq. (4.13) with $\alpha(t) = 1 + t^3$, $f(t, x) = C \sin(At + B)$, $A = 10$, $B = 0$ and $C = 10^4$ (left), $C = 10^6$ (right)

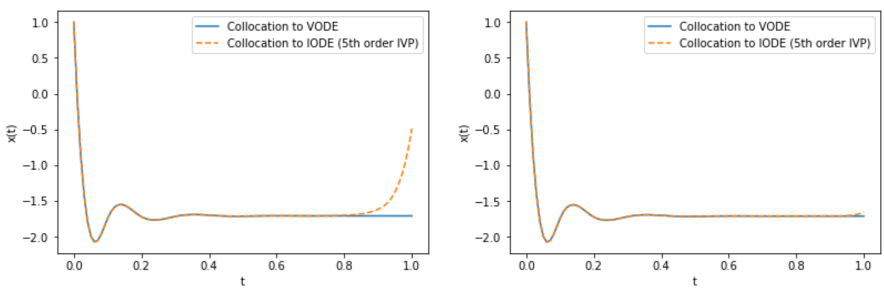

Figure 5: Comparison between the solutions of the VODE Eq. (4.1) and the approximate fifth order ODE Eq. (4.13) with $\alpha(t) = 1 + t^3$, $\beta(t) = 2 - t^3$, $f(t, x) = C \sin(At + B)$, $A = 0$, $B = 10$ and $C = 10^6$ (left), $C = 10^7$ (right)

The numerical procedure to approximate the VODE equations are described in section **4.**

ELECTRIC CURRENT DYNAMICS IN A CIRCUIT WITH TIME VARIABLE COMPONENTS

Let us consider an electric circuit with concentrated components: electric resistance R, electric capacitance C, inductance L and loss. The evolution of the electric current $i(t)$ through this circuit is given by a space averaged

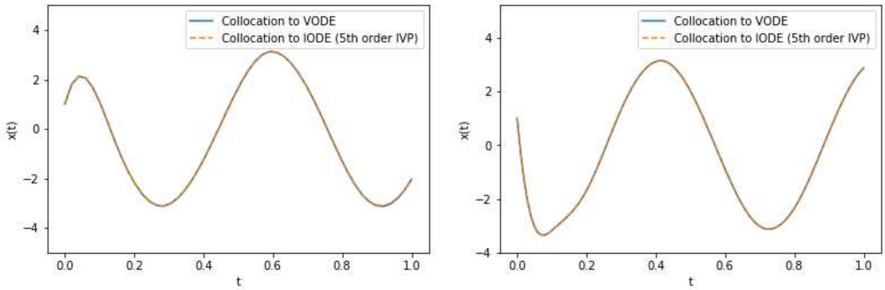

Figure 6: Numerical solution of the VODE equation Eq. (4.1) on $t \in [0,1]$ for $f(t,x) = C\sin(At+B)$, $\alpha(t) = 1+t^3$, $\beta(t) = 2-t^3$, $A = B = 10, C = 10^6$ (left) and $C = -10^7$ (right).

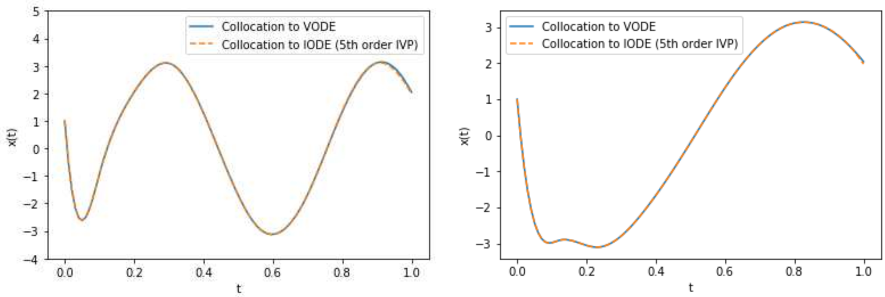

Figure 7: Numerical solution of the VODE equation Eq. (4.1) on $t \in [0,1]$ for $f(t,x) = C\sin(At+B)$, $\alpha(t) = 1+t^3$, $\beta(t) = 2-t^3$, $C = 10^6$, $(A = 10, B = 5$ left) and $(A = 5, B = 10$ right).

version of Eq. (2.2)

$$LC\frac{d^2 i}{dt^2} + RC\frac{di}{dt} + i + i_{loss} = 0, \tag{5.1}$$

where i_{loss} can play as dielectric loss or, if negative, as external current pumped in the circuit. If the electric components have a known time evolution, then Eq. (5.1) is not anymore valid, and new terms taking into account the time variation of the coefficients must be introduced. Such situations occur for example, in the case of ultra high magnetic field generators for plasma installations, or for controlled nuclear fusion [23]. In such systems the initial inductance of the circuit is reduced to a very small value in a finite amount of time and by the magnetic flux conservation theorem the intensity of electric current blows-up to extreme values. However, if such a simplified model

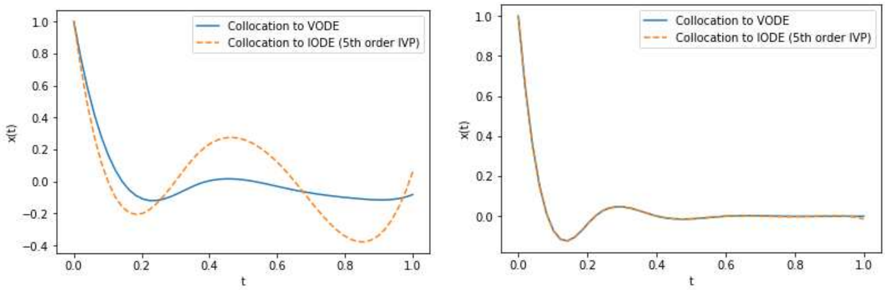

Figure 8: Numerical solution of the VODE equation Eq. (4.1) and its approximate 5th order IODE Eq. (4.13) for $f(t,i) = -\lambda(1-e^{-t/(RC)})$, $\alpha(t) = 1+t^3$, $\beta(t) = 2-t^3$ with $\tau LC = 5\times 10^4, \tau RC = 10^4$ and $\lambda = 10\pi$ (left) and $\lambda = 1000\pi$ (right). $\tau = 10^4$ is a time scaling reducing the physical duration of the electric process of $100\mu s$ to dimensionless time interval of integration $(0,1)$.

like the one in Eq. (2.2) or Eq. (5.1) is used for the description of the hot and dense plasma inside the reaction chamber there is no physical equivalence between the plasma space extended parameters and the electric concentrated components. In such a situation it is impossible to model the system with time variable R, L, C components, or in other words to manage the terms in the differential equation with time variable coefficients. Such coefficients could not be delivered by the physical model or first principles. In such situations our VODE model can take over the phenomenon and provide a good description of variation of the physical state of the system. The VODE model equation is given by Eq. (4.1) and reads for this system

$$LC\frac{d^{\alpha(t)}i}{dt^{\alpha(t)}} + RC\frac{d^{\beta(t)}i}{dt^{\beta(t)}} + i = f(t,i). \tag{5.2}$$

The right hand side term in Eq. (5.1) plays again the role of external generated current component, but in this case we have the freedom to consider it time and circuit current dependent. The variable orders of differentiation α and β manage the type of dynamic of this system. We integrated this equation along a time interval $t \in [0,1]$ and choose the functions $\alpha(t) = 1 + t^3$ and $\beta(t) = 2 - t^3$ for the variable orders of the derivative.

With this choice for orders, and initial condition $x(t_0) = 1$, we note that around the initial moment of time we have the first term in Eq. (4.1) close to a first order derivative, and the second term close to a second order time derivative. It this regime the system has the damping (resistance) term dominant and the electric current is expected decay exponentially, which is very well modeled by the VODE/IODE solutions in Fig. (8). In the second part of the interval, the time derivatives switch the orders: the first term becomes a second order derivative, and the second term, a first order time derivative. In this case the system becomes inductively dominant and based on the energy accumulated up to that moment (around $t = 0.2$) the circuit enters into electric oscillation. We choose for the loss term in the equation the dynamics of discharging a capacitor electric energy into the circuit:

$$f(t, i(t)) = i_{loss}\left(1 - e^{-\frac{t}{CR}}\right).$$

Such a non-homogeneous term will take over in time and will cancel the current in the circuit. One can note that the behavior of the solution $i(t)$ is independent of the strength of the non-homogeneous imposed current. In Fig. (8-left) the asymptotic current $i_{loss} = -\lambda < 0$ has a smaller value, while in Fig. (8-right) it is two orders of magnitude larger, yet the qualitative aspect of the solution is exactly the same. Moreover, the IODE approximation method gives perfect results with this injected current, i.e. the strength of the non-homogeneous term is larger. This example demonstrates that a VODE equation can model a very complex change of behavior between two different regimes, without the need of managing the coefficients.

ACOUSTIC WAVES IN VISCOUS FLUIDS

In this example we use the Navier-Stokes Eq. (2.10) in order to model acoustic waves in a fluid medium with loss, a very general situation occurring in the ocean or atmosphere. The acoustical wave equation can be derived from the Navier-Stokes equation as conservation of momentum, and from the equation of continuity as conservation of mass. Constitutive relations between material stress and strain or the related pressure and density values are obtained from the Kelvin-Voigt constitutive relations [24]. According to this study, the resulting wave equation for a visco-elastic medium is given by

$$\triangle\vec{v} - \frac{1}{c_0^2}\frac{\partial^2\vec{v}}{\partial t^2} + \tau_\sigma^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\triangle\vec{v} - \frac{\tau_\epsilon^\beta}{c_0^2}\frac{\partial^{\beta+2}}{\partial t^{\beta+2}}\vec{v} = 0, \tag{5.3}$$

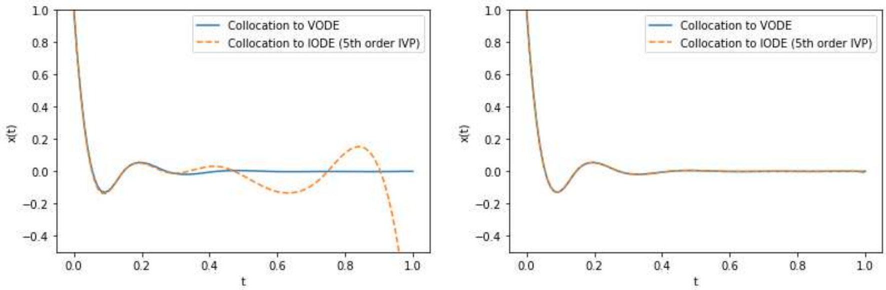

Figure 9: Numerical solution of the VODE equation Eq. (4.1) and its approximate 5th order IODE Eq. (4.13) for $f = 0$, $\alpha(t) = 1 + t^3$, $\beta(t) = 2 - t^3$, $A = 10, B = 60, C = 5460\lambda$ with $\lambda = 10$ (left) and $\lambda = 10^6$ (right).

where the superscripts placed on the relaxation times τ represent algebraic powers, while in the derivatives they represent fractional order. In order to generalize the constitutive relations, the authors in [24] incorporate such fractional time derivatives in this wave equation. Such a fractional derivative model for this wave equation can be linked to the hypothesis that multiple relaxation phenomena give rise to the attenuation measured in complex media. In Eq. (5.3) the constant c_0 is the speed of sound for small amplitude linear acoustical waves in the medium. The parameter τ_σ is a time constant that characterizes the medium, namely the retardation time which is proportional to the viscosity of the fluid. This parameter controls the loss through absorption. The parameter τ_ϵ is the relaxation time and it is responsible for the acoustic wave dispersion in the fluid.

In order to apply our VODE model to the system described by Eq. (5.3) we consider a one-dimensional model $\vec{v} = (0, 0, v)$ which obeys a Helmholtz equation $\triangle\vec{v} = \lambda\vec{v}$ occurring from a separation of variables in the wave equation. By allowing the fractional orders α, β to be time dependent, Eq. (5.3) generates our VODE model and equation similar to Eq. (4.1) in the form

$$\lambda\vec{v} - \frac{1}{c_0^2}\frac{\partial^2\vec{v}}{\partial t^2} + A\lambda\frac{\partial^{\alpha(t)}}{\partial t^{\alpha(t)}}\vec{v} - B\frac{\partial^{\beta(t)}}{\partial t^{\beta(t)}}\vec{v} = f(t, \vec{v}), \tag{5.4}$$

where f represents like before the non-homogeneous term. Solutions for this wave model Eq. (5.4) in the one-dimensional case ($\vec{v} = (v, 0, 0)$) are obtained by solving Eqs. (4.1,4.13) and are presented in Figs. (9, 10). The physical

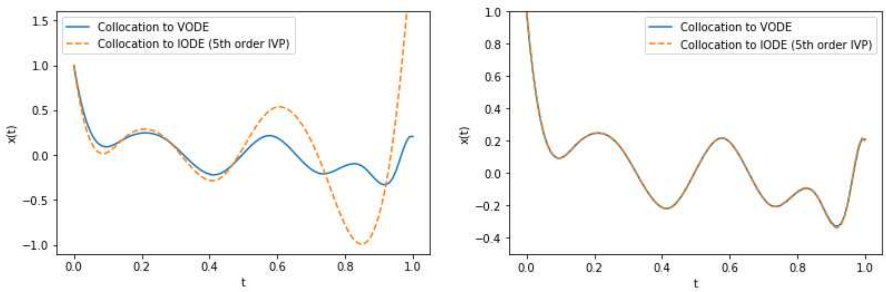

Figure 10: Numerical solution of the VODE equation Eq. (4.1) and its approximate 5th order IODE Eq. (4.13) for $f(t,i) = -1200\lambda\cos(60t)$, $\alpha(t) = 1 + t^3$, $\beta(t) = 2 - t^3$, $A = 10, B = 60, C = 5460\lambda$ with $\lambda = 10\pi$ (left) and $\lambda = 10^1 0\pi$ (right).

interpretation is similar to the one discussed in section **2.** The acoustic waves in this VODE model change their type of dynamics with time: from the initial phase ($t \simeq t_0$) to the final stage ($t \simeq 1$) transforming from damping waves to sustained oscillations. In this model the time variable orders of differentiation manage the balance between loss by viscosity, dispersion and linear effects.

In the cases presented in Figs. (9) the equation is maintained homogeneous and the wave decays to zero at the end of time interval. In a manner similar with all previous cases, the IODE approximation mimics very good the behavior of the fully integrated equation (left frame in the figure) with exception of some scale effects, and blow-up towards $t \simeq 1$ as expected from the previous analysis, Fig. (3). For larger values of the constant C in the equation, which represent negative eigenvalues of larger magnitude, the approximate IODE solution overlaps with the full numerical VODE solutions, right frame in the figure.

In Figs. (10) we present results for the same wave equation model with non-homogeneous forced oscillations term. The initial exponential decay behavior at the beginning is reproduced, the sustained oscillations have larger and more stable amplitude, and the solution acquires a symmetrical form between the two limits for α and β at $t = 0$ and 1 respectively.

6. CONCLUSIONS

In this paper we introduced and solved numerically a new type of dynamical ordinary differential equations (VODE) whose order of differentiation is time dependent. We review two physical situations of interest in which such variable order of differentiation can model a complex phenomenon. We show that the VODE can be represented in terms of a generalization of fractional derivatives whose order of differentiation are functions of time. Based on our previous results we demonstrated that solving this VODE reduces to solve a Volterra integral equation of second kind with singular integrable kernel, which is shown to have unique solution for appropriately chosen initial conditions. In order to solve the resulting integral equation we introduced a new method based on a Taylor expansion of the solution and identification of the series coefficients plus a collocation procedure. In this way the critical problem of the initial conditions, when the order of the differential equation varies from one to two, is solved in an efficient way. We introduced in our equation two variable order of differentiation with respect to time in two different terms, in order to manage the dynamics, loss and dispersion in the model equation. The model is applied to two examples from physics: electric circuit and acoustic waves in a loss-fluid. We present the solutions for various initial conditions, and the results are in agreement with the expected behavior of such systems.

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