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ESTIMATION OF $y(0)$ FOR THE BOUNDARY-VALUE PROBLEM: $y''(z) = y(z)^2 - zy(z)y'(z)$, $y'(0) = -\sqrt{3}$, $y(\infty) = 0$

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ABSTRACT: We investigate the construction of an approximation to the solution of a second-order ODE arising in the problem of the thermal explusion of a fluid from a long, slender heated tube. This boundary-value problem makes requirements on $y(z)$ at $z = 0$ and $z = \infty$, i.e., $y'(0) = -\sqrt{3}$ and $y(\infty) = 0$. Our major task is to determine the value $y(0)$. Using a rational approximation for $y(z)$, we are able to calculate an accurate estimation for $y(0)$. We also derive a number of important properties for the exact solution, $y(z)$.

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1. INTRODUCTION

An important topic in the subject of heat transfer is the thermal explusion of a fluid from a long, slender heated tube. One of the original mathematical modeling contributions to this area was done by Dresner [1, 2] and his work has appeared in many of the standard advanced textbooks on nonlinear partial differential equations [3]. The mathematics can be formulated as the following boundary-value problem:

$$y''(z) = y(z)[y(z) - zy'(z)], \quad 0 < z < \infty, \tag{1a}$$

$$y'(0) = -\sqrt{3}, \quad y(\infty) = 0, \tag{1b}$$

with the requirements

$$y(z) > 0, \quad y'(z) < 0, \tag{1c}$$

where $y'(z) = dy(z)/dz$, etc. The ultimate goal is to determine $y(z)$ and then evaluate $y(0)$. The results in Eq. (1c) are consequences of the fact that $y(z)$ is generally related to a physical temperature, which for the absolute Kelvin scale is non-negative, and the original physical system satisfies classical, continuum thermodynamics for which diffusion is a major factor, i.e., this implies that $y'(z) < 0$. The details of the physical system and what is actually measured, and the interpretation of the results is given in Dresner [1, 2].

As may have been expected, it turns out that, to date, no exact analytical solution $y(z)$ has been found. This means that one must use either numerical methods to obtain numerical solutions and/or construct analytical approximations to $y(z)$. Using the shooting method, within the MATHEMATICA© software, Logan found the following value for $y(0)$ [3]

$$y(0) = 1.5111. \tag{2}$$

The main goals of this paper are to determine some of the basic properties of the boundary-value problem given in Eqs. (1), construct an analytical approximation for $y(z)$, and use the (approximate) representation for $y(z)$ to obtain an estimate for $y(0)$.

This paper is organized as follows: Section 2 gives the derivation of Eqs. (1) from the original nonlinear partial differential equation using similarity methods [4]. In Section 3, a number of the fundamental properties of $y(z)$ are derived. Section 4 provides the details for our specific rational approximation for $y(z)$ and calculates from it a value for $y(0)$. Finally, in Section 5, we discuss possible generalizations of our methods to possibly obtain more accurate representation for $y(z)$ and $y(0)$.

2. DERIVATION OF EQUATIONS (??) AND (??)

The following is a very brief overview of the genesis of Eqs. (1a) and (1b). The problem of the thermal explusion of a fluid from a long, slender heated tube is discussed in detail by Dresner [1, 2]. The fundamental physical equation modeling this situation is the nonlinear partial differential equation [1]

$$uu_t = u_{xx}, \quad x > 0, \quad t > 0, \quad u = u(x,t) \tag{3a}$$

where

$$u(x,t) > 0 \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty, \tag{3b}$$

and the following initial- and boundary-value conditions are required to hold

$$u(x,0) = 0, \quad x > 0, \tag{4a}$$

$$u(\infty,t) = 0, \quad t > 0, \tag{4b}$$

$$u_x(0,t) = -1. \tag{4c}$$

Applying the stretching transformation [4]

$$x \to \bar{x} = \epsilon^a x, \quad t \to \bar{t} = \epsilon^b t, \quad u \to \bar{u} = \epsilon^c u, \tag{5}$$

and using the initial- and boundary-conditions, from Eqs. (4), allows us to conclude that $u(x,t)$ has the structure

$$u(x,t) = t^{1/3} y(z), \quad z = \frac{x}{\sqrt{3}\, t^{1/3}}, \tag{6}$$

where $y(z)$ satisfies the second-order ODE

$$y''(z) = y(z)[y(z) - zy'(z)] \tag{7a}$$

with boundary conditions

$$y'(0) = -\sqrt{3}, \quad y(\infty) = 0, \tag{7b}$$

and (a,b,c) have the values

$$a = 1, \quad b = 3, \quad c = 1. \tag{8}$$

Note that the factor, $\sqrt{3}$, is introduced into the definition of z for convenience; any constant will do, but this particular one is that which is used in the research literature [1, 2].

The fundamental importance of the similarlity transformation, given in Eq. (5), is that it allows the original nonlinear PDE and its associated initial- and boundary-conditions, Eqs. (4), to be transformed into a second-order, nonlinear, ODE boundary-value problem, i.e., given in Eqs. (7); see Birkhoff [4] for a deep discussion of this methodology and the conditions under which it can be applied.

3. GENERAL PROPERTIES OF $Y(Z)$

Many of the important features and properties of the general behavior of the solutions to the ODE

$$y''(z) = y(z)^2 - zy(z)y'(z) \tag{9}$$

may be found without the knowledge of its exact general solution. The purpose of this Section is to provide such derivations:

(i) One trivial, but important exact solution to Eq. (9) is

$$y(z) = \bar{y}(z) = 0. \tag{10}$$

Another exact solution can be found by seeing if there are power law solutions, i.e., where $y(z)$ takes the form

$$y(z) = az^{\alpha}. \tag{11}$$

Substitution of Eq. (11) into Eq. (9) and simplifying the resulting expression gives

$$\alpha(\alpha - 1)z^{\alpha-2} = a(1 - \alpha)z^{2\alpha} \tag{12}$$

and this implies that

$$\alpha = -2, \quad a = 2, \tag{13}$$

and, as a consequence,

$$y(z) = y_s(z) = \frac{2}{z^2} \tag{14}$$

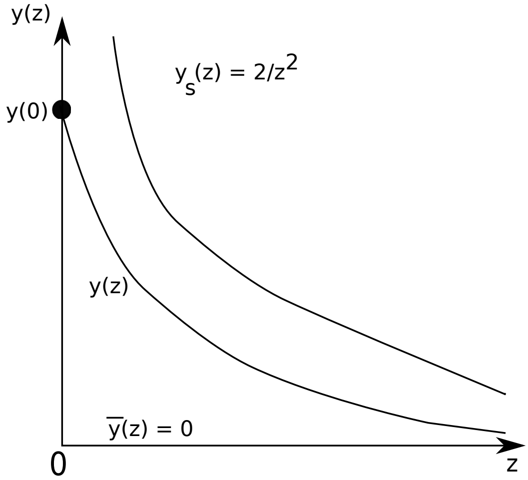

Figure 1: $y_s(z) = \frac{2}{z^2}$ is the singular solution. $\bar{y}(z) = 0$ is the nontrivial, zero solution.

is a solution. Note that $y_s(z)$ is a singular solution in the sense that

$$y_s(0) = \infty, \quad y_s'(0) = -\infty. \tag{15}$$

Close inspection of Figure 1, using the general methodology of the qualitative theory of ODE's [5] allows the following conclusions to be reached: (i) There are three classes of solutions, those whose solution curves lie above $y_s(z)$, $y_s(z)$ itself, and those that lie between $\bar{y}(z)$ and $y_s(z)$. Since a solution curve $y(z)$, such that $y(z) > y_s(z)$, must have $y(0) = \infty$, it follows that the solutions we seek satisfy the restrictions

$$\bar{y}(z) < y(z) < y_s(z), \tag{16}$$

or

$$0 < y(z) < \frac{2}{z^2}, \quad 0 < z < \infty. \tag{17}$$

(ii) Setting $z = 0$ in Eq. (9) gives

$$y''(0) = y(0)^2, \tag{18}$$

a relationship that will prove to be useful in Section 4.

(iii) We now prove that the boundary-value problem given in Eqs. (7) produces a solution satisfying the integral relation

$$\int_0^\infty y(z)^2 dz = \frac{2}{\sqrt{3}}. \tag{19}$$

We start by observing that $zy(z)y'(z)$ can be rewritten to the form

$$zy(z)y'(z) = \left(\frac{1}{2}\right)[zy(z)^2]' - \left(\frac{1}{2}\right)y(z)^2, \tag{20}$$

where, as a reminder, the prime denotes differentiation with respect to z. Substitution of this into the second term on the right-side of Eq. (7a) gives

$$y''(z) = \left(\frac{3}{2}\right)y(z)^2 - \left(\frac{1}{2}\right)[zy(z)^2]'. \tag{21}$$

Integrating this expression from $z = 0$ to $z = \infty$ yields

$$y'(\infty) - y'(0) = \left(\frac{3}{2}\right)\int_0^\infty y(z)^2 dz - \left(\frac{1}{2}\right)[zy(z)^2]\Big|_0^\infty. \tag{22}$$

From our previous results, it follows that

$$y'(\infty) = 0, \quad \operatorname*{Lim}_{z\to 0}[zy^2(z)] = 0, \quad \operatorname*{Lim}_{z\to\infty}[zy^2(z)] = 0, \tag{23}$$

and we conclude, using $y'(0) = -\sqrt{3}$, that the relation given in Eq. (19) holds.

(iv) If Eq. (1a) is differentiated twice, and if the requirements of Eq. (1c) are imposed, then it follows that

$$y''(z) > 0, \quad y'''(z) < 0, \quad y^{(4)}(z) > 0. \tag{24}$$

The higher-order derivatives do not appear to maintain any particular sign pattern.

4. APPROXIMATION TO $Y(Z)$ AND $Y(0)$

A mathematical function which is consistent with all of the properties of $y(z)$, presented in Sections 1 and 3, is the following rational form

$$y_a(z) = \frac{A}{1 + Bz + Cz^2}, \tag{25}$$

where the parameters (A, B, C) are all positive and the notation $y_a(z)$ indicates that it is an approximation to the exact solution, $y(z)$. The positivity requirement follows from $y(0) > 0$ and the fact that $y(z)$ is to be a bounded function for $0 < z < \infty$.

It should be pointed out that for (A, B, C) positive, $y_a(z)$ as given by Eq. (25) satisfies the inequalities expressed in Eq. (24). Also, note that Eq. (11) to Eq. (13) allows the determination of a singular solution and this function provides a barrier between bounded and unbounded solutions to Eq. (9). The singular solution is an exact solution to Eq. (9).

The parameter A is determined by requiring $y_a(0) = y_0$; this gives

$$A = y(0). \tag{26}$$

Further, a direct and rather easy calculation shows that the requirement

$$y_a'(0) = -\sqrt{3} \tag{27}$$

implies that

$$B = \frac{\sqrt{3}}{y(0)}. \tag{28}$$

Finally, the asymptotic relation

$$y_a(z) = \frac{2}{z^2} - O\left(\frac{1}{z^3}\right), \tag{29}$$

where the correction has a minus sign, is fulfilled if

$$C = \frac{y(0)}{2}. \tag{30}$$

The substitution of these results into the expression presented in Eq. (25) gives the following rational approximation to the boundary-value problem given by Eqs. (1),

$$y_a(z) = \frac{y(0)}{1 + \left[\frac{\sqrt{3}}{y(0)}\right] z + \left[\frac{y(0)}{2}\right] z^2}. \tag{31}$$

Observe, however, that $y(0)$ appears on the right-side and its evaluation is one of the primary purposes pursuing the work of this paper.

One approach to estimate $y(0)$ is to use the integral relation of Eq. (19), namely, require

$$\int_0^\infty y_a(z)^2 dz = \frac{2}{\sqrt{3}}. \tag{32}$$

The integral on the left-side can be evaluated [6], and will depend only on $y(0)$, i.e.,

$$\int_0^\infty y_a(z)^2 dz = F(y(0)). \tag{33}$$

However, $F(y(0))$ is an extremely complicated function, depending on logarithmic, arctangent, and radical functions of $y(0)$. These complications indicate that one has to be extremely careful to insure that a unique solution exists.

An alternative method for estimating $y(0)$, using the ansatz of Eq. (31), is to substitute it into Eq. (1a) and use the resulting expression to solve for $y(0)$. After a long, but straightforward calculation, this expression is found to be the following cubic equation

$$y_0^3 + y_0^2 - 6 = 0, \quad y_0 = y(0). \tag{34}$$

This equation has one real, positive root and two complex conjugate roots. The real root is

$$y_0 = y(0) = 1.537656171\ldots, \tag{35}$$

and this is to be compared with the value $y_L(0) = 1.5111$ obtained by Logan [3] using a numerical method. Note that absolute and percentage errors between our estimation and that of Logan are, respectively,

$$\text{Absolute Error} = |(1.5377 - 1.5111)| = 0.0266, \tag{36a}$$

$$\%\ \text{Error} = \left|\frac{(1.5377 - 1.5111)}{1.5111}\right| \cdot 100 = 1.76\% \tag{36b}$$

Finally, we determine the asymptotics of $y_a(z)$ for large values of z. Expanding $y_a(z)$, Eq. (31), in powers of $1/z^2$ gives

$$y_a(z) = \left(\frac{2}{z^2}\right) - \left[\frac{4\sqrt{3}}{y(0)^2}\right]\left(\frac{1}{z^3}\right) + O\left(\frac{1}{z^4}\right), \tag{37}$$

a result which is consistent with the inequality

$$0 < y_a(z) < y_s(z). \tag{38}$$

5. DISCUSSION

To summarize, we have provided a methodology to determine an approximation to the solution of the initial-value, boundary-value problem presented in

Eq. (1). We then used this expression to estimate the value of $y(0)$. A comparison of our value with that obtained by a shooting numerical method indicates the accuracy of our procedure.

A reason why the results of this paper are important is that engineers wish to have simple functional forms for the desired solutions of complex nonlinear PDE's. However, in general, the exact solutions do not have this property and, as a consequence, the determination of a simple and accurate analytical expression is of great value and adequate for their engineering applications. Further, in many of these situations some parameter value or function evaluation is required and is obtainable only from the solution or (in most cases) an approximation to the solution. The results of this paper show that such can be done for the problem of the thermal expulsion of fluid from a long, slender heated tube [1, 2]. In general, it should be expected that the general procedures of our work can be extended to similar problems in nonlinear heat transfer (see Chapter 4, Dresner [1]).

A possible generalization of the methodology of this paper is to consider rational approximations within the framework of the Pade approximant formulation [7]; thus, for this situation, $y_a(z)$ takes the form

$$y_a(z) = \frac{P_N(z)}{Q_{N+2}(z)}, \tag{39}$$

where $P_{N+2}(z)$ and $Q_{N+2}(z)$ are N-th and $(N+2)$-th degree polynomials. Since for given $y(0)$ and $y'(0)$ allows the calculation of the Taylor series for $y(z)$, see Eq. (1a), this higher-order rational form should be successful.

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