AN INVENTORY MODEL FOR DETERIORATING PHARMACEUTICAL ITEMS WITH TIME DEPENDENT DEMAND UNDER COMPLETE BACKLOGGING

R. UTHAYAKUMAR\textsuperscript{1} AND S. THARANI\textsuperscript{2}
\textsuperscript{1,2}Department of Mathematics
The Gandhigram Rural Institute - Deemed University
Gandhigram, Tamilnadu, INDIA

\textbf{ABSTRACT:} A high level of service for medical supplies and effective inventory policies are essential objectives for all health care industries. Medicine shortages and improper use of pharmaceuticals can not only lead to financial losses but also have a significant impact on patients. Many health systems and hospitals experience difficulties in achieving these goals as they have not addressed how medicines are managed, supplied, and used to save lives and improve health. Studies are essential to understand operations in health care industries and to offer decision support tools that improve health policy, public health, patient safety, and strategic decision-making in the pharmaceutical inventory model. We present a deterministic inventory model for pharmaceutical items having time dependent quadratic demand under the effect of deterioration. We develop a procedure for determining optimal solutions for inventory and the number of deliveries to achieve hospital (Customer Service Level) CSL targets with a minimum total cost of the system. The necessary and sufficient conditions for the existence and uniqueness of the optimal solution which could minimize the average total cost per unit time has been discussed. A numerical example illustrates the model application and behavior using MATLAB.
1. INTRODUCTION

Pharmaceuticals represent a significant part of health care costs, account for approximately 10% of annual health care expenditure in the USA and about $600 billion globally in 2009. Pharmaceutical products can be expensive to purchase and distribute, but shortages of essential medicines, improper use of medicines, and spending on unnecessary or low-quality medicines also have a high costs in terms of wasted resources and preventable illness and death. Management of pharmaceutical inventory has become a major challenge for health care industries as they simultaneously try to reduce costs and improve CSL in an increasingly competitive business environment. Almarsdttir and Traulsen identified a number of reasons why pharmaceutical deserve special consideration in the control of inventory [3]. In the current economic crisis, increasing attention is being focused on the rising costs of health care and specifically pharmaceuticals.

Careful management of pharmaceutical is directly related to a country’s ability to address public health concerns. Aptel and Pourjalali stated that management of pharmaceutical supplies is one of the most important managerial issues in health care industries [4]. However, many health care industries experience difficulty in managing their pharmaceutical products.

Unlike many industries, hospital administrators and pharmacy managers have to manage very complicated distribution networks and inventory management problems without proper guidance on efficient practices. This is because most hospital administrators and pharmacy managers are doctors with expert knowledge in medicine, and are not inventory professionals [40]. Hence, given the high costs, coordination, constraints, and perishability of pharmaceuti-
INVENTORY MODEL FOR PHARMACEUTICAL ITEMS

Pharmaceutical products can be more expensive than other products to purchase and distribute, and shortages and improper use of essential medicines can have a high cost in terms of wasted resources and preventable illness and death. Therefore, special care should be taken in pharmaceutical inventory decisions to ensure 100% product availability at the right time, at the right cost, and in good condition to the right customers. The quality of health care industries strongly depends on the availability of pharmaceuticals on time. If a shortage occurs at a hospital, an emergency delivery is necessary, which is very costly and can be implications for patient health. Inventory management strategies that are unsuitable for health care industries may lead to large financial losses and a significant impact on patients. Hence, inventory strategies for pharmaceutical products are more critical than those for other products. Thus, a specific inventory model is necessary for control of pharmaceutical products to save patient lives and reduce unnecessary inventory costs.

The next section reviews the literature on deteriorating issues from the perspective of health care industries.

2. LITERATURE REVIEW

Management of the procurement, storage, and distribution of pharmaceutical supplies is crucial for hospitals and pharmaceutical companies from economic and organizational points of view.

In the traditional inventory models, one of the assumptions was that the items preserved their physical characteristics while they were kept stored in the inventory. This assumption is evidently true for most items, but not for all. However, the deteriorating items are subject to a continuous loss in their masses or utility throughout their lifetime due to decay, damage, spoilage, and penalty of other reasons. Owing to this fact, controlling and maintaining the
inventory of deteriorating items becomes a challenging problem for decision makers.


Norris investigated cost reductions for hospitals by considering the total delivered cost of a product rather than just the unit cost [25]. This involves quantifying every cost associated with a product, including the unit cost and costs related to ordering, inventory, distribution, preparation and use, and paperwork.

Inventory problems involving time variable demand patterns have received the attention of several researchers in recent years. Silver and Meal [31] constructed an approximate solution procedure for the general case of a deterministic, time varying demand pattern.

The classical no-shortage inventory problem for a linear trend in demand over a finite time horizon was analytically solved by Donaldson [8]. However, Donaldson’s solution procedure was computationally complicated. Silver [30] derived a heuristic for the special case of positive, linear trend in demand and applied it to the problem Donaldson. Ritchie [28] obtained an exact solution, having the simplicity of the EOQ formula, for Donaldson’s problem for linear, increasing demand. Mitra et al. [23] presented a simple procedure for adjusting the economic order quantity model for the cases of increasing or decreasing linear trend in demand.

In all these models, the possibilities of shortages and deterioration in inventory were left out of consideration. Harris [13] developed the first inventory model, Economic Order Quantity, which was generalized by Wilson [39] who introduced a formula to obtain EOQ. Whitin [38] considered the deterioration of the fashion goods at the end of prescribed shortage period. Dave and Patel [6] studied a deteriorating inventory with linear increasing demand when shortages are not allowed.

Ghare and Schrader [11] addressed the inventory lot-sizing problem with constant demand and deterioration rate. With the help of some mathematical
approximations, they developed a simple Economic Order Quantity, EOQ, model. Then, Covert and Philip [5] and Tadikamalla [33] extended Ghare and Schrader’s work by considering variable rate of deterioration. Shah [29] provided a further generalization of all these models by allowing shortages and using a general distribution for the deterioration rate.

Other authors [9], [19], [16], [26] readjusting Ghare and Schrader’s model by relaxing the assumption of infinite replenishment rate. All these inventory models were formulated in a static environment where the demand is assumed to be constant and steady over a finite planning horizon. However, in a realistic product life cycle, demand is increasing with time during the growth phase. Naddor [24] assumed a demand function that increasing in linear proportion with time during the growth phase and analyzed the cost performances of three inventory policies. Mandal [17] studied a EOQ model for Weibull distributed deteriorating items under ramp-type demand and shortages.

Alamri and Balkhi [2] studied the effects of learning and forgetting on the optimal production lot size for deteriorating items with time varying demand and deterioration rates. Dye [7] gave an inventory model to determine optimal selling price and lot size with a varying rate of deterioration and exponential partial backlogging and deterministic inventory model for deteriorating items with capacity constraint and time-proportional backlogging rate. Teng et al. [34] gave a comparison between two pricing and lot-size models with partial backlogging and deteriorated items. Roy [27] developed an inventory model for deteriorating items with time varying holding cost and demand is price dependent.

Mishra and Singh [20], [21] constructed an inventory model for ramp-type demand, time dependent deteriorating items with salvage value and shortages and deteriorating inventory model for time dependent demand and holding cost and with partial backlogging. Hung [15] investigated an inventory model with generalized type demand, deterioration and back order rates. In this paper, we made the work of Mishra et al. [22] more realistic by considering time dependent demand and developed an inventory model for deteriorating items where deterioration rate and holding cost are constants. Shortages are allowed and partially backlogged. An inventory model for deteriorating items with time dependent linear demand under partial backlogging is discussed by Milu and Smrutirekha [18].
3. NOTATIONS AND ASSUMPTIONS

The fundamental assumptions and notations used in this paper are given as follows:

3.1. Notations

i. \( I(t) \) is the level of inventory at time \( t \), \( 0 \leq t \leq T \).

ii. \( T \) is the length of the cycle.

iii. \( T_1 \) is the time when the inventory level reaches zero.

iv. \( T_1^* \) and \( T^* \) are the optimal points.

v. \( A \) is the fixed ordering cost per order.

vi. \( Dc \) is the cost of each deteriorated item.

vii. \( h \) is the inventory holding cost per unit per unit of time.

viii. \( s \) is the shortage cost per unit per unit of time.

ix. \( Q \) is the maximum inventory level for the ordering cycle, such that \( Q = I(0) \).

x. \( TC \) is the average total cost per unit time under the condition \( T_1 \leq T \).

3.2. Assumptions

i. The demand rate is time dependent and quadratic, i.e. \( D(t) = a + bt + ct^2 \); \( a, b, c > 0 \) and are constant.

ii. The replenishment rate is infinite, thus replenishment is instantaneous.

iii. The rate of deterioration is constant with parameter \( \theta \) where \( 0 < \theta < 1 \).

iv. Shortages are allowed.
4. MATHEMATICAL FORMULATION

Here we consider the deteriorating inventory model with quadratic time dependent demand rate. Replenishment occurs at time $t = 0$ when the inventory level attains its maximum. From $t = 0$ to $T_1$, the inventory level reduces due to demand and deterioration.

At $T_1$, the inventory level reaches zero, then the shortages are allowed to occur during the time interval $(T_1, T)$ is completely backlogged. The total number of backlogged items is replaced by the next replenishment.

Figure 1: Graphical representation of inventory system.

According to the notations and assumption mentioned above, the behavior of the inventory system at any time can be described by the following differential equations:

\[
\frac{dI(t)}{dt} = -D(t) - \theta I(t), \quad 0 \leq t \leq T_1 \tag{1}
\]

\[
\frac{dI(t)}{dt} = -D(t), \quad T_1 \leq t \leq T \tag{2}
\]

With the boundary conditions $I(0) = S$, $I(T_1) = 0$.

The solutions of the equations (1) and (2) with the boundary conditions
are as follows:

\[ I(t) = \left( \frac{a + bT_1 + cT_1^2}{\theta} - \frac{b + 2cT_1}{\theta^2} + \frac{2c}{\theta^3} \right) e^{\theta(T_1 - t)} \]
\[ - \left( \frac{a + bt + ct^2}{\theta} - \frac{b + 2ct}{\theta^2} + \frac{2c}{\theta^3} \right), \quad 0 \leq t \leq T_1 \]

(3)

\[ I(t) = a(T_1 - t) + \frac{b}{2}(T_1^2 - t^2) + \frac{c}{3}(T_1^3 - t^3), \quad T_1 \leq t \leq T \]

(4)

The beginning inventory level can be computed as

\[ Q = \left( \frac{a}{\theta} - \frac{b}{\theta^2} + \frac{2c}{\theta^3} \right) (e^{\theta T_1} - 1) + \left( \frac{b\theta - 2c}{\theta^2} \right) T_1 e^{\theta T_1} + \frac{c}{\theta} T_1^2 e^{\theta T_1} \]

(5)

The total number of items which deteriorates in the interval \([0, T_1]\), say \(D_T\) is

\[ D_T = Q - \int_0^{T_1} D(t) \, dt \]
\[ = Q - \int_0^{T_1} (a + bt + ct^2) \, dt \]
\[ = Q - \left( aT_1 + \frac{b}{2}T_1^2 + \frac{c}{3} T_1^3 \right) \]

(6)

The total number of inventory carried during the interval \([0, T_1]\), say \(H\) is

\[ H = \int_0^{T_1} I(t) \, dt \]
\[ = \left( \frac{a + bT_1 + cT_1^2}{\theta^2} - \frac{b + 2cT_1}{\theta^3} + \frac{2c}{\theta^4} \right) (e^{\theta T_1} - 1) \]
\[ - \left( \frac{a\theta^2 - b\theta + c}{\theta^3} T_1 + \frac{b\theta - 2c}{2\theta^2} T_1^2 + \frac{c}{3\theta} T_1^3 \right) \]

(7)

The total shortage quantity during the interval \([T_1, T]\), say \(B_T\) is

\[ B_T = - \int_{T_1}^{T} I(t) \, dt \]
\[ = \frac{a}{2} (T^2 - 2TT_1 + T_1^2) + \frac{b}{6} (T^3 - 3T_1^2 T + 2T_1^3) + \frac{c}{12} (T^4 - 4T_1^3 T + 3T_1^4) \]
Then, the average total cost per unit time under the condition $T_1 \leq T$ can be given by

$$TC = \frac{1}{T} [A + D_cDT + hH + sBT]$$  \hspace{1cm} (9)

The first order derivatives of $TC$ with respect to $T_1$ and $T$ are as follows

$$\frac{\partial TC}{\partial T_1} = \frac{1}{T} \left( D_c \frac{\partial DT}{\partial T_1} + h \frac{\partial H}{\partial T_1} + s \frac{\partial BT}{\partial T_1} \right)$$  \hspace{1cm} (10)

$$\frac{\partial TC}{\partial T} = -\frac{1}{T^2} A - \frac{1}{T^2} D_cDT - \frac{1}{T^2} hH - \frac{1}{T^2} sBT + \frac{1}{T} s \frac{\partial BT}{\partial T}$$  \hspace{1cm} (11)

Since $\frac{\partial DT}{\partial T} = \frac{\partial H}{\partial T} = 0$. To find the above derivatives, we find the following

$$\frac{\partial DT}{\partial T_1} = (a + bT_1 + cT^2_1)(e^{\theta T_1} - 1)$$  \hspace{1cm} (12)

$$\frac{\partial H}{\partial T_1} = \left( \frac{a + bT_1 + cT^2_1}{\theta} \right)(e^{\theta T_1} - 1)$$  \hspace{1cm} (13)

$$\frac{\partial BT}{\partial T_1} = (a + bT_1 + cT^2_1)(T_1 - T)$$  \hspace{1cm} (14)

$$\frac{\partial BT}{\partial T} = a(T - T_1) + \frac{b}{2}(T^2 - T^2_1) + \frac{c}{3}(T^3 - T^3_1)$$  \hspace{1cm} (15)

On substituting the Equations (12),(13),(14) in (10) and (15) in (11), we get

$$\frac{\partial TC}{\partial T_1} = \frac{1}{T}(a + bT_1 + cT^2_1) \left( (e^{\theta T_1} - 1) \left(D_c + \frac{h}{\theta} \right) + (T_1 - T)s \right)$$  \hspace{1cm} (16)

$$\frac{\partial TC}{\partial T} = -\frac{1}{T^2} (A + D_cDT + hH + sBT) + \frac{1}{T} s (a(T - T_1))$$

$$+ \frac{b}{2}(T^2 - T^2_1) + \frac{c}{3}(T^3 - T^3_1)$$  \hspace{1cm} (17)

$$\frac{\partial^2 TC}{\partial T^2} = \frac{2}{T^3} (A + D_cDT + hH + sBT) - \frac{2}{T^2} s \left( a(T - T_1) + \frac{b}{2}(T^2 - T^2_1) \right)$$

$$+ \frac{c}{3}(T^3 - T^3_1) - \frac{1}{T} s (a + bT + cT^2)$$  \hspace{1cm} (18)

$$\frac{\partial^2 TC}{\partial T_1 \partial T} = -(a + bT_1 + cT^2_1) \left\{ \frac{1}{T^2} \left( (e^{\theta T_1} - 1) \left(D_c + \frac{h}{\theta} \right) + s(T_1 - T) \right) \right\}$$  \hspace{1cm} (19)
The optimal values of $T_1$ and $T$ as $T_1^*$, $T^*$ can be obtained by satisfying the necessary condition for minimization of the cost function

$$\frac{\partial TC}{\partial T_1} = 0, \quad \frac{\partial TC}{\partial T} = 0,$$

from the Eqs (16) and (17) and provided the following sufficient conditions are satisfied

$$\frac{\partial^2 TC}{\partial T_1^2} > 0, \quad \frac{\partial^2 TC}{\partial T^2} > 0,$$

$$\left(\frac{\partial^2 TC}{\partial T_1^2}\right) \left(\frac{\partial^2 TC}{\partial T^2}\right) - \left(\frac{\partial^2 TC}{\partial T_1 \partial T}\right)^2 > 0$$

from the Eqns (18) and (19) and (20). Since the nature of the cost function is highly non-linear thus the optimality of the function is shown graphically in the next section.

5. NUMERICAL ILLUSTRATION

Here we incorporated the more practical numerical example for the support of our model verification. Consider an inventory system with parameters as $A = $200 per order, $D_c = $3 per item, $h = $10 per unit per unit of time, $s = $7 per unit per unit of time, $a = 100, \quad b = 50, \quad c = 2, \quad \theta = 0.001.$

The numerical values of $T_1^*$, $T^*$ and $TC$ have been calculated as $T_1^* = 0.3367$, $T^* = 0.8078$ and $TC = $451.5546 by using Eq. (21) for solution of system of non-linear equation with the help of MATLAB.

5.1. Algorithm

**Step1:** Set $T = 1$ in the equation $\frac{\partial TC}{\partial T_1} = 0$ and find $T_1$.

**Step2:** Substitute $T_1$ in the equation $\frac{\partial TC}{\partial T} = 0$ and find $T$.

**Step3:** Substitute $T$ in the equation $\frac{\partial TC}{\partial T_1} = 0$ and find $T_1$. 
**Step 4:** Repeat the steps 2 and 3 until there is no change in the successive values of $T$ and $T_1$.

**Step 5:** Compute $TC$ by substituting those optimal values of $T$ and $T_1$.

### 6. SENSITIVITY ANALYSIS

We now study the effects of changes in the values of the system parameters $\theta$, $a$, $b$, $c$, $D_c$, $h$ and $s$ on the optimal replenishment policy of the above Example. We change one parameter at a time keeping the other parameters unchanged. The results are summarized in Table 1.

Based on our numerical results, we obtain the following managerial phenomena:

1. As the value of $\theta$, $a$, $b$, $c$, $h$ and $s$ increases the values of $T$ and $T_1$ decrease but at the same time the value of $TC$ increases.

2. As the value of $D_c$ increases the values of $T$ and $T_1$ are stable but the value of $TC$ increases.

### 7. CONCLUSION

The present paper develops a mathematical model of a pharmaceutical inventory system which has quadratic time dependent demand for deteriorating items with backlogging and gives the condition to minimize the total average cost of the system.

The deterioration factor taken into account in the present model, almost all pharmaceutical products undergo either direct decay or physical decay in due course of time. The model is very practical for the healthcare industries in which demand rate is depending upon time. The sensitivity of the model has checked with respect to the various parameter of the system.

The model is helpful for taking the decision in the sense that to determine the appropriate time i.e. $T^*_1$ in order to avoid the deterioration and backlog so
Table 1: Effect of changes in the parameter of the inventory model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variation</th>
<th>$T_1$</th>
<th>$T$</th>
<th>$TC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.001</td>
<td>0.3367</td>
<td>0.8078</td>
<td>451.5546</td>
</tr>
<tr>
<td></td>
<td>0.002</td>
<td>0.3366</td>
<td>0.8078</td>
<td>451.5838</td>
</tr>
<tr>
<td></td>
<td>0.003</td>
<td>0.3364</td>
<td>0.8077</td>
<td>451.6160</td>
</tr>
<tr>
<td></td>
<td>0.004</td>
<td>0.3363</td>
<td>0.8076</td>
<td>451.6483</td>
</tr>
<tr>
<td></td>
<td>0.005</td>
<td>0.3362</td>
<td>0.8076</td>
<td>451.6806</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.3367</td>
<td>0.8078</td>
<td>451.5546</td>
</tr>
<tr>
<td></td>
<td>101</td>
<td>0.3357</td>
<td>0.8056</td>
<td>453.2171</td>
</tr>
<tr>
<td>$a$</td>
<td>102</td>
<td>0.3347</td>
<td>0.8033</td>
<td>454.8738</td>
</tr>
<tr>
<td></td>
<td>103</td>
<td>0.3337</td>
<td>0.8011</td>
<td>456.5248</td>
</tr>
<tr>
<td></td>
<td>104</td>
<td>0.3327</td>
<td>0.7989</td>
<td>458.1700</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.3367</td>
<td>0.8078</td>
<td>451.5546</td>
</tr>
<tr>
<td></td>
<td>51</td>
<td>0.3360</td>
<td>0.8062</td>
<td>452.1804</td>
</tr>
<tr>
<td>$b$</td>
<td>52</td>
<td>0.3352</td>
<td>0.8044</td>
<td>452.8113</td>
</tr>
<tr>
<td></td>
<td>53</td>
<td>0.3345</td>
<td>0.8028</td>
<td>453.4387</td>
</tr>
<tr>
<td></td>
<td>54</td>
<td>0.3338</td>
<td>0.8011</td>
<td>454.0580</td>
</tr>
<tr>
<td>$c$</td>
<td>2</td>
<td>0.3367</td>
<td>0.8078</td>
<td>451.5546</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.3338</td>
<td>0.7964</td>
<td>454.4746</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.3312</td>
<td>0.7857</td>
<td>457.2797</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.3288</td>
<td>0.7750</td>
<td>459.9695</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.3267</td>
<td>0.7671</td>
<td>462.5362</td>
</tr>
<tr>
<td>$D_c$</td>
<td>3</td>
<td>0.3367</td>
<td>0.8078</td>
<td>451.5546</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.3367</td>
<td>0.8079</td>
<td>451.5624</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.3367</td>
<td>0.8079</td>
<td>451.5702</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.3367</td>
<td>0.8079</td>
<td>451.5774</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.3367</td>
<td>0.8079</td>
<td>451.5852</td>
</tr>
<tr>
<td>$h$</td>
<td>10</td>
<td>0.3367</td>
<td>0.8078</td>
<td>451.5546</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>0.3132</td>
<td>0.7966</td>
<td>458.8387</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>0.2929</td>
<td>0.7871</td>
<td>465.2331</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>0.2752</td>
<td>0.7790</td>
<td>470.8702</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>0.2594</td>
<td>0.7718</td>
<td>475.8864</td>
</tr>
<tr>
<td>$s$</td>
<td>7</td>
<td>0.3367</td>
<td>0.8078</td>
<td>451.5546</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.3512</td>
<td>0.7805</td>
<td>468.2095</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>0.3636</td>
<td>0.7582</td>
<td>482.6171</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.3746</td>
<td>0.7399</td>
<td>495.2069</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>0.3841</td>
<td>0.7243</td>
<td>506.3190</td>
</tr>
</tbody>
</table>
that the health care centres can be run smoothly without facing the problem of deterioration and backlog.

This model can further be extended by considering more realistic assumptions such as finite replenishment rate, probabilistic demand, variable deterioration rate, time dependent holding cost etc.

Figure 2: Graphical representation of Optimal solutions when the value of c varies from 2 to 6

REFERENCES


Figure 3: Graphical representation of Optimal solutions when the value of a varies from 100 to 104


Figure 4: Graphical representation of Optimal solutions when the value of b varies from 50 to 54


Figure 5: Graphical representation of Optimal solutions when the value of $\theta$ varies from 0.001 to 0.005


[21] V.K. Mishra, L.S. Singh, Deteriorating inventory model for time dependent demand and holding cost with partial backlogging, *Internationa-
Figure 6: Graphical representation of Optimal solutions when the value of $D_c$ varies from 3 to 7


[27] A. Roy, An inventory model for deteriorating items with price dependent
Figure 7: Graphical representation of Optimal solutions when the value of $s$ varies from 7 to 11


Figure 8: Graphical representation of Optimal solutions when the value of h varies from 10 to 14


