EXISTENCE OF MULTIPLE POSITIVE RADIAL SOLUTIONS TO ELLIPTIC EQUATIONS IN AN ANNULUS

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ABSTRACT: In this paper, we use Leggett-Williams multiple fixed point theorems to obtain sufficient conditions for the existence of at least one or two positive radial solutions of the equation

$$-\Delta u = \lambda g(|x|)f(u), \ \ R_1 < |x| < R_2,$$

$x \in R^N$, $N \geq 2$ subject to a linear mixed boundary condition at R_1 and R_2.

1. INTRODUCTION

In this article, we investigate the existence of multiple positive radial solutions of the elliptic equation

$$\Delta u + \lambda g(|x|) f(u) = 0, \ R_1 < |x| < R_2, \tag{1.1}$$

where $x \in R^N$, $N \geq 2$, along with the following linear boundary conditions at R_1 and R_2:

$$\left. \begin{array}{l} u = 0 \text{ on } |x| = R_1 \text{ and } |x| = R_2, \\[2mm] u = 0 \text{ on } |x| = R_1 \text{ and } \dfrac{\partial u}{\partial r} = 0 \text{ on } |x| = R_2, \\[2mm] \dfrac{\partial u}{\partial r} = 0 \text{ on } |x| = R_1 \text{ and } u = 0 \text{ on } |x| = R_2, \end{array} \right\} \tag{1.2}$$

where $x \in R^N$, $N \geq 2$, $r = |x|$ and $\frac{\partial}{\partial r}$ denotes the differentiation in the radial direction, and $0 < R_1 < R_2 < \infty$.

Equation (1.1) appears in several applications in mechanics and physics, and in particular, it can be an equation of equilibrium states in thin films. Equations of the form

$$u_t = -\nabla.(f(u)\nabla u) - \nabla.(g(u)\nabla u) \tag{1.3}$$

have been used in modelling the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The coefficient $f(u)$ reflects surface tension effects; a typical choice is $f(u) = u^3$. For a detailed background on the equation (1.3), we refer to [3, 4, 5, 13, 15, 16, 17] and the references cited there in.

Boundary value problems of the form (1.1) with $\lambda = 1$ and

$$u = 0 \text{ and } |x| = R_1 \text{ and } |x| = R_2 \tag{1.4}$$

has been studied by many authors. Wang [34] used cone compression and cone expansion method to find the existence of at least one positive radial solution of (1.1) and (1.4). Arcoya [1] used Mountain Pass method to find the existence of positive radial solutions of (1.1) and (1.4).

A particular case of (1.1) is the semilinear elliptic equation

$$\Delta u + \lambda f(u) = 0. \tag{1.5}$$

Lin [19] used supersolution and subsolution method to find the existence of at least one positive solution of (1.5) and (1.4). Hai and Smith [12] studied the existence and uniqueness of the positive solutions for the boundary value problem (1.5) and

$$u = 0 \text{ on } \partial\Omega, \tag{1.6}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $f : (0, \infty) \to$ $(0, \infty)$ is possibly singular at 0 and λ is a positive parameter. Dancer and Schmitt [6] used supersolution and subsolution method to find the existence of positive solutions of (1.5) and (1.6). Wang [35] used Mountain Pass theorem to find the existence of positive solutions of (1.1) and (1.6) with $\lambda = 1$. Garzier [11] proved that if f satisfies

(A_1) $f \in [0, \infty)$, $f(u) > 0$ for $u \neq 0$ and $f(0) = 0$;

(A_2) $d_1 u^k \leq f(u) \leq d_2 u^k$ for large u, where $d_1, d_2 > 0$, $k > -1$; and

(A_3) $f_0 = 0$ and $k > 1$ or $f_0 = \infty$ and $k < 1$, where

$$f_0 = \lim_{u \to 0+} \frac{f(u)}{u} \text{ and } f_\infty = \lim_{u \to \infty} \frac{f(u)}{u},$$

then there exists a radial solution to (1.1) and (1.4) with $\lambda = 1$.

Bandle, Coffman and Marcus [2] proved that, let $f \in C^1[0, \infty)$ satisfying (A_1) and

(A_4) f is nondecreasing on $(0, \infty)$.

If f is superlinear at 0 and ∞, or $f_0 = 0$ and $f_\infty = \infty$, then there exists a positive, radial symmetric solution to (1.1) and (1.4) with $\lambda = 1$.

Moussaoui and Precup [25] used Leray-Schauder alternative method to find the existence result for semilinear equation

$$-\Delta u = f(x, u) \tag{1.7}$$

and (1.6), where $\Omega \subset R^N$, $(N \geq 2)$ is a nonempty bounded open set with smooth boundary $\partial\Omega$ and $f : \overline{\Omega} \times R \to R$ is a continuous function. Lu and Bai [21] used supersolution and subsolution method to find the existence of positive solutions of

$$-\Delta u = f(|x|, u), \text{ in } B,$$
$$u > 0 \text{ for } x \in B,$$
$$\text{and } u = 0 \text{ on } x \in \partial B,$$

where B is the unit open ball centered at the origin in R^n, that is, $B = \{x \in R^n; |x| < 1\}$ and the function f is allowed to change the sign.

From the works due to Wang [34] and Bandle et al. [2], it has been observed that the obtained sufficient conditions on the existence of at least one positive radial solutions of (1.1) subject to either one of the boundary conditions given in (1.2) is satisfied. Our aim of this paper is to obtain sufficient conditions for the existence of multiple positive solutions of (1.1) subjected to all the boundary conditions given in (1.2) are satisfied. Erbe and Wang [8] used cone expansion and cone compression method to find the existence of at least two positive radial solutions of (1.1) when all the boundary conditions in (1.2) are satisfied. It has been observed that very few papers exist in the literature on the existence of at least two positive solutions of (1.1) and (1.2). Maya and Robinson [22] used supersolution and subsolution method to obtain sufficient conditions for the existence of at least three positive solutions of the equation

$$-\Delta u = \phi g(u) u^{-\alpha} \text{ on } \Omega$$

with the boundary condition (1.6), where $\Omega \in R^n$ is a bounded domain, ϕ is a nonnegative function in $L^\infty(\Omega)$ such that $\phi > 0$ on some subset of Ω of positive measure, $g : [0, \infty) \to \infty$ is continuous and $\alpha > 0$. In another attempt, Maya and Shivaji [23] studied the existence of two positive classical solutions of (1.1) and (1.6), when $\lambda > 0$ is a parameter, Ω is a bounded region in R^n with smooth boundary $\partial\Omega$. They obtained their result with the additional condition that $f(0) = 0$, $f'(0) < 0$, there exists a $\beta > 0$ such that $f(u) < 0$ for $u \in (0, \beta)$, $f(u) > 0$ for $u > \beta$, f is eventually increasing and $\lim_{u \to \infty} \frac{f(u)}{u} = 0$. Marcos do Ó et al. [24] used a fixed point theorem of Guo and Lakshmikantham [7] to find the existence of at least three positive radial solution of (1.1) along with the boundary condition

$$u > 0 \text{ in } \Omega \text{ and } u = a \text{ in } \partial\Omega,$$

where Ω is the ball of radious R_0 centered at the origin, λ, a are positive parameters, $f \in C([0, \infty), [0, \infty))$ is an increasing function and $k \in C([0, R_0], [0, \infty))$ is not identically zero in any interval of $[0, R_0]$. The use of Leggett-Williams multiple fixed point theorem for the existence of multiplicity of positive solutions of (1.1) is scarce in the literature. In this paper, by using Leggett-Williams multiple fixed point theorem, we provide some sufficient conditions on the existence of three solutions to (1.1) with the boundary conditions given in (1.2), at least two of which are positive radial solutions.

The method used in this paper are motivated from the works by the author in [27, 28, 29, 30, 31, 32, 33]. A detailed use of Leggett-Williams multiple fixed point theorem has been used in the monograph due to Padhi, Graef and Srinivasu in [26].

2. PRELIMINARIES

Because of the radial symmetry, the existence of positive radial solutions of (1.1) is equivalent to the existence of positive solutions of the second order equation

$$u''(r) + \frac{N-1}{r}u'(r) + \lambda g(r)f(u(r)) = 0, \ R_1 \le r \le R_2. \tag{2.1}$$

By the change of variables $v(s) = u(r(s))$, the transformation

$$s = -\int_r^{R_2} \frac{1}{t^{N-1}} \, dt$$

transforms (2.1) into

$$v''(s) + \lambda r^{2(N-1)}(s)g(r(s))f(v(s)) = 0, \ m < s < 0, \tag{2.2}$$

where $m = -\int_{R_1}^{R_2} \frac{1}{t^{N-1}} \, dt$. Again the transformation $t = \frac{m-s}{m}$ and $y(t) = v(s)$ transforms (2.2) into the equation

$$y''(t) + \lambda h(t)f(y(t)) = 0, \ 0 < t < 1, \tag{2.3}$$

where

$$h(t) = m^2 r^{2(N-1)}(m(1-t))g(r(m(1-t))), \tag{2.4}$$

and the set of boundary conditions in (1.2) reduces to

$$\left.\begin{array}{r} \alpha y(0) - \beta y'(0) = 0 \\ \gamma y(1) + \delta y'(1) = 0 \end{array}\right\} \tag{2.5}$$

where with suitable choices of α, β, γ and δ are nonnegative reals with

$$\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0. \tag{2.6}$$

The boundary value problem (2.3)-(2.5) has been studied by many authors. One may refer to Erbe and Wang [9], which has been extended to (1.1)-(1.2) in [8] using the same method.

Let $g(|x|) > 0$ for all $R_1 \leq |x| \leq R_2$ and $t \geq 0$. It is clear that $h(t) \geq 0$ for $t \geq 0$. Let $f(u) > 0$ for $u > 0$ and both f and h are continuous in its respective domain. Then it follows from the series of transformations used above, that the existence of a positive radial solution of the boundary value problem (1.1)-(1.2) is equivalent to the existence of a positive solution of the boundary value problem (2.3)-(2.5), subject to the condition that (2.6) is satisfied.

The boundary value problem (2.3) and (2.5) is equivalent to the integral equation

$$y(t) = \lambda \int_0^1 G(t,s)h(s)f(y(s))\,ds,$$

where $G(t,s)$ is the Green's function for the problem $-y'' = 0$ subject to the boundary condition (2.5), which is given by

$$G(t,s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1 \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.7}$$

and ρ is given in (2.6).

The following lemmas give some estimates on the Green's kernel $G(t,s)$.

Lemma 2.1. $G(t,s) \leq G(s,s)$, $s,t \in [0,1]$.

Lemma 2.2. $G(t,s) > \sigma_1 G(s,s)$ for $t \in [1/4, 3/4]$ and $s \in [0,1]$, where

$$\sigma_1 = \min\left\{ \frac{\gamma + 4\delta}{4(\gamma + \delta)}, \frac{\alpha + 4\beta}{4(\alpha + \beta)} \right\}.$$

Proof. Let $0 \leq s \leq t \leq 1$. Then

$$\frac{G(t,s)}{G(s,s)} = \frac{\gamma + \delta - \gamma t}{\gamma + \delta - \gamma s} > \frac{\gamma + \delta - \frac{3\gamma}{4}}{\gamma + \delta} = \frac{\gamma + 4\delta}{4(\gamma + \delta)}.$$

Again, if $0 \leq t \leq s \leq 1$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{\beta + \alpha t}{\beta + \alpha s} > \frac{\beta + \frac{\alpha}{4}}{\beta + \alpha} = \frac{\alpha + 4\beta}{4(\alpha + \beta)}.$$

Now, the proof of the lemma follows from the above two inequalities.

Let $X = C[0,1]$ be a space of continuous functions under the sup. norm

$$\|y\| = \sup_{t \in [0,1]} |y(t)|,$$

and define a cone K on X by

$$K = \left\{ y \in X; \ \min_J y(t) \geq \sigma_1 \|y\| \right\},$$

where $J = [1/4, 3/4]$. Further, we define an operator A on X by

$$(Ay)(t) = \lambda \int_0^1 G(t,s)h(s)f(y(s)) \, ds.$$

Let $y \in K$. Then $y \in X$ and

$$
\begin{aligned}
\min_J Ay(t) &= \min_{t \in J} \lambda \int_0^1 G(t,s)h(s)f(y(s)) \, ds \\
&\geq \lambda \sigma_1 \int_0^1 G(s,s)h(s)f(y(s)) \, ds \\
&\geq \lambda \sigma_1 \max_{t \in [0,1]} \int_0^1 G(s,s)h(s)f(y(s)) \, ds \\
&= \sigma_1 \|Ay\|,
\end{aligned}
$$

that is, $Ay \in K$. Hence $A : K \to K$ and it is easy to check that this mapping is completely continuous.

The following concept from Leggett-Williams multiple fixed point theorem [18] is needed for our use. Let X be a Banach space and K be a cone in X. For $a > 0$, define $K_a = \{x \in K; \|x\| < a\}$. A mapping ψ is said to be a *concave nonnegative continuous functional* on K if $\psi : K \to [0, \infty)$ is continuous and

$$\psi(\mu x + (1 - \mu)y) \geq \mu \psi(x) + (1 - \mu)\psi(y), \quad x, y \in K, \mu \in [0, 1].$$

Let $b, c > 0$ be constants with K and X as defined above. Define

$$K(\psi, b, c) = \{x \in K; \psi(x) \geq b, \|x\| \leq c\}.$$

Theorem 2.3. *(Theorem 3.5, [18]): Let $c_3 > 0$ be a constant. Assume that $A : \overline{K}_{c_3} \to K$ is completely continuous, there exists a concave nonnegative functional ψ with $\psi(x) \leq \|x\|, x \in K$ and numbers c_1 and c_2 with $0 < c_1 < c_2 < c_3$ satisfying the following conditions:*

(i) $\{x \in K(\psi, c_2, c_3); \psi(x) > c_2\} \neq \phi$ and $\psi(Ax) > c_2$ if $x \in K(\psi, c_2, c_3)$;
(ii) $\|Ax\| < c_1$ if $x \in \overline{K}_{c_1}$;
and
(iii) $\psi(Ax) > \frac{c_2}{c_3}\|Ax\|$ for each $x \in \overline{K}_{c_3}$ with $\|Ax\| > c_3$.
Then A has at least two fixed points x_1, x_2 in $\overline{K}_{c_3}$. Furthermore, $\|x_1\| \leq c_1 <$
$\|x_2\| < c_3$.

Theorem 2.4.   *(Theorem 3.3, [18]) Let $X = (X, \|.\|)$ be a Banach space
and $K \subset X$ a cone, and $c_4 > 0$ a constant. Suppose there exists a concave
nonnegative continuous function ψ on K with $\psi(x) \leq \|x\|$ for $x \in \overline{K}_{c_4}$ and let
$A : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a continuous compact map. Assume that there are numbers
c_1, c_2 and c_3 with $0 < c_1 < c_2 < c_3 \leq c_4$ such that
(i) $\{x \in K(\psi, c_2, c_3); \psi(x) > c_2\} \neq \phi$ and $\psi(Ax) > c_2$ for all $x \in K(\psi, c_2, c_3)$;
(ii) $\|Ax\| < c_1$ for all $x \in \overline{K}_{c_1}$;
(iii) $\psi(Ax) > c_2$ for all $x \in K(\psi, c_2, c_4)$ with $\|Ax\| > c_3$.
Then A has at least three fixed points x_1, x_2 and x_3 in $\overline{K}_{c_4}$. Furthermore, we
have $x_1 \in \overline{K}_{c_1}$, $x_2 \in \{x \in K(\psi, c_2, c_4); \psi(x) > c_2\}$, $x_3 \in \overline{K}_{c_4} \backslash \{K(\psi, c_2, c_4) \cup
\overline{K}_{c_1}\}$.*

It is well known that the Leggett-Williams multiple fixed point Theorems
2.3 and 2.4 has been used by many authors for the existence of multiple so-
lutions of boundary value problems. Once the problem is transformed to an
equivalent integral operator, then it is easy to study the existence of fixed point
of the operator by using different fixed point theorems which is equivalent to
the existence of periodic solution of the problem. The use of the Theorems
2.3 and 2.4 can be found in [27, 28, 29, 30, 31, 32, 33].

3. MAIN RESULTS

In this section, we shall prove the main results of this paper by using the
Theorems 2.3 and 2.4. Denote

$$f_0 = \lim_{y \to 0+} \frac{f(y)}{y} \text{ and } f_\infty = \lim_{y \to \infty} \frac{f(y)}{y}.$$

Theorem 3.1.   *Let $f_0 = 0$ and $f_\infty = 0$. Let $g(r) \not\equiv 0$ in some neighborhood
of $1/2$. Then there exists a $\lambda_1 > 0$ such that the boundary value problem (2.3)*

and (2.5) has three solutions for $\lambda > \lambda_1$, at least two of which are positive.

Proof. For $y \in K$, we have

$$
\begin{aligned}
(Ay)(1/2) =& \lambda \int_0^1 G(1/2, s)h(s)f(y(s))\, ds \\
\geq & \lambda \int_{1/4}^{3/4} G(1/2, s)h(s)f(y(s))\, ds.
\end{aligned}
\tag{3.1}
$$

Define, for any $p > 0$,

$$
m(p) = \min \left\{ \int_{1/4}^{3/4} G(1/2, s)h(s)f(y(s))\, ds,\ y \in K,\ \|y\| < p \right\}.
\tag{3.2}
$$

Clearly, $m(p) > 0$. Let $0 < p_1 < c_2 < p_2$ be arbitrary constants, and define

$$
\lambda_1 = \max \left\{ \frac{p_1}{m(p_1)},\ \frac{c_2}{\overline{m}(c_2)},\ \frac{p_2}{m(p_2)} \right\},
\tag{3.3}
$$

where

$$
\overline{m}(c_2) = \min \left\{ \sigma_1 \int_0^1 G(s, s)h(s)f(y(s))\, ds;\ c_2 \leq y(s) \leq \frac{c_2}{\sigma_1}, s \in [0, 1] \right\}.
$$

Then, for $\lambda \geq \lambda_1$, we have

$$
(Ay)(1/2) \geq \lambda \int_{1/4}^{3/4} G(1/2, s)h(s)f(y(s))\, ds \geq \lambda_1\, m(p).
\tag{3.4}
$$

Hence, in particular, for $p = p_1$, we have

$$
(Ay)(1/2) \geq \lambda_1\, m(p_1) = \lambda_1 \frac{m(p_1)}{p_1}.p_1 \geq p_1,
\tag{3.5}
$$

and for $p = p_2$, we have

$$
(Ay)(1/2) \geq \lambda_1\, m(p_2) = \lambda_1 \frac{m(p_2)}{p_2}.p_2 \geq p_2.
\tag{3.6}
$$

Thus, from (3.1)-(3.6), it follows that $\|Ay\| \geq \|y\|$ for $\|y\| \leq p_1$ and $\|y\| \leq p_2$.

Since $f_0 = 0$ for any $\lambda \geq \lambda_1$, then we can choose constants $q_1 > 0$ and $\eta > 0$ such that $2q_1 < p_1$ and such that $f(y) \leq \eta y$ for $0 < y \leq q_1$, where η satisfies the property

$$
\eta\lambda \int_0^1 G(s, s)h(s)\, ds < 1.
\tag{3.7}
$$

Set $q_1 = c_1$. Then

$$c_1 = q_1 < 2q_1 < p_1 < c_2 < p_2.$$

Now, for $x \in \overline{K}_{c_1}$, we have

$$
\begin{aligned}
\|(Ay)(t)\| &= \lambda \int_0^1 G(t,s)h(s)f(y(s))\, ds \\
&\leq \lambda \int_0^1 G(s,s)h(s)f(y(s))\, ds \\
&\leq \lambda \eta \int_0^1 G(s,s)h(s)y(s)\, ds \\
&\leq \lambda \eta \int_0^1 G(s,s)h(s)\|y\|\, ds \\
&\leq c_1 \lambda \eta \int_0^1 G(s,s)h(s)\, ds \\
&< c_1.
\end{aligned}
$$

Let $c_3 = \frac{c_2}{\sigma_1}$ and since $\sigma_1 < 1$, we have $c_1 < c_2 < c_3$. Define a nonnegative continuous concave functional ψ on K by

$$\psi(y) = \min_{t \in [0,1]} y(t).$$

Then $\psi(y) \leq \|y\|$. Clearly,

$$\{y : y \in K(\psi, c_2, c_3);\ \psi(y) > c_2\} \neq \phi$$

Further, for $y \in K(\psi, c_2, c_3)$, we have

$$
\begin{aligned}
\psi(Ay) &= \lambda \min_{t \in [0,1]} \int_0^1 G(t,s)h(s)f(y(s))\, ds \\
&> \lambda \sigma_1 \int_0^1 G(s,s)h(s)f(y(s))\, ds \\
&> \lambda_1 \overline{m}(c_2) \\
&\geq \frac{c_2}{\overline{m}(c_2)} \overline{m}(c_2) \\
&= c_2.
\end{aligned}
$$

Since $f_\infty = 0$, we may choose $q_2 > 2p_2$ such that $f(y) \leq \eta y$ for $y \geq q_2$, where η satisfies (3.7). Set $c_4 > \max\left\{\frac{c_2}{\sigma_1}, q_2\right\}$. Then clearly,

$$0 < c_1 = q_1 < 2q_1 < p_1 < c_2 < p_2 < 2p_2 < q_2 < c_4.$$

If f is bounded, say $f(y) \leq N$ for all $y \in (0, \infty)$, then we may suppose that

$$N\lambda \int_0^1 G(s, s)h(s)\, ds < c_4.$$

In this case, we have

$$(Ay)(t) = \lambda \int_0^1 G(t, s)h(s)f(y(s))\, ds$$
$$\leq N\lambda \int_0^1 G(s, s)h(s)\, ds$$
$$< c_4.$$

If f is unbounded, then $c_4 > q_2 > 2p_2$ is chosen so that $f(y) \leq f(c_4)$ for $0 < y \leq c_4$, and $f(c_4) < \eta c_4$ where η satisfies (3.7). Hence, for $y \in \overline{K}_{c_4}$, we have

$$(Ay)(t) = \lambda \int_0^1 G(t, s)h(s)f(y(s))\, ds$$
$$\leq \lambda f(c_4) \int_0^1 G(t, s)h(s)\, ds$$
$$\leq \lambda \eta c_4 \int_0^1 G(t, s)h(s)\, ds$$
$$< c_4,$$

that is, $Ay \in \overline{K}_{c_4}$, whenever $y \in \overline{K}_{c_4}$.

Next, suppose that $y \in K(\psi, c_2, c_4)$ with $\|Ay\| > c_3$. Then

$$c_3 < \|Ay\| \leq \lambda \int_0^1 G(s, s)h(s)f(y(s))\, ds$$

implies that

$$\psi(Ay) = \min_{t \in [0,1]} \lambda \int_0^1 G(t, s)h(s)f(y(s))\, ds$$
$$> \sigma_1 \lambda \int_0^1 G(s, s)h(s)f(y(s))\, ds$$
$$\geq \sigma_1 c_3 = c_2.$$

Hence, by Theorem 2.4, the boundary value problem (2.3) and (2.5) has at least three solutions. Since $f_0 = 0$ implies $f(0) = 0$, we have at least two positive solutions.

Theorem 3.2.   *Let $\lambda \equiv 1$. Suppose that $f_0 = 0$ and $f_\infty = \infty$. Then the boundary value problem (2.3) and (2.5) has at least two positive solutions.*

Proof. $f_\infty = \infty$ implies that there is a $c_2 > 0$ such that $f(y) > My$ for $c_2 \leq y \leq \frac{c_2}{\sigma_1}$, where M satisfies the property

$$\sigma_1 M \int_{1/4}^{3/4} G(1/2, s) h(s)\, ds > 1.$$

Set $c_3 = \frac{c_2}{\sigma_1}$. Consider a nonnegative function $\psi(y)$ by $\psi(y) = \min\limits_{t \in [0,1]} y(t)$. Clearly, $c_2 < c_3$, and the set $\{y : y \in K(\psi, c_2, c_3); \psi(y) > c_2\}$ is nonempty. For $y \in K(\psi, c_2, c_3)$, we have

$$\psi(Ay) = \min_{t \in [0,1]} \int_0^1 G(t, s) h(s) f(y(s))\, ds$$

$$\geq M c_2 \min_{t \in [0,1]} \int_0^1 G(t, s) h(s)\, ds$$

$$\geq M c_2 \sigma_1 \min_{t \in [0,1]} \int_0^1 G(s, s) h(s)\, ds$$

$$> c_2.$$

Now, for any $y \in K$, we have

$$(Ay)(t) = \max_{t \in [0,1]} \int_0^1 G(t, s) h(s) f(y(s))\, ds$$

$$\leq \int_0^1 G(s, s) h(s) f(y(s))\, ds.$$

From the condition $f_0 = 0$, we can find a $\epsilon > 0$ and a constant $\xi \in (0, c_2/2)$ such that $f(y) < \epsilon y$ for $0 \leq y \leq \xi$, where ϵ is chosen so that

$$\epsilon \int_0^1 G(s, s) h(s)\, ds < 1$$

holds. Set $\xi = c_1$; then, $0 < c_1 < c_2$. For $y \in \overline{K}_{c_1}$, we have

$$\|Ay\| \leq \epsilon \int_0^1 G(s, s) h(s) y(s)\, ds$$

$$\leq \epsilon c_1 \int_0^1 G(s, s) h(s)\, ds$$

$$< c_1.$$

Finally, suppose that $y \in \overline{K}_{c_3}$ with $\|Ay\| > c_3$. Then

$$\psi(Ay) = \min_{t \in [0,1]} \int_0^1 G(t,s)h(s)f(y(s))\,ds$$

$$\geq \sigma_1 \int_0^1 G(s,s)h(s)f(y(s))\,ds.$$

Thus we have,

$$c_3 < \|Ay\| = \max_{t \in [0,1]} \int_0^1 G(t,s)h(s)f(y(s))\,ds$$

$$\leq \int_0^1 G(s,s)h(s)f(y(s))\,ds$$

$$\leq \frac{1}{\sigma_1}\psi(Ay)$$

showing that

$$\psi(Ay) \geq \sigma_1\|Ay\| = \frac{c_2}{c_3}\|Ay\|.$$

Hence, by Theorem 2.3, the boundary value problem (2.3) and (2.5) has at least two positive solutions.

Remark 3.3. Note that the Theorem 3.2 can be extended to any $\lambda \in (\lambda_*, \lambda^*)$ such that $0 < \lambda_* < 1 < \lambda^* < \infty$ for appropriate λ_* and λ^* which satisfy

$$\sigma_1 M\lambda_* \int_{1/4}^{3/4} G(1/2,s)h(s)\,ds > 1 \text{ and } \epsilon\lambda^* \int_0^1 G(s,s)h(s)\,ds < 1.$$

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