OPTIMAL SEARCH FOR NONLINEAR DISCRETE SYSTEMS CYCLES

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ABSTRACT: We consider a classical problem of stabilization of a priori unknown unstable periodic orbits in nonlinear autonomous discrete dynamical systems. A new approach was suggested in [11], where a nonlinear delay feedback control (DFC) scheme with apparently optimal gain was introduced. The optimality criteria in [11] were stated in terms of the size of the convergence region for the system multipliers. In numerical simulations it turns out that the optimal coefficients (in the above sense) produce slowly convergent recurrences. In this paper we suggest a generalization of the formulas from [11] to improve the rate of convergence while preserving stability. The subtlety of the problem is illustrated in numerous numerical simulations examples.

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1. INTRODUCTION

Chaotic regimes are typical for many nonlinear dynamical systems that simulate processes in various areas of research, e.g. in physics, economics, ecology, electronics, etc. (c.f. [3]). A characterization of such regimes is the existence of infinitely many unstable periodic orbits. Stabilization of such orbits is one of the important tasks in nonlinear control theory [2].

If a periodic orbit is known then the most popular method of stabilization is the OGY method [14]. Various delay feedback control schemes have been developed (c.f. [15, 18, 13, 7, 8, 11]) for stabilizing a priori unknown periodic orbits. Certain limitations of DFC schemes were mentioned in [17, 5].

One of the methods, at least in theory, that allows stabilizing an unstable orbit is a generalization of the method of nonlinear delay feedback control (NDFC) by Viera-Licheberg [18]. This method's main deficiencies are related to having a narrow basin of attraction for stabilized periodic solutions as well as to slow rates of convergence of perturbed solutions to the periodic one.

Let us consider the vectorial nonlinear discrete dynamical system

$$x_{n+1} = F(x_n), \quad x_n \in \mathbb{R}^m, \quad n = 1, 2, \dots \quad (1.1)$$

It is assumed that the system (1.1) has an invariant convex set A , i.e. if $\xi \in A$ then $F(\xi) \in A$. It is also assumed that this system has an unstable T -cycle $(\eta_1, \dots, \eta_T)$, where all vectors $\eta_1, \dots, \eta_T$ are pairwise distinct and belong to the invariant set A , i.e. $\eta_{j+1} = F(\eta_j)$, $j = 1, \dots, T-1$, $\eta_1 = F(\eta_T)$.

In previous work [7, 8, 10, 11] we suggested a possible modification of the NDFC scheme that allows one to increase the size of the stability region (the region of the associated multiplier's location that allows stabilization) and we defined the cycle multipliers $\{\mu_1, \dots, \mu_m\}$, computed as eigenvalues of a product of Jacobi matrices $\prod_{j=1}^T F'(\eta_j)$. The T -cycle is asymptotically stable if and only if all multipliers are in the open unit disc of the complex plane. In the current work the control coefficients are chosen such that the cycle multipliers are contained in the central disc of radius $\rho < 1$. Thus, the rate of convergence will be of order at least ρ^n .

It is clear that we have to pay a price for that: the limitation on the range of stability. Namely, in our construction we use a nonlinear DFC/mixing with the decreasing of the size of the region of stability. Moreover, if $\rho = 1$ then

for any region size there exists a certain prehistory length that allows us to stabilize the cycle [6, 11]. That is not the case anymore when $\rho < 1$. For a given size of the diameter of the multiplier's region there exists a limited value ρ_0 that does not guarantee the convergence at a rate of order smaller than ρ_0^n regardless of the length of the prehistory N .

In other words, there is a trade-off between stability and the speed of convergence. For a given dynamical system with ρ decreasing, the length of prehistory used in the control increases to infinity when $\rho \rightarrow \rho_0$. In this article, our main goal is to determine the minimal prehistory length N and the optimal control coefficients for a given cycle length T and multipliers localization $\rho < 1$. We provide a solution for this problem and the examples provided in this paper clearly show the effectiveness of the modified control.

The rest of the article is organized as follows. We give preliminary results the current work relies on in Section 2. The main result of the paper is introduced in Section 3. In Section 4 we show the differences between the *standard* method in [11] and the *modified* method introduced here. A number of examples illustrating the effectiveness of the proposed solution are presented in Section 5. We discuss necessary conditions for applying our method in Section 6 and conclude in Section 7.

2. SETTINGS AND PRELIMINARY RESULTS

We are interested in efficiently finding cycles of arbitrary lengths for a vectorial nonlinear discrete dynamical system (1.1). Let us remark that even in the simplest case of a scalar polynomial function F , detecting a cycle of the length T by performing T self-iterations of function F does not work when T is large. Indeed, the T -iterated function will be a polynomial of order the initial degree risen to the power T . The second problem consists of extraneous cycles obtained that way. We have to increase the length of the prehistory. It is impossible to keep the length of prehistory one as in the case of Viera-Lightenebrg or Pyragas controls.

To avoid the aforementioned obstacles, we suggest to construct a new system *which has same cycles as the original one, but in such a way that they are*

stable cycles. Namely, we modify the system (1.1) either to the form

$$x_{n+1} = \sum_{i=1}^N a_i F(x_{n-iT+T}), \quad (2.1)$$

or

$$x_{n+1} = F\left(\sum_{i=1}^N a_i x_{n-iT+T}\right), \quad (2.2)$$

where $a_1 + \dots + a_N = 1$.

Let us note that the system (2.1) is obtained from the system

$$x_{n+1} = F(x_n) + u_n,$$

where u_n is control based on the nonlinear feedback with delay, i.e.

$$u_n = - \sum_{j=1}^{N-1} \epsilon_j (F(x_{n-jT+T}) - F(x_{n-jT}))$$

while the control in the system (2.2) is organized on the mixing principle. c.f. [9].

Our goal is to make the T -cycles of the systems (2.1) and (2.2) asymptotically locally stable by choosing the control coefficients $a_1, \dots, a_N$ in a specific way, preferably optimal, where the meaning of optimality will be later specified. It is important that the convex set A is still invariant for the systems (2.1) and (2.2) as well. This follows from the definition of the convex combination of the vectors. On a top of that the systems (2.1) and (2.2) have same T -cycles that are in the system (1.1).

The characteristic equation for the cycle of systems (2.1) and (2.2) is found in [6]

$$\prod_{j=1}^m \left(\lambda^{(N-1)T+1} - \mu_j (a_1 \lambda^{N-1} + \dots + a_N) \right) = 0. \quad (2.3)$$

The stability condition is that the roots of the characteristic equation (2.3) lie in the unit disc. Thus it is required to find a number N and coefficients $(a_1, \dots, a_N)$ such that all polynomials of the one-parametric family

$$\left\{ \lambda^{(N-1)T+1} - \mu (a_1 \lambda^{N-1} + \dots + a_N) : \mu \in M \right\} \quad (2.4)$$

are Schur stable. Here M represents a set of multipliers locations.

Consequently we formulate the following problem: *for a given cycle length T and a given set of multipliers localization find the control coefficients $a_i, i = 1, \dots, N$ such that the cycle of length T will be locally asymptotically stable and while minimizing the length of the prehistory used.*

Clearly, the solution of problem depends on the localization of the set of multipliers $\{\mu_1, \dots, \mu_m\}$.

We will consider two possibilities: either all multipliers are real negative

$$\{\mu_1, \dots, \mu_m\} \subset \{\mu \in \mathbb{R} : \mu \in (-\mu^*, 0)\}, \mu^* > 1,$$

or are complex and located in the left half-plane

$$\{\mu_1, \dots, \mu_m\} \subset \{\mu \in \mathbb{C} : |\mu + R| < R\}, R > 1/2.$$

Finding optimal values for the coefficients $a_1, \dots, a_N$ turns to be a difficult problem. It was completely solved for $T = 1, 2$ for real multipliers [7, 8] and for $T = 1$ in the case of complex multipliers with negative real part [6]. There is strong numerical evidence that the coefficients suggested in [11] are optimal for all T values and the multipliers above. These optimal coefficients are computed in a number of steps, as follows [11]:

a) compute the nodes:

$$\psi_j = \frac{\pi(\sigma + T(2j - 1))}{\sigma + (N - 1)T},$$

$$j = 1, 2, \dots, \frac{N - 2}{2}, \text{ if } N \text{ is even; } \frac{N - 1}{2}, \text{ if } N \text{ is odd;}$$

In the case $\{\mu_1, \dots, \mu_m\} \subset \{\mu \in \mathbb{R} : \mu \in (-\mu^*, 0)\}$ we pick $\sigma = 2$, while in the case $\{\mu_1, \dots, \mu_m\} \subset \{\mu \in \mathbb{C} : |\mu + R| < R\}$ we pick $\sigma = 1$;

b) construct the auxiliary polynomials:

$$\eta_N(z) = z(z + 1) \prod_{j=1}^{\frac{N-2}{2}} (z - e^{i\psi_j})(z - e^{-i\psi_j}), \quad N \text{ even,}$$

$$\eta_N(z) = z \prod_{j=1}^{\frac{N-1}{2}} (z - e^{i\psi_j})(z - e^{-i\psi_j}), \quad N \text{ odd;}$$

c) compute the coefficients of auxiliary polynomials:

$$\eta_N(z) = \sum_{j=1}^N c_j z^j;$$

d) construct the *standard* coefficients:

$$a_j = \frac{\left(1 - \frac{1+(j-1)T}{2+(N-1)T}\right) c_j}{\sum_{k=1}^N \left(1 - \frac{1+(k-1)T}{2+(N-1)T}\right) c_k}, \quad j = 1, \dots, N; \quad (2.5)$$

e) in the case $\{\mu_1, \dots, \mu_m\} \subset \{\mu \in \mathbb{R} : \mu \in (-\mu^*, 0)\}$, compute values:

$$I_N^{(T)} = - \left[\frac{T}{2 + (N-1)T} \prod_{k=1}^{\frac{N-2}{2}} \cot^2 \frac{\pi(2 + T(2k-1))}{2(2 + (N-1)T)} \right]^T, \quad N \text{ even}$$

$$I_N^{(T)} = - \left[\prod_{k=1}^{\frac{N-1}{2}} \cot^2 \frac{\pi(2 + T(2k-1))}{2(2 + (N-1)T)} \right]^T, \quad N \text{ odd};$$

The optimal value of N is computed as minimal positive integer that satisfies the inequality

$$\mu^* \cdot |I_N^{(T)}| < 1; \quad (2.6)$$

f) in the case $\{\mu_1, \dots, \mu_m\} \subset \{\mu \in \mathbb{C} : |\mu + R| < R\}$, compute values:

$$I_N^{(T)} = - \left[\frac{T}{1 + (N-1)T} \prod_{k=1}^{\frac{N-2}{2}} \cot^2 \frac{\pi(1 + T(2k-1))}{2(1 + (N-1)T)} \right]^T, \quad N \text{ even},$$

$$I_N^{(T)} = - \left[\prod_{k=1}^{\frac{N-1}{2}} \cot^2 \frac{\pi(1 + T(2k-1))}{2(1 + (N-1)T)} \right]^T, \quad N \text{ odd};$$

The optimal value of N is computed as minimal positive integer that satisfies the inequality

$$R \cdot 2 |I_N^{(T)}| < 1. \quad (2.7)$$

It was found in [7, 8] that in case of real multipliers we have

$$|I_N^{(1)}| = \tan^2 \frac{\pi}{2(N+1)}, \quad |I_N^{(2)}| = \frac{1}{N^2}. \quad (2.8)$$

In other words, the minimal value N should satisfy the inequality

$$\mu^* < \cot^2 \frac{\pi}{2(N+1)} \quad \text{or} \quad \mu^* < N^2, \quad (2.9)$$

In the case of complex multipliers we have

$$|I_N^{(1)}| = \frac{1}{N} \quad (2.10)$$

and the minimal value N should satisfy the inequality

$$2R < N. \quad (2.11)$$

For example, in the case e) above the *standard* coefficients $(a_1, \dots, a_N)$ can be computed as following

$$a_j = 2 \tan \frac{\pi}{2(N+1)} \left(1 - \frac{j}{N+1} \right) \sin \frac{\pi j}{N+1}, \quad j = 1, \dots, N \quad (2.12)$$

for the case $T = 1$ and as following

$$a_j = \frac{2}{N} \left(1 - \frac{2j-1}{2N} \right) \quad j = 1, \dots, N \quad (2.13)$$

for the case $T = 2$.

In the case f) above the *standard* coefficients are computed as following

$$a_j = \frac{2}{N} \left(1 - \frac{j}{N+1} \right) \quad j = 1, \dots, N. \quad (2.14)$$

We will call these coefficients optimal. Let us emphasize one more time that the coefficients defined by the formula (2.5), in both real and complex cases, are called *standard*.

They are optimal in the sense of widest range for multipliers, but they might be far from optimal in the sense of rate of convergence. Indeed, in the case of real multipliers and $T = 1$ if the value of μ^* is close to $\cot^2 \frac{\pi}{2(N+1)}$ then the roots of the polynomial $f(\lambda) = \lambda^N + \mu^*(a_1 \lambda^{N-1} + \dots + a_N)$ might be close to the boundary of the unit disc, therefore the convergence of the iterative procedures (2.1) and (2.2) will be quite slow.

Furthermore, if the roots are in the disc of radius less than one the rate of convergence might be slow because the *standard* coefficients are designed to serve the worse case scenario. This situation is very well illustrated by the green lines in Figures 1 and 2 in Section 4.

A natural question emerges - how to increase the rate of convergence? Let us formulate the following problem: find a positive integer N and the coefficients $(a_1, \dots, a_N)$ such that all polynomials of the one-parametric family (2.4) have all roots inside the disc of radius $\rho < 1$.

3. FAST STABILIZATION RESULT

Let us consider the polynomials of the family (2.4). When using the *standard* coefficients $a_1, \dots, a_N$ then all roots of these polynomials are inside the unit disc $\mathbb{D}$. In that case the value N determining the prehistory length is minimal. Let us require the roots of the considered polynomials be inside the disc of radius $\rho < 1$. It is clear that the value N will be larger than the *standard*. Let us find the corresponding coefficients $b_1, \dots, b_N$, which we will call *modified*.

First, let us consider the case of real multipliers and $T = 1$. To solve this problem let us make the substitution $\lambda = z\rho$. It is clear that $|\lambda| < \rho$ if and only if $|z| < 1$. Denote $p(\lambda) = a_1\lambda^{N-1} + \dots + a_N$. The equation $\lambda^N - \mu p(\lambda) = 0$ is equivalent to the equation

$$z^N - \mu \frac{p(\rho)}{\rho^N} \frac{p(z\rho)}{p(\rho)} = 0,$$

or

$$z^N - \hat{\mu} \hat{p}(z) = 0, \quad (3.1)$$

where $\hat{\mu} = \mu \frac{p(\rho)}{\rho^N}$ and $\hat{p}(z) = \frac{p(z\rho)}{p(\rho)}$.

The roots of the polynomial (3.1) have to be in the unit disc. By (2.9) this happens if $\hat{\mu}^* < \cot^2 \frac{\pi}{2(N+1)}$ and $\hat{p}(z) = a_1^{(N)} z^{N-1} + \dots + a_N^{(N)}$, where $a_1^{(N)}, \dots, a_N^{(N)}$ are the *standard* coefficients. Since $\hat{p}(z) = \frac{p(z\rho)}{p(\rho)}$ then

$$\frac{p\left(\frac{1}{\rho}\rho\right)}{p(\rho)} = \hat{p}\left(\frac{1}{\rho}\right).$$

Since $p(1) = 1$, then $\frac{1}{p(\rho)} = \hat{p}\left(\frac{1}{\rho}\right)$.

Further

$$\mu^* = \hat{\mu}^* \frac{\rho^N}{p(\rho)} = \hat{\mu}^* \rho^N \hat{p}\left(\frac{1}{\rho}\right) < \rho \hat{q}(\rho) \cot^2 \frac{\pi}{2(N+1)},$$

where $z\hat{q}(z) = z^N \hat{p}(\frac{1}{z})$, i.e. $\hat{q}(z) = a_1^{(N)} + \dots + a_N^{(N)} z^{N-1}$.

Furthermore, $\frac{p(\lambda)}{p(\rho)} = \hat{p}\left(\frac{\lambda}{\rho}\right)$, from there

$$p(\lambda) = p(\rho) \hat{p}\left(\frac{\lambda}{\rho}\right) = \frac{1}{\hat{p}\left(\frac{1}{\rho}\right)} \hat{p}\left(\frac{\lambda}{\rho}\right).$$

Therefore, the solution is the following:

i. The minimal value of N should satisfy the inequality

$$\mu^* < (a_1^{(N)} \rho + \dots + a_N^{(N)} \rho^N) \cot^2 \frac{\pi}{2(N+1)},$$

where the coefficients $(a_1^{(N)}, \dots, a_N^{(N)})$ are *standard*, i.e. determined by (2.12);

ii. The optimal polynomial is

$$p_O(\lambda) = \frac{1}{\hat{p}\left(\frac{1}{\rho}\right)} \hat{p}\left(\frac{\lambda}{\rho}\right) = \frac{\rho^N}{\rho \hat{q}(\rho)} \hat{p}\left(\frac{\lambda}{\rho}\right) = b_1^{(N)} \lambda^{N-1} + \dots + b_N^{(N)}.$$

where the optimal modified coefficients are

$$b_j^{(N)} = \frac{a_j^{(N)} \rho^j}{\sum_{k=1}^N a_k^{(N)} \rho^k}. \quad (3.2)$$

A similar approach allows us to consider the T cycles in the real and in complex cases, where μ might belong to $(-\mu^*, 0)$ and also $|\mu + R| < R$. Namely, the equation $\lambda^{(N-1)T+1} - \mu[p(\lambda)]^T = 0$ is equivalent to the following

$$z^{(N-1)T+1} - \mu \frac{[p(\rho)]^T}{\rho^{(N-1)T+1}} \left[\frac{p(z\rho)}{p(\rho)} \right]^T = 0$$

or

$$z^{(N-1)T+1} - \hat{\mu}[\hat{p}(z)]^T = 0, \quad (3.3)$$

where $\lambda = z\rho$, $\hat{\mu} = \mu \frac{[p(\rho)]^T}{\rho^{(N-1)T+1}}$ and $\hat{p}(z) = \frac{p(z\rho)}{p(\rho)}$.

The roots of equation (3.3) have to be in the central unit disc which means that $\hat{\mu}^* < \frac{1}{|I_T^{(N)}|}$ in the real case, and that $\hat{\mu}^* \in \left\{ z \in \mathbb{C} : \left| z + \frac{1}{2|I_T^{(N)}|} \right| < \frac{1}{2|I_T^{(N)}|} \right\}$

in the complex case, with $p(z) = a_1^{(N)}z^{N-1} + \dots + a_N^{(N)}$, where $a_j^{(N)}$ are the *standard* coefficients for the general case. Since $\hat{p}(z) = \frac{p(z\rho)}{p(\rho)}$ then $\frac{1}{p(\rho)} = \hat{p}\left(\frac{1}{\rho}\right)$.

Hence

$$\mu^* < \frac{\rho [\hat{q}(\rho)]^T}{|I_N^{(T)}|},$$

or

$$\mu \in \left\{ z \in \mathbb{C} : \left| z + \frac{\rho [\hat{q}(\rho)]^T}{2|I_N^{(T)}|} \right| < \frac{\rho [\hat{q}(\rho)]^T}{2|I_N^{(T)}|} \right\},$$

where $\hat{q}(z) = a_1^{(N)} + \dots + a_N^{(N)}z^{N-1}$ and $z\hat{q}(z)$ is inverse to $\hat{p}(z)$.

Furthermore,

$$p(\lambda) = \frac{1}{\hat{p}\left(\frac{1}{\rho}\right)} \hat{p}\left(\frac{\lambda}{\rho}\right) = \frac{\rho^N}{\rho \hat{q}(\rho)} \hat{p}\left(\frac{\lambda}{\rho}\right) = b_1^{(N)}\lambda^{N-1} + \dots + b_N^{(N)},$$

where the optimal modified coefficients are again given by (3.2).

4. ROOTS VISUALIZATION FOR THE STANDARD AND THE MODIFIED POLYNOMIALS

In this section we consider the problem of location of the zeros of the *standard* polynomial $\hat{f}(\lambda) = \lambda^{(N-1)T+1} - \hat{\mu}\hat{p}(\lambda)$ and the *modified* polynomials $f_O(\lambda) = \lambda^{(N-1)T+1} - \mu\hat{p}_O(\lambda)$ for various T and coefficients (the *standard* $(a_1^{(N)}, \dots, a_N^{(N)})$ and the *modified* ones $(b_1^{(N)}, \dots, b_N^{(N)})$, computed in the case of real as well as complex multipliers).

4.1. Homothecy property

The stability conditions of polynomials $\hat{f}(\lambda)$ and $f_O(\lambda)$ in the space of parameters $\hat{\mu}$ and μ , correspondingly, are given by the inclusions [11]

$$\hat{\mu} \in \left\{ \left(\bar{\mathbb{C}} \setminus z \left(\sum_{j=1}^N a_j^{(N)} z^{j-1} \right)^T \right)^*, z \in \mathbb{D} \right\} \quad (4.1)$$

$$\mu \in \left\{ \left(\bar{\mathbb{C}} \setminus z \left(\sum_{j=1}^N b_j^{(N)} z^{j-1} \right)^T \right)^*, z \in \mathbb{D} \right\} \quad (4.2)$$

where $\mathbb{D}$ denotes the unit disc and $*$ denotes inversion, i.e. $z^* = \frac{1}{\bar{z}}$, $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Let us show some examples, for different parameters T , N and ρ , of the inverse images of the central unit circle under the polynomial maps

$$z \left(\sum_{j=1}^N a_j^{(N)} z^{j-1} \right)^T \quad \text{and} \quad z \left(\sum_{j=1}^N b_j^{(N)} z^{j-1} \right)^T$$

from the definitions of the sets (4.1) and (4.2), respectively. These maps are represented in Figures 1 and 2 in green and red, respectively.

Figure 1

Figure 1 displays the inverse image of the unit disc under the *standard* map (green) and under the *modified* map (red) for $T = 1, N = 8, \rho = 0.9$. Figure 2 shows similar images for $T = 3, N = 8, \rho = 0.9$ (the maps are slightly expanded). The multipliers are real on the left and complex on the right.

Let us note that the following identity is valid:

$$\frac{1}{e^{it}} \left(\sum_{j=1}^N a_j^{(N)} e^{i(j-1)t} \right)^T = \frac{1}{\rho \left(\sum_{j=1}^N a_j^{(N)} \rho^{j-1} \right)^T} \frac{1}{\frac{1}{\rho} e^{it} \left(\sum_{j=1}^N b_j^{(N)} \left(\frac{1}{\rho} e^{it} \right)^{j-1} \right)^T}.$$

Figure 2

It means that the inverse image of the unit disc under the *standard* polynomial map is homothetic to the inverse image of the disc with radius $\frac{1}{\rho}$ under the *modified* polynomial map with the coefficient of homothecy

$$\frac{1}{\rho \left(\sum_{j=1}^N a_j^{(N)} \rho^{j-1} \right)^T} = \frac{1}{\rho (\hat{q}(\rho))^T}.$$

We illustrate this phenomenon in Figures 3 and 4.

Figure 3

Figures 3 and 4 illustrate the inverse image of the unit disc under the *standard* map (green) and the inverse image of the disc of radius $\frac{1}{\rho}$ under the

Figure 4

modified map (blue) for $T = 1$ (and $T = 3$, respectively), $N = 8, \rho = 0.9$ (the multipliers are real on the left and complex on the right). Indeed, the inverse images of the disc of radius $\frac{1}{\rho}$ under the modified map (the blue lines) are homothetic to the inverse images of the unit disc under the standard map (the green lines).

4.2. Roots location for various multiplier values

We show next how the location of the polynomial roots changes with the multipliers values. More precisely, we show that for small values of μ more roots are likely located away from the boundary (and very few roots are possibly located closer to the boundary) of the unit disc. As the value of μ approaches the critical value μ^* , more roots are likely to be positioned closer and closer to the boundary. First, let us note that in the particular case $m = 1$ (the scalar case) the degree of the equation (2.3) is $1 + (N - 1)T$, e.g. if $N = 8, T = 1$ the degree is 8, if $N = 8, T = 2$ it is 15, and when $N = 8, T = 3$ it is 22.

In the following examples we will illustrate, for both standard and modified polynomials, how the roots location depends on the multipliers values for $T = 1$ and $T = 3$ and for both real and complex multipliers. For each case we

will distinctively show the case when the multiplier is very close to the critical value (and hence the roots are likely to be located very close to the border).

4.2.1. The standard polynomials case

We will show roots location when $T = 1$ and $T = 3$, for both real and complex multipliers. We illustrate roots location for various values of μ and distinctively show, for each case, the location of roots when μ is close to the critical value μ^* .

- The case of real multipliers, $T = 1, N = 8$, and critical value $\mu^* = \cot^2 \frac{\pi}{18} = 32.16343764\dots$ (see (2.9)). The roots corresponding to choices $\mu = -2$ (red), $\mu = -10$ (green), $\mu = -20$ (blue) are well inside the unit circle in Figure 5 (left) while for $\mu = -31$ (brown) close to the critical value the corresponding roots are closer to the boundary, as it can be seen for clarity in Figure 5 (right).

Figure 5: $T=1$, $\mu = -2, -10, -20, -31$

$T=1$, $\mu = -31$

- The case of real multipliers, $T = 3, N = 8$, and critical value $|I_N^{(3)}|^{-1} = 89.72584369\dots$. As before, the roots corresponding to choices $\mu = -2$ (red), $\mu = -30$ (green), $\mu = -60$ (blue) are well inside the unit circle in Figure 6 (left), while for $\mu = -89$ (brown) close to the critical value the corresponding roots are closer to the boundary, as distinctively shown in Figure 6 (right).

Figure 6: $T=3$, $\mu = -2, -30, -60, -89$ $T=3$, $\mu = -89$

• The case of complex multipliers, $T = 1$, $N = 8$, and critical value constraint $2 \cdot R = 8$, that is $R = 4$ (see (2.11)). The roots corresponding to choices $\mu = -4 + 4 \exp(0.3\pi i)$ (red), $\mu = -4 + 4 \exp(0.5\pi i)$ (green), $\mu = -4 + 4 \exp(0.7\pi i)$ (blue), $\mu = -4 + 4 \exp(0.9\pi i)$ (brown) all correspond to the critical value of R and as a result there are roots of each of these colors close to the boundary in Figure 7 (left). This situation is very well visible in Figure 7 (right), where the case $\mu = -4 + 4 \exp(0.9\pi i)$ was singled out.

Figure 7: $T=1$, μ complex

$$\mu = -4 + 4 \exp(0.9\pi i)$$

Figure 8: $T=3$, μ complex $T=3$, $\mu = -5.9 + 5.8 \exp(0.7\pi i)$

• The case of complex multipliers, $T = 3$, $N = 8$, and critical value constraint $2 \cdot R = 11.79242673\dots$ For this case we have chosen the multiplier values smaller than the critical value (in absolute value), namely $\mu = -5.9 + 5.8 \exp(0.3\pi i)$, $\mu = -5.9 + 5.8 \exp(0.5\pi i)$, $\mu = -5.9 + 5.8 \exp(0.7\pi i)$, and $\mu = -5.9 + 5.8 \exp(0.9\pi i)$ (red, green, blue, and brown, respectively, in Figure 8, left). In this case one can also observe the existence of roots of each color close to the boundary in Figure 8, left. In Figure 8 (right) we singled out the case $\mu = -5.9 + 5.8 \exp(0.9\pi i)$ (brown) with some roots almost on the boundary.

4.2.2. The modified polynomials case

As for the standard polynomials, we will show examples of the roots location for $T = 1$ and $T = 3$ cases, both for real and complex multipliers. For each case we illustrate how for values of μ no larger than the critical value μ^* the roots location boundary shrinks to a disc of radius $\rho < 1$ (and hence the rate of convergence increases). For smaller μ values, the roots are completely contained in such a disc. Even for large μ values most of the roots are likely located inside the disc of radius ρ and all roots still located inside the unit disc.

Figure 9: $T=1$, $\mu = -2, -10, -20, -22.2$. $T=1$, $\rho = 0.9, \mu = -2$

• In the real multipliers case $T = 1$, $N = 8$, and $\rho = 0.9$ we have the critical value estimate

$$\mu^* < (a_1^{(8)} 0.9 + \dots + a_8^{(8)} 0.9^8) \cot^2 \frac{\pi}{18} = 22.17436354\dots$$

Figure 9 (left) shows roots location for $\mu = -2$ (red), $\mu = -10$ (green), $\mu = -20$ (blue), and $\mu = -22.2$ (brown) cases. All the roots, except brown, lie inside a disc of radius 0.9 (the yellow circle). For clarity, the case $\mu = -2$ (red), when all roots lie inside the disc of radius 0.9, is shown separately in Figure 9 right. If the multiplier value is increased, then even the modified polynomial can have roots outside a disc of radius 0.9. It happens to the brown roots for which $\mu = -22.2$ is slightly smaller than the critical value -22.17 (notice the brown root on the x -axis, left, in Figure 9 right). Note that in this case $\cot^2 \frac{\pi}{18} = 32.16343748\dots$ and therefore the brown root lies inside the unit circle.

• The real multipliers case $T = 3$, $N = 8$, $\rho = 0.9$. In this case the critical value for the multipliers is $-47.82046491\dots$. As one can observe from Figure 10 (left) all the roots, except brown ($\mu = -48$), are inside the yellow disc of radius 0.9. Especially it is very well visible in Figure 10 (right) where the case $\mu = -2$ is displayed separately. One of the brown roots lies outside the circle of radius 0.9 because the brown multiplier $\mu = -48.8$ is slightly smaller than the critical value -47.8 .

Figure 10: $T=3$, $\mu = -2, -20, -40, -48$. $T=3$, $\rho = 0.9$, $\mu = -2$

• The complex multipliers case $T = 1$, $N = 8$, $\rho = 0.9$. In this situation the critical value for the multipliers is $-5.748409779\dots$. As one can observe from Figure 11 (left) all the roots $\mu = -2.9 + 2.9 \exp(\frac{\pi i}{5})$, $\mu = -2.9 + 2.9 \exp(\frac{2\pi i}{5})$, $\mu = -2.9 + 2.9 \exp(\frac{4\pi i}{5})$, except brown ($\mu = -6.5$), are inside the yellow disc of radius 0.9. This is clearly visible in Figure 11 (right) where the case $\mu = -2.9 + 2.9 \exp(\frac{\pi i}{5})$ is displayed separately. One of the brown roots lies outside the yellow circle of radius 0.9 because the brown multiplier $\mu = -6.5$ is slightly smaller than the critical value -5.7 but still inside the unit disc.

• The complex multipliers case $T = 3$, $N = 8$, $\rho = 0.9$. In this situation the critical value for the multipliers is $7.362286563\dots$. As one can observe from Figure 12 (left) all the roots $\mu = -3.7 + 3.7 \exp(\frac{\pi i}{5})$, $\mu = -3.7 + 3.7 \exp(\frac{2\pi i}{5})$, $\mu = -3.7 + 3.7 \exp(\frac{4\pi i}{5})$, except brown ($\mu = -3.9 + 3.9 \exp(\frac{3\pi i}{5})$), are inside the yellow disc of radius 0.9. Figure 12 (right) shows distinctively the case $\mu = -3.6 + 3.6 \exp(\frac{3\pi i}{5})$, for clarity. As before, one of the brown roots lies outside the circle of radius 0.9 because the brown multiplier $\mu = -3.9 + 3.9 \exp(\frac{3\pi i}{5})$ is slightly larger than the critical value 7.36 (in absolute value). However, it is still inside the unit disc.

Figure 11: $T=1$, μ complex; $T=1$, $\rho = 0.9, \mu = -5.9 + 5.8 \exp(0.7\pi i)$

5. EXAMPLES

5.1. Example 1 [12]

Let $\hat{\mu}^* < 3$. Let us investigate the problem of constructing the optimal control in the system (2.1) for $N = 2$.

In this case $\hat{p}(\lambda) = \frac{2}{3}\lambda + \frac{1}{3}$, $\lambda\hat{q}(\lambda) = \frac{2}{3}\lambda + \frac{1}{3}\lambda^2$. Therefore, ρ should satisfy the inequality $\mu^* < \left(\frac{2}{3}\rho + \frac{1}{3}\rho^2\right) 3 = 2\rho + \rho^2$, or $\rho > -1 + \sqrt{1 + \mu^*}$. For example, for $\hat{\mu}^* = 2$ the minimally possible value ρ is equal to $-1 + \sqrt{3} \approx 0.73$. The optimal polynomial is

$$p_O(\lambda) = \frac{1}{\frac{2}{3}\rho + \frac{1}{3}} \left(\frac{2}{3}\frac{\lambda}{\rho} + \frac{1}{3} \right) = \lambda \frac{2}{2 + \rho} + \frac{\rho}{2 + \rho}.$$

If we want to increase the rate of convergence to exceed 0.73 we need to chose $N > 2$ in the control system (2.1).

Figure 12: $T=3$, μ complex $T=3$, $\mu = -3.9 + 3.9 \exp(\frac{3\pi i}{5})$

5.2. Example 2

The well-known Allee effect [1] in biology (c.f. [4]) is modeled by a bell shaped equation, e.g.

$$x_{n+1} = F(x_n), \quad F(x) = \frac{e^{-5(2x-1)^2} - e^{-5}}{1 - e^{-5}} \tag{5.1}$$

Figure 13: The Allee effect

Equation (5.1) describes the dynamics of vanishing population, i.e. for any initial value $x_0 \in (0, 1)$ that is different from the equilibrium or cycles of the system (5.1), we have $x_n \rightarrow 0, n \rightarrow \infty$.

Figure 14: The Allee effect modeling, $N = 2$

It is not difficult to see that for equilibrium $x^* \approx 0.647$ and the multiplier being negative we have $\mu^* \approx 3.84$, while the second multiplier is positive. To stabilize the equilibrium in the system (2.1) one should choose $N \geq 3$. For $N = 3$ one can find that the *standard* coefficients $a_1^{(3)} \approx 0.439$, $a_2^{(3)} \approx 0.414$, $a_3^{(3)} \approx 0.146$. The rate of convergence is determined by the largest (in absolute value) root of the equation $z^3 + \mu^*(a_1^{(3)}z^2 + a_2^{(3)}z + a_3^{(3)}) = 0$. This value is approximately 0.969. As we can see, it is close to 1, therefore the convergence will be quite slow.

To increase the rate of convergence let us apply an algorithm from the previous section. We start with the root of the equation

$$(a_1^{(3)}\rho + a_2^{(3)}\rho^2 + a_3^{(3)}\rho^3) \cot^2 \frac{\pi}{8} = \mu^*.$$

The root is $\rho_1 \approx 0.766$. Let us find the new *modified* control coefficients:

$$b_j^{(3)} = \frac{a_j^{(3)}\rho_1^j}{a_1^{(3)}\rho_1 + a_2^{(3)}\rho_1^2 + a_3^{(3)}\rho_1^3} : b_1^{(3)} \approx 0.516, b_2^{(3)} \approx 0.379, b_3^{(3)} \approx 0.105.$$

To determine the rate of convergence let us find the maximal (in absolute value) root of the equation:

$$z^3 + \mu^*(b_1^{(3)}z^2 + b_2^{(3)}z + b_3^{(3)}) = 0.$$

The root is 0.761.

It is possible to increase the rate of convergence even more if we take $N = 4$. Then the *standard* coefficients are $a_1^{(4)} \approx 0.306$, $a_2^{(4)} \approx 0.371$, $a_3^{(4)} \approx 0.247$,

$a_4^{(4)} \approx 0.076$. Let us find the root of the equation:

$$(a_1^{(4)} \rho + a_2^{(4)} \rho^2 + a_3^{(4)} \rho^3 + a_4^{(4)} \rho^4) \cot^2 \frac{\pi}{10} = \mu^*.$$

The root is $\rho_2 \approx 0.623$. Then

$$b_j^{(4)} = \frac{a_j^{(4)} \rho_2^j}{a_1^{(4)} \rho_2 + a_2^{(4)} \rho_2^2 + a_3^{(4)} \rho_2^3 + a_4^{(4)} \rho_2^4}$$

and

$$b_1^{(4)} \approx 0.466, b_2^{(4)} \approx 0.356, b_3^{(4)} \approx 0.149, b_4^{(4)} \approx 0.029.$$

The rate of convergence is about 0.618, as the maximal (in absolute value) root of the equation

$$z^4 + \mu^* (b_1^{(4)} z^3 + b_2^{(4)} z^2 + b_3^{(4)} z + b_4^{(4)}) = 0.$$

The solutions of the equation (2.1) with the choice of the controls, i.e. corresponding to the system

$$x_{n+1} = f \left(\sum_{j=1}^3 a_j^{(3)} x_{n-j+1} \right), \quad x_{n+1} = f \left(\sum_{j=1}^3 b_j^{(3)} x_{n-j+1} \right),$$

$$x_{n+1} = f \left(\sum_{j=1}^4 b_j^{(4)} x_{n-j+1} \right)$$

are displayed in Figure 15 in pink, blue, and red colors, respectively.

The difference is clear.

5.3. Example 3

Figure 16 and Figure 17 illustrate the differences in the convergence rate of detecting of 3-cycles in the *standard* logistic equation.

5.4. Example 4

Let us consider the equation of a sudden occurrence of chaos (SOC) [16]

$$x_{n+1} = F(x_n), \quad F(x) = (1 + \sqrt{2}) \left(\frac{1}{2} - \left| x - \frac{1}{2} \right| \right) + x.$$

Figure 15: The Ally effect: controlled system with different ρ

Figure 16: Logistic equation: 3-cycle and equilibrium, *standard* control for $n = 500, \dots, 800$

To define the 2-cycle let us apply the system (2.1) with $N = 4$ and $T = 2$. The mixing coefficients are computed for $\rho = 1$ and $\rho = 0.8$:

$$\{0.4375, 0.3125, 0.1875, 0.0625\}, \quad \{0.5211, 0.2978, 0.1429, 0.0381\}$$

The dynamics are displayed in Figure 18. It is clearly visible that the control works faster for $\rho = 0.8$.

Figure 17: Logistic equation: 3-cycle and equilibrium, $r = 0.9$ control for $n = 500, \dots, 800$

Figure 18: SOC $T = 2, N = 4$ SOC $T = 2, N = 4, \rho = 0.8$

6. NECESSARY CONDITIONS FOR APPLYING THE METHOD

In theory, the standard method of stabilization should stabilize a cycle of any length T for any value of the cycle multiplier. This follows from asymptotic estimates of the quantities $I_N^{(N)}$ in real and complex cases [11]

$$|I_N^{(T)}| \sim \frac{1}{N^2} \quad \text{for large } N \text{ and } T, \{\mu_1, \dots, \mu_m\} \in \{\mu \in \mathbb{R} : \mu \in (-\mu^*, 0)\}$$
$$|I_N^{(T)}| \sim \frac{1}{N} \quad \text{for large } N \text{ and } T, \{\mu_1, \dots, \mu_m\} \in \{\mu \in \mathbb{C} : |\mu + R| < R\}$$

If we want not only to stabilize the cycle but also guarantee the proper rate

of convergence, then the quantities μ^* and R characterize the diameter of the region of multipliers localizations, which cannot be made arbitrarily large regardless of the choice of N .

Let us first justify it for $T = 1$ with real multipliers. Let us estimate the right hand side in the inequality:

$$\mu^* < \rho \sum_{k=1}^N a_k^{(N)} \rho^{k-1} \cdot \frac{1}{|I_N^{(1)}|},$$

where $|I_N^{(1)}| = \tan^2 \frac{\pi}{2(N+1)}$ and $a_j^{(N)}$ satisfies (2.12). We have

$$\begin{aligned} \sum_{k=1}^N a_k^{(N)} \rho^k \cdot \frac{1}{|I_N^{(1)}|} &< 2 \cot \frac{\pi}{2(N+1)} \sum_{j=1}^N \frac{\pi j}{N+1} \rho^j = \frac{2\pi}{N+1} \cot \frac{\pi}{2(N+1)} \sum_{j=1}^N j \rho^j = \\ &= \frac{2\pi}{N+1} \cot \frac{\pi}{2(N+1)} \frac{\rho}{(1-\rho)^2} \cdot (1 - \rho^{N+1} - (N+1)\rho^N + (N+1)\rho^{N+1}). \end{aligned}$$

Therefore for a given radius ρ the value μ^* characterizing the size of admissible set of multipliers location does not exceed $\frac{4\rho}{(1-\rho)^2}$ regardless of N .

We post the dependence of the value μ^* on the radii $\rho \in \{\frac{1}{2}, \frac{2}{3}, \frac{9}{10}, 1\}$ and $N = 1, \dots, 10$ in Table 1 at the end of the article.

Let us consider the case $T = 2, \mu$ -real. In this case, the inequality for the diameter of the sets of localization of multipliers has the form

$$\mu^* < \rho \left(\sum_{k=1}^N a_k^{(N)} \rho^{k-1} \right)^2 \cdot \frac{1}{|I_N^{(2)}|},$$

where $|I_N^{(2)}| = \frac{1}{N^2}$ and $a_j^{(N)}$ are defined by (2.13).

The sum $\sum_{k=1}^N a_k^{(N)} \rho^k$ can be computed explicitly as

$$\begin{aligned} \sum_{k=1}^N a_k^{(N)} \rho^k &= \left(\frac{2}{N} + \frac{1}{N^2} \right) \frac{\rho(1-\rho^N)}{1-\rho} - \frac{2}{N^2} \frac{\rho}{(1-\rho)^2} \cdot \\ &\quad (1 - \rho^{N+1} - (N+1)\rho^N + (N+1)\rho^{N+1}). \end{aligned}$$

Then, by taking $N \rightarrow \infty$

$$\mu^* < \frac{N^2}{\rho} \left(\sum_{k=1}^N a_k^{(N)} \rho^k \right)^2 \sim \frac{4\rho}{(1-\rho)^2}.$$

The value μ^* is also bounded for all N if $\rho < 1$. We post the dependence of the value μ^* on the radii $\rho \in \{\frac{1}{2}, \frac{2}{3}, \frac{9}{10}, 1\}$ and $N = 1, \dots, 10$ in Table 2 at the end of the article.

The complex case is considered in a similar way. For $T = 1$

$$R < \rho \sum_{k=1}^N a_k^{(N)} \rho^{k-1} \cdot \frac{1}{2|I_N^{(1)}|}$$

where $|I_N^{(1)}| = \frac{1}{N}$ and $a_k^{(N)}$ are defined by (2.14).

Let us compute

$$\begin{aligned} \sum_{k=1}^N a_k^{(N)} \rho^k &= \frac{2}{N} \frac{\rho(1 - \rho^N)}{1 - \rho} - \frac{2}{N(N+1)} \frac{\rho}{(1 - \rho)^2} \cdot \\ &\quad (1 - \rho^{N+1} - (N+1)\rho^N + (N+1)\rho^{N+1}). \end{aligned}$$

From there

$$R < \frac{\rho(1 - \rho^N)}{1 - \rho} - \frac{1}{N+1} \frac{\rho}{(1 - \rho)^2} (1 - \rho^{N+1} - (N+1)\rho^N + (N+1)\rho^{N+1}).$$

Asymptotically, when $N \rightarrow \infty$

$$R < \frac{\rho}{1 - \rho}.$$

Thus, in the case of complex multipliers the diameter of the region of multiplier locations is bounded by a value independent of N , for $\rho < 1$. Tables of computed values of R and μ^* for several practical values of ρ and N are given at the end of the article (Tables 1, 2, and 3).

Thus, it is shown that dramatic improvement of the rate of convergence is possible only for relatively small regions of multipliers. Conversely, if the region of multipliers location is large enough then placing the roots of the characteristic polynomials in a small radius disc is impossible for arbitrary N .

7. CONCLUSION

In this paper we consider a generalization of the nonlinear delay feedback control (NDFC) developed in [6, 7, 9, 10, 11]. We show that one can modify

the control coefficients of NDFC to increase the rate of convergence for T -cycles of interest. The price we pay for the acceleration of convergence is the limitation of the stability region. In case $T = 1, 2$ we have estimated the limitation of the stability range under the modified control coefficients. For several values of ρ and N we tabulated the optimal values of R and μ^* , which are given in the tables at the end of the article. Our experimental results show that the rate of convergence increases when the control multipliers are within the range of limitation, whereas outside the range of limitation the rate of convergence is similar to the case of standard control coefficients.

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N	$\mu_N^*(1)$	$\mu_N^*(0.9)$	$\mu_N^*(2/3)$	$\mu_N^*(1/2)$
1	1.	.9	.666666667	.500000000
2	3.000000000	2.610000000	1.777777778	1.250000000
3	5.828427123	4.882347562	3.032995295	1.990577650
4	9.472135954	7.631892583	4.327466156	2.663259414
5	13.92820323	10.79130682	5.601974199	3.252984336
6	19.19566935	14.30194674	6.822475375	3.761456200
7	25.27414236	18.11176689	7.970659385	4.196794604
8	32.16343748	22.17436353	9.038213110	4.568918033
9	39.86345818	26.44832483	10.02310080	4.887514234
10	48.37415005	30.89670701	10.92711594	5.161240519

Table 1: Critical values for $T=1$, $\mu \in (-\mu^*, 0)$

N	$\mu_N^*(1)$	$\mu_N^*(0.9)$	$\mu_N^*(2/3)$	$\mu_N^*(1/2)$
1	1.	.9	.6666666667	.5000000000
2	4.000000000	3.422250000	2.240740741	1.531250000
3	9.000000000	7.242009999	4.105166897	2.531250000
4	16.00000000	12.08735331	5.963020122	3.363769533
5	25.00000000	17.73121275	7.684296098	4.025703125
6	36.00000000	23.98439167	9.223014722	4.546997075
7	49.00000000	30.68965059	10.57385227	4.959843852
8	64.00000000	37.71670341	11.74969591	5.290775772
9	81.00000000	44.95798306	12.77026032	5.559896679
10	100.0000000	52.32505773	13.65652485	5.781992354

Table 2: Critical values for $T=2$, $\mu \in (-\mu^*, 0)$

N	$R(1)$	$R(0.9)$	$R(2/3)$	$R(1/2)$
1	.5000000000	.4500000000	.3333333334	.2500000000
2	1.0000000000	.8700000000	.5925925925	.4166666667
3	1.5000000000	1.262250000	.7962962965	.5312500000
4	2.0000000000	1.628820000	.9580246915	.6125000000
5	2.5000000000	1.971615000	1.087791495	.6718750000
6	3.0000000000	2.292388715	1.193023712	.7165178570
7	3.5000000000	2.592756112	1.279263832	.7509765625
8	4.0000000000	2.874204890	1.350674864	.7782118050
9	4.5000000000	3.138105961	1.410404918	.8001953120
10	5.0000000000	3.385723059	1.460851464	.8182705965

Table 3: Critical values for $T=1$, $R > 0$.

REFERENCES

- [1] W.C. Allee, E. Bowen, *Studies in animal aggregations: mass protection against colloidal silver among goldfishes*. Journal of Experimental Zoology. 61 (2): 1932, pp. 185-207.
- [2] B. R. Andrievskii and A. L. Fradkov, *Control of Chaos: Methods and Applications*. Automation and Remote Control, Vol. 64, No. 5, 2003, pp. 673-713.
- [3] G. Chen and X. Dong, *From Chaos to Order: Methodologies, Perspectives and Applications*, World Scientific, Singapore (1999), 776 pp.
- [4] F. Courchamp, L. Berec, and J. Gascoigne, *Allee Effects in Ecology and Conservation*. Oxford University Press, New York, 2008, 272 pp.
- [5] D. Dmitrishin, P. Hagelstein, A. Khamitova, and A. Stokolos, *Limitations of Robust Stability of a Linear Delayed Feedback Control*, SIAM J. Control Optim., 56(1), 2018, pp. 148-157.
- [6] D. Dmitrishin, P. Hagelstein, A. Khamitova, A. Stokolos, *On the stability of cycles by delayed feedback control*, Linear and Multilinear Algebra, Vol. 64, Iss. 8, 2016, pp. 1538-1549.
- [7] D.V. Dmitrishin and A. D. Khamitova, *Methods of Harmonic Analysis in Control of Nonlinear Discrete Systems*, C.R. Acad. Sci. Paris, 351 (2013), pp 357-370.
- [8] D. Dmitrishin, A. Khamitova and A. Stokolos, *Fejér polynomials and Chaos*, Springer Proceedings in Mathematics and Statistics, 108 (2014), pp 49-75.
- [9] D. Dmitrishin, I.M. Skrinnik and A. Stokolos, *From chaos to order through mixing*, arXiv:1607.05493 [nlin.CD], 2016.
- [10] D. Dmitrishin, I.M. Skrinnik, A. Stokolos and E. Franzheva *Generalization of nonlinear control for nonlinear discrete systems*, arXiv:1709.10410 [nlin.CD], 2017.

- [11] D.Dmitrishin, A. Khamitova, A. Stokolos and M. Tohaneanu, *Finding cycles in nonlinear autonomous discrete dynamical systems*. In: Harmonic Analysis, Partial Differential Equations, Banach Spaces, and Operator Theory (Volume 2). Association for Women in Mathematics Series, vol 5(2017). Springer, Cham.
- [12] E.D. Franzheva, *Optimization of parameters in self-organizing systems*. Informatics and Mathematical Methods in Simulations, ONPU, Odessa. (Submitted)
- [13] Ö. Morgül, *Further stability results for a generalization of delayed feedback control*, Nonlinear Dynamics, 1 August 2012, pp. 1-8.
- [14] E. Ott, C. Grebogi, J. A. Yorke, *Controlling chaos*. Phys. Rev. Lett. 64(11)(1990), 1196-1199.
- [15] K. Pyragas, *Continuous control of chaos by self-controlling feedback*, Physics Letters A 170 (1992), 421–428.
- [16] L. Tian, G. Dong, *Predictive control of sudden occurrence of chaos*, Int. J. Nonlinear Science. 5(2) (2008), 99-105.
- [17] T. Ushio, *Limitation of delayed feedback control in nonlinear discrete-time systems*, IEEE Transactions on Circuits and Systems - I: Fundamental Theory and Applications 43 (1996), 815–816.
- [18] d.S.M. Vieira, A.J. Lichtenberg, *Controlling chaos using nonlinear feedback with delay*. Phys. Rev. E 54 (1996), 1200-1207.