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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A MULTI-TERM NONLINEAR SEQUENTIAL FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH TREE-POINT BOUNDARY CONDITIONS

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ABSTRACT: We study a new class of boundary value problems of multi-term nonlinear sequential fractional integro-differential equations with three-point nonlocal fractional boundary conditions. Some existence and uniqueness results are obtained by using standard fixed point theorems. An example is given to illustrate our results.

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1. INTRODUCTION

Boundary value problems for fractional differential equations have been extensively studied in the recent years. The study of fractional differential equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. A strong motivation for studying fractional differential equations comes from the fact that they have been proved to be valuable tools in the modeling of many phenomena in engineering and sciences such as physics, mechanics, chemistry, economics and biology, etc. [9, 10, 11, 13]. Some recent results on fractional boundary value problems can be found in [2, 3, 5, 6, 7, 8, 12, 14, 16, 17] and references therein.

In [1] Ahmed Alsaedi and Bachir Ahmad investigated the existence of solutions for a nonlinear fractional integro-differential equations, with fractional nonlocal integral boundary conditions given by

$$\begin{cases} D^q x(t) + f(t, x(t), (\Phi x)(t), (\Psi x)(t)) = 0, \quad 0 < t < 1, 1 < q \leq 2, \\ D^{\frac{q-1}{2}} x(0) = 0, \quad aD^{\frac{q-1}{2}} x(1) + x(\eta) = 0, \quad 0 < \eta < 1, \end{cases}$$

where D^q denotes the Riemann-Liouville fractional derivative of order q, f:$[0;1]\times X \times X \times X \to X$ is a continuous, for $\gamma, \delta : [0,1] \times [0,1] \to [0, \infty)$,

$$(\Phi x)(t) = \int_0^t \gamma(t,s)x(s)ds, \quad (\Psi x)(t) = \int_0^t \delta(t,s)x(s)ds,$$

and $a \in \mathbb{R}$ satisfies the condition $a\Gamma(q) + \eta^{q-1}\Gamma(\frac{q+1}{2}) \neq 0$. Here, $(X, \|.\|)$ is a Banach space and $C = C([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by$\|.\|$.

In [4] the authors considered the Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions

$$\begin{cases} D^\alpha u(t) = f(t, u(t), (\Phi x)(t), (\Psi x)(t)), \quad t \in [0,T], \alpha \in (1,2], \\ D^{\alpha-2} u(0^+) = \nu I^{\alpha-1} u(\eta), \quad 0 < \eta < T, \end{cases}$$

ν is a constant, D^α denotes the Riemann-Liouville fractional derivative of order α, f: $[0;T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and

$$(\Phi x)(t) = \int_0^t \gamma(t,s)x(s)ds, \quad (\Psi x)(t) = \int_0^t \delta(t,s)x(s)ds,$$

with γ and δ being continuous functions on $[0,T] \times [0,T]$.

Very recently the authors Ying Wang and Lishan Liu in [15] have investigated existence and uniqueness of positive solutions for the following fractional integro-differential equation

$$\begin{cases} {}^{C}D^{\alpha}u(t) + f(t, u(t), Tu(t), Su(t)) = 0, \quad t \in [0,1], \\ u(0) = b_0, \quad u'(0) = b_1, ..., \quad u^{(n-3)} = b_{n-3}, \\ u^{(n-1)}(0) = b_{n-1}, u(1) = \mu \int_0^1 u(s)ds, \end{cases}$$

where $n-1 < \alpha \leq n, \quad 0 \leq \mu < n-1, \quad n \geq 3, \quad b_i \geq 0 (i = 1, 2, ..., n-3, n-1)$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative. f: $[0;T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and

$$(Tu)(t) = \int_0^t K(t,s)u(s)ds, \quad (Su)(t) = \int_0^t H(t,s)u(s)ds.$$

Motivated by the above works, we consider in this paper the existence and uniqueness of solutions for multi-term nonlinear sequential fractional integro-differential equations with three-point nonlocal fractional boundary conditions.

$$\begin{cases} Lx(t) = f(t, x(t), (\Phi x)(t), (\Psi x)(t), D^{\beta_1}x(t), ..., D^{\beta_n}x(t)), \quad t \in [0,1], \\ D^{\frac{(\alpha-\beta)}{2}}x(0) = 0, \quad aD^{\frac{(\alpha-\beta)}{2}}x(1) + x(\eta) = 0, \quad 0 < \eta < 1, \end{cases} \tag{1}$$

where L the operator defined by $L = D^{\alpha} + kD^{\beta}$, $\alpha \in (1,2]$, $0 < \beta \leq 1$, D^{α} denotes the Riemann-Liouville fractional derivative of order α, k is are real constant, $f : [0;1] \times \mathbb{R}^{n+3}$ is a continuous function and $\gamma, \delta : [0,1] \times [0,1] \to [0, \infty)$,

$$(\Phi x)(t) = \int_0^t \gamma(t,s)x(s)ds, \quad (\Psi x)(t) = \int_0^t \delta(t,s)x(s)ds,$$

$0 < \beta_i \leq 1$, $\alpha - \beta - \beta_i > 0$, $(i = 1, 2, ..., n)$ and $a \in \mathbb{R}$ satisfies the condition $a\Gamma(\alpha) + \eta^{\alpha-1}\Gamma(\frac{\alpha+\beta}{2}) \neq 0$.

2. PRELIMINARIES

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later [9, 10, 11].

Definition 2.1. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a continuous function $g : (0, \infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order α of a function $g:(0,\infty)\to\mathbb{R}$ is defined by

$$D^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^{n}\int_{0}^{t}g(s)(t-s)^{n-\alpha-1}ds,\quad n-1<\alpha\leq n,\quad \alpha>0,$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.3. *For $\alpha>0$, let x, $D^{\alpha}x\in C(0,1)\cap L(0,1)$, then*

$$I^{\alpha}D^{\alpha}x(t)=x(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+...+c_nt^{\alpha-n},$$

where $c_i\in\mathbb{R}$, $i=1,2,...,n$(n is smallest integer such that $n\geq\alpha$).

Remark 2.4. Let $x\in L(0,1)$, then

- (i) $D^{\beta}I^{\alpha}x(t)=I^{\alpha-\beta}x(t),\quad \alpha>\beta>0.$
- (ii) $D^{\beta}t^{\alpha-1}=\dfrac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}.$

Lemma 2.5. *For a given $h\in C(0,1)\cap L(0,1)$, then the unique solution of the boundary value problem*

$$(D^{\alpha}+kD^{\beta})x(t)=h(t),\quad 0<t<1,\quad 1<\alpha\leq 2, 0<\beta\leq 1, \tag{2}$$

$$D^{\frac{(\alpha-\beta)}{2}}x(0)=0,\quad aD^{\frac{(\alpha-\beta)}{2}}x(1)+x(\eta)=0,\quad 0<\eta<1, \tag{3}$$

is given by

$$\begin{aligned}x(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds-\frac{\lambda_1at^{\alpha-1}}{\Gamma(\frac{\alpha+\beta}{2})}\int_0^1(1-s)^{\frac{\alpha+\beta}{2}-1}h(s)ds\\&-\frac{\lambda_1t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}h(s)ds-\frac{k}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds\\&+\frac{\lambda_1akt^{\alpha-1}}{\Gamma(\dfrac{\alpha-\beta}{2})}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}x(s)ds+\frac{\lambda_1kt^{\alpha-1}}{\Gamma(\alpha-\beta)}\int_0^{\eta}(\eta-s)^{\alpha-\beta-1}x(s)ds.\end{aligned}$$

Proof. We begin by solving equation (2) for D^{α} and applying the integral operator I^{α}_{0+} to both sides

$$I^{\alpha}_{0+}D^{\alpha}x(t)=-kI^{\alpha}_{0+}D^{\beta}x(t)+I^{\alpha}_{0+}h(t),\quad 0<t<1, \tag{4}$$

from lemma 2.3 and remark 2.4 the equation 4 simplifies to

$$x(t) = -kI^{\alpha-\beta}x(t) + b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + I^{\alpha}h(t), \quad 0 < t < 1, \tag{5}$$

for some constants b_1, b_2.

Applying the boundary condition $D^{\frac{\alpha-\beta}{2}}x(0) = 0$ we find that $b_2 = 0$. By the boundary condition $aD^{\frac{\alpha-\beta}{2}}x(1) + x(\eta) = 0$ we have

$$b_1 = \frac{\Gamma(\frac{\alpha+\beta}{2})}{a\Gamma(\alpha) + \eta^{\alpha-1}\Gamma(\frac{\alpha+\beta}{2})} \times \left[akI^{\frac{\alpha-\beta}{2}}x(1) + kI^{\alpha-\beta}x(\eta) - aI^{\frac{\alpha+\beta}{2}}h(1) - I^{\alpha-\beta}h(\eta)\right].$$

Substituting the values of b_1, b_2 in (5), we obtain $x(t)$. □

Theorem 2.6. *(Schauder fixed point theorem) Let E be a closed, convex and bounded subset of the Banach space X, and let $F : E \to E$ be a continuous mapping such that $F(E)$ is a relatively compact subset of X. Then F has a fixed point in E.*

3. MAIN RESULTS

In this section, we give some existence and uniqueness results for the problem (1) and then give an example illustrating the usefulness of our results.

Let $C(I)$ be the space of all continuous real-valued functions on $I = [0, 1]$ and

$$X = \left\{x : x \in C(I) \quad and \quad D^{\beta_i}x(t) \in C(I), \quad 0 < \beta_i < 1, \quad i = 1, 2, ..., n\right\}$$

endowed with the norm $\|x\|_X = \max_{t\in I}|x(t)| + \Sigma_{i=1}^{n}\max_{t\in I}|D^{\beta_i}x(t)|$. It is known that $(X, \|.\|)$ is a Banach space.

In view of Lemma 2.5 we define an operator $F : X \to X$ by

$$\begin{aligned}(Fx)(t) \quad = \quad & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), (\phi x)(s), (\psi x)(s), D^{\beta_1}x(s), ..., D^{\beta_n}x(s))ds \\ & -\frac{\lambda_1 a t^{\alpha-1}}{\Gamma(\frac{\alpha+\beta}{2})}\int_0^1 (1-s)^{\frac{\alpha+\beta}{2}-1}\end{aligned}$$

$$
\begin{aligned}
&\times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),D^{\beta_2}x(s),...,D^{\beta_n}x(s))ds\\
&-\frac{\lambda_1 t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}\\
&\times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),D^{\beta_2}x(s),...,D^{\beta_n}x(s))ds\\
&-\frac{k}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}x(s)ds\\
&+\frac{\lambda_1 akt^{\alpha-1}}{\Gamma(\frac{\alpha-\beta}{2})}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}x(s)ds\\
&+\frac{\lambda_1 kt^{\alpha-1}}{\Gamma(\alpha-\beta)}\int_0^\eta(\eta-s)^{\alpha-\beta-1}x(s)ds,
\end{aligned}
$$

It is clear that the problem (1) has solutions if and only if the operator equation $Fx=x$ has fixed points. For any $x\in X$. Therefore

$$
\begin{aligned}
D^{\beta_i}(Fx)(t) &= \frac{1}{\Gamma(\alpha-\beta_i)}\int_0^t(t-s)^{\alpha-\beta_i-1}\\
&\times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))ds\\
&-\frac{\lambda_1 a\Gamma(\alpha)t^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)\Gamma(\frac{\alpha+\beta}{2})}\int_0^1(1-s)^{\frac{\alpha+\beta}{2}-1}\\
&\times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),D^{\beta_2}x(s),...,D^{\beta_n}x(s))ds\\
&-\frac{\lambda_1\Gamma(\alpha)t^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}\\
&\times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),D^{\beta_2}x(s),...,D^{\beta_n}x(s))ds\\
&-\frac{k}{\Gamma(\alpha-\beta-\beta_i)}\int_0^t(t-s)^{\alpha-\beta-\beta_i-1}x(s)ds\\
&+\frac{\lambda_1 k\Gamma(\alpha)t^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta_i)}\int_0^\eta(\eta-s)^{\alpha-\beta-1}x(s)ds\\
&+\frac{\lambda_1 ak\Gamma(\alpha)t^{\alpha-\beta_i-1}}{\Gamma(\frac{\alpha-\beta}{2})\Gamma(\alpha-\beta_i)}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}x(s)ds,
\end{aligned}
$$

where

$$\lambda_1=\frac{\Gamma(\frac{\alpha+\beta}{2})}{a\Gamma(\alpha)+\eta^{\alpha-1}\Gamma(\frac{\alpha+\beta}{2})}$$

Now we present our main results. The methods used to prove the existence results are standard, however their exposition in the framework of the problem (1) is new.

Theorem 3.1. *Assume that there exist $l\in(0,\alpha-1)$ and $\mu(t)\in L^{\frac{1}{l}}([0,1],(0,$*

$\infty)$) *such that*

$$\begin{aligned} & |f(t,u,v,w,x_1,x_2,...,x_n) - f(t,u',v',w',y_1,y_2,...,y_n)| \\ \le\ & \mu(t)\left(|u-u'|+|v-v'|+|w-w'|+|x_1-y_1|+|x_2-y_2|+...+|x_n-y_n|\right), \end{aligned}$$

for all $t \in [0,1]$ *and* $u,v,w,u',v',w',x_1,x_2,...,x_n,y_1,y_2,...,y_n \in \mathbb{R}$. *Then problem (1) has a unique solution whenever*

$$\begin{aligned} \Delta &= \frac{|1+\lambda_1||k|}{\Gamma(\alpha-\beta+1)} + \sum_{i=1}^{n}\left(\frac{|k|}{\Gamma(\alpha-\beta-\beta_i+1)} + \frac{|\lambda_1 ak|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)}\right. \\ & \left. + \frac{|\lambda_1 k|\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)\Gamma(\alpha-\beta+1)}\right) + \frac{|\lambda_1 ak|}{\Gamma(\frac{\alpha-\beta}{2}+1)} + (1+\gamma_0+\lambda_0)\Delta_1 < 1, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= \frac{|1+\lambda_1|\mu^*}{\Gamma(\alpha)}\left(\frac{1-l}{\alpha-l}\right)^{1-l} + \frac{|\lambda_1 a|\mu^*}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l} \\ & + \sum_{i=1}^{n}\frac{\mu^*}{\Gamma(\alpha-\beta_i)}\left(\left(\frac{1-l}{\alpha-\beta_i-l}\right)^{1-l} + \frac{|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l}\right. \\ & \left. + |\lambda_1|\left(\frac{1-l}{\alpha-l}\right)^{1-l}\right), \end{aligned}$$

and

$$\gamma_0 = \sup_{t\in I}\left|\int_0^t \gamma(t,s)ds\right|, \lambda_0 = \sup_{t\in I}\left|\int_0^t \lambda(t,s)ds\right|, \mu^* = \left(\int_0^t (\mu(s))^{\frac{1}{l}}ds\right)^l.$$

Proof. For any $x, y \in X$ and for each $t \in [0,1]$, by using the Holder inequality, we have

$$\begin{aligned} & |(Fx)(t) - (Fy)(t)| \\ &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\right. \\ & \times\Big(f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s)) \\ & - f(s,y(s),(\phi y)(s),(\psi y)(s),D^{\beta_1}y(s),...,D^{\beta_n}y(s))\Big)ds \\ & - \frac{\lambda_1 a t^{\alpha-1}}{\Gamma(\frac{\alpha+\beta}{2})}\int_0^1 (1-s)^{\frac{\alpha+\beta}{2}-1} \end{aligned}$$

$$
\begin{aligned}
&\times\Big(f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))\\
&-f(s,y(s),(\phi y)(s),(\psi y)(s),D^{\beta_1}y(s),...,D^{\beta_n}y(s))\Big)ds\\
&-\frac{\lambda_1 t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}\\
&\times\Big(f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))\\
&-f(s,y(s),(\phi y)(s),(\psi y)(s),D^{\beta_1}y(s),...,D^{\beta_n}y(s))\Big)ds\\
&-\frac{k}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}(x(s)-y(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda_1 akt^{\alpha-1}}{\Gamma(\frac{\alpha-\beta}{2})}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}(x(s)-y(s))ds\\
&+\frac{\lambda_1 kt^{\alpha-1}}{\Gamma(\alpha-\beta)}\int_0^\eta(\eta-s)^{\alpha-\beta-1}(x(s)-y(s))ds\Big|\\
\leq\ &\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mu(s)\Big(|x-y|+|\phi x-\phi y|\\
&+|\psi x-\psi y|+|D^{\beta_1}x-D^{\beta_1}y|+...+|D^{\beta_n}x-D^{\beta_n}y|\Big)ds\\
&+\frac{|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2})}\int_0^1(1-s)^{\frac{\alpha+\beta}{2}-1}\mu(s)\Big(|x-y|+|\phi x-\phi y|\\
&+|\psi x-\psi y|+|D^{\beta_1}x-D^{\beta_1}y|+...+|D^{\beta_n}x-D^{\beta_n}y|\Big)ds\\
&+\frac{|\lambda_1|}{\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}\mu(s)\Big(|x-y|+|\phi x-\phi y|\\
&+|\psi x-\psi y|+|D^{\beta_1}x-D^{\beta_1}y|+...+|D^{\beta_n}x-D^{\beta_n}y|\Big)ds\\
&+\frac{|k|}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}|x(s)-y(s)|ds\\
&+\frac{|\lambda_1 ak|}{\Gamma(\frac{\alpha-\beta}{2})}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}|x(s)-y(s)|ds\\
&+\frac{|\lambda_1 k|}{\Gamma(\alpha-\beta)}\int_0^\eta(\eta-s)^{\alpha-\beta-1}|x(s)-y(s)|ds,\\
\leq\ &\frac{(1+\gamma_0+\lambda_0)\|x-y\|}{\Gamma(\alpha)}\left(\int_0^t((t-s)^{\alpha-1})^{\frac{1}{1-l}}ds\right)^{1-l}\left(\int_0^t|\mu(s)|^{\frac{1}{l}}ds\right)^l\\
&+\frac{|a\lambda_1|(1+\gamma_0+\lambda_0)\|x-y\|}{\Gamma(\frac{\alpha+\beta}{2})}\left(\int_0^1((1-s)^{\frac{\alpha+\beta}{2}-1})^{\frac{1}{1-l}}ds\right)^{1-l}\left(\int_0^t|\mu(s)|^{\frac{1}{l}}ds\right)^l
\end{aligned}
$$

$$+\frac{|\lambda_1|(1+\gamma_0+\lambda_0)\|x-y\|}{\Gamma(\alpha)}\left(\int_0^\eta((\eta-s)^{\alpha-1})^{\frac{1}{1-l}}ds\right)^{1-l}\left(\int_0^t|\mu(s)|^{\frac{1}{l}}ds\right)^l$$
$$+\frac{|k|}{\Gamma(\alpha-\beta+1)}\|x-y\|+\frac{|\lambda_1ak|}{\Gamma(\frac{\alpha-\beta}{2}+1)}\|x-y\|+\frac{|\lambda_1k|}{\Gamma(\alpha-\beta+1)}\|x-y\|$$

$$\leq \left\{\left(\frac{|1+\lambda_1|}{\Gamma(\alpha)}\left(\frac{1-l}{\alpha-l}\right)^{1-l}+\frac{|a\lambda_1|}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l}\right)(1+\gamma_0+\lambda_0)\mu^*\right.$$
$$\left.+\frac{|1+\lambda_1||k|}{\Gamma(\alpha-\beta+1)}+\frac{|\lambda_1ak|}{\Gamma(\frac{\alpha-\beta}{2}+1)}\right\}\|x-y\|.$$

Also we have

$$|D^{\beta_i}(Fx)(t)-D^{\beta_i}(Fy)(t)|$$
$$= \left|\frac{1}{\Gamma(\alpha-\beta_i)}\int_0^t(t-s)^{\alpha-\beta_i-1}\right.$$
$$\times\Big(f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))$$
$$-f(s,y(s),(\phi y)(s),(\psi y)(s),D^{\beta_1}y(s),...,D^{\beta_n}y(s))\Big)ds$$
$$-\frac{\lambda_1a\Gamma(\alpha)t^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)\Gamma(\frac{\alpha+\beta}{2})}\int_0^1(1-s)^{\frac{\alpha+\beta}{2}-1}$$
$$\times\Big(f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))$$
$$-f(s,y(s),(\phi y)(s),(\psi y)(s),D^{\beta_1}y(s),...,D^{\beta_n}y(s))\Big)ds$$
$$-\frac{\lambda_1\Gamma(\alpha)t^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}$$
$$\times\Big(f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))$$
$$-f(s,y(s),(\phi y)(s),(\psi y)(s),D^{\beta_1}y(s),...,D^{\beta_n}y(s))\Big)ds$$
$$-\frac{k}{\Gamma(\alpha-\beta-\beta_i)}\int_0^t(t-s)^{\alpha-\beta-\beta_i-1}(x(s)-y(s))ds$$
$$+\frac{\lambda_1ak\Gamma(\alpha)t^{\alpha-\beta_i-1}}{\Gamma(\frac{\alpha-\beta}{2})\Gamma(\alpha-\beta_i)}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}(x(s)-y(s))ds$$
$$\left.+\frac{\lambda_1k\Gamma(\alpha)t^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta_i)}\int_0^\eta(\eta-s)^{\alpha-\beta-1}(x(s)-y(s))ds\right|$$
$$\leq \frac{1}{\Gamma(\alpha-\beta_i)}\int_0^t(t-s)^{\alpha-\beta_i-1}\mu(s)\Big(|x-y|+|\phi x-\phi y|$$

$$+|\psi x-\psi y|+|D^{\beta_1}x-D^{\beta_1}y|+...+|D^{\beta_n}x-D^{\beta_n}y|\Big)ds$$

$$\begin{aligned}
&+\frac{|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)\Gamma(\frac{\alpha+\beta}{2})}\int_0^1(1-s)^{\frac{\alpha+\beta}{2}-1}\mu(s)\Big(|x-y|+|\phi x-\phi y|\\
&+|\psi x-\psi y|+|D^{\beta_1}x-D^{\beta_1}y|+...+|D^{\beta_n}x-D^{\beta_n}y|\Big)ds\\
&+\frac{|\lambda_1|}{\Gamma(\alpha-\beta_i)}\int_0^\eta(\eta-s)^{\alpha-1}\mu(s)\Big(|x-y|+|\phi x-\phi y|\\
&+|\psi x-\psi y|+|D^{\beta_1}x-D^{\beta_1}y|+...+|D^{\beta_n}x-D^{\beta_n}y|\Big)ds\\
&+\frac{|k|}{\Gamma(\alpha-\beta-\beta_i)}\int_0^t(t-s)^{\alpha-\beta-\beta_i-1}|x(s)-y(s)|ds\\
&+\frac{|\lambda_1 ak|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2})\Gamma(\alpha-\beta_i)}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}|x(s)-y(s)|ds\\
&+\frac{|\lambda_1 k|\Gamma(\alpha)}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta_i)}\int_0^\eta(\eta-s)^{\alpha-\beta-1}|x(s)-y(s)|ds\\
\leq\ &\left\{\frac{(1+\gamma_0+\lambda_0)\mu^*}{\Gamma(\alpha-\beta_i)}\left[\left(\frac{1-l}{\alpha-\beta_i-l}\right)^{1-l}+\frac{|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l}\right.\right.\\
&\left.+|\lambda_1|\left(\frac{1-l}{\alpha-l}\right)^{1-l}\right]+\frac{|k|}{\Gamma(\alpha-\beta-\beta_i+1)}+\frac{|\lambda_1 ak|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)}\\
&\left.+\frac{|\lambda_1 k|\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\beta_i)}\right\}\|x-y\|,
\end{aligned}$$

for $i=1,2,...,n$. Hence, we get

$$\begin{aligned}
\|Fx-Fy\| \ \leq\ &\left\{\frac{|1+\lambda_1||k|}{\Gamma(\alpha-\beta+1)}+\frac{|\lambda_1 ak|}{\Gamma(\frac{\alpha-\beta}{2}+1)}+(1+\gamma_0+\lambda_0)\Delta_1\right.\\
&+\sum_{i=1}^n\left(\frac{|k|}{\Gamma(\alpha-\beta-\beta_i+1)}+\frac{|\lambda_1 ak|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)}\right.\\
&\left.\left.+\frac{|\lambda_1 k|\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)\Gamma(\alpha-\beta+1)}\right)\right\}\|x-y\|=\Delta\|x-y\|.
\end{aligned}$$

Since $\Delta<1$, F is a contraction mapping, therefore, by using the Banach contraction principle, F has a unique fixed point, which is the unique solution of problem (1). □

Corollary 3.2. *Assume that there exists $L > 0$ such that*

$$|f(t,u,v,w,x_1,x_2,...,x_n) - f(t,u',v',w',y_1,y_2,...,y_n)|$$
$$\leq L(|u-u'| + |v-v'| + |w-w'| + |x_1-y_1| + |x_2-y_2| + ... + |x_n-y_n|),$$

for all $t \in [0,1]$ and $u,v,w,u',v',w',x_1,x_2,...,x_n,y_1,y_2,...,y_n \in \mathbb{R}$. Then problem (1) has a unique solution whenever

$$\Delta' = \sum_{i=1}^{n}\left(\frac{|k|}{\Gamma(\alpha-\beta-\beta_i+1)} + \frac{|\lambda_1 ak|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)} + \frac{|\lambda_1 k|\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)\Gamma(\alpha-\beta+1)}\right)$$
$$+ \frac{|1+\lambda_1||k|}{\Gamma(\alpha-\beta+1)} + \frac{|\lambda_1 ak|}{\Gamma(\frac{\alpha-\beta}{2}+1)} + (1+\gamma_0+\lambda_0)L\left[\frac{|1+\lambda_1|}{\Gamma(\alpha+1)} + \frac{|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2}+1)}\right.$$
$$\left. + \sum_{i=1}^{n}\frac{1}{\Gamma(\alpha-\beta_i)}\left(\frac{1}{\alpha-\beta_i} + \frac{|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\frac{\alpha+\beta}{2}+1)} + \frac{|\lambda_1|}{\alpha}\right)\right] < 1,$$

where $\gamma_0 = \sup_{t\in I}|\int_0^t \gamma(t,s)ds|, \lambda_0 = \sup_{t\in I}|\int_0^t \lambda(t,s)ds|$.

Now, we restate the Schauder's fixed point theorem, which is needed to prove next result.

Theorem 3.3. *Let $f : [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ be a continuous function such that there exists a constant $l \in (0, \alpha-1)$ and a real-valued function $m(t) \in L^{\frac{1}{l}}([0,1],(0,\infty))$ such that*

$$|f(t,u,v,w,x_1,x_2,...,x_n)| \leq$$

$$m(t) + d|u|^{\rho} + d'|v|^{\rho'} + d''|w|^{\rho''} + d_1|x_1|^{\rho_1} + ... + d_n|x_n|^{\rho_n}), \qquad (*)$$

where $d, d', d'', d_i > 0$ and $\rho, \rho', \rho'', \rho_i > 1$ for $i = 1,2,...,n$. Then problem (1) has at least one solution.

Proof. Suppose that f satisfy condition $(*)$. Define $B_r = \{x \in X, \|x\| \leq r\}$, where

$$r \geq \frac{1}{1-C}\max\{((n+4)Bd)^{\frac{1}{1-\rho}}, ((n+4)Bd')^{\frac{1}{1-\rho'}}, ((n+4)Bd'')^{\frac{1}{1-\rho''}},$$
$$((n+4)Bd_1)^{\frac{1}{1-\rho_1}}, ..., ((n+4)Bd_n)^{\frac{1}{1-\rho_n}}, (n+4)\},$$

$$A=\sum_{i=1}^{n}\frac{\|m\|}{\Gamma(\alpha-\beta_i)}\left(\left(\frac{1-l}{\alpha-\beta_i-l}\right)^{1-l}+\frac{|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l}\right.$$
$$\left.+|\lambda_1|\left(\frac{1-l}{\alpha-l}\right)^{1-l}\right)+\frac{|1+\lambda_1|\|m\|}{\Gamma(\alpha)}\left(\frac{1-l}{\alpha-l}\right)^{1-l}+\frac{|\lambda_1 a|\|m\|}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l},$$

$$B = \frac{|1+\lambda_1|}{\Gamma(\alpha+1)}+\frac{|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2})}+\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha-\beta_i)}\left(\frac{1}{\alpha-\beta_i}+\frac{|\lambda_1 a|\Gamma(\alpha)}{\frac{\alpha+\beta}{2}+1}+\frac{|\lambda_1|}{\alpha}\right),$$
$$C = \sum_{i=1}^{n}\left(\frac{|k|}{\Gamma(\alpha-\beta-\beta_i+1)}+\frac{|k\lambda_1 a|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)}\right.$$
$$\left.+\frac{|k\lambda_1|\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\beta_i)}\right)+\frac{|k||1+\lambda_1|}{\Gamma(\alpha-\beta+1)}+\frac{|k\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2}+1)}.$$

Note that B_r is closed, bounded and convex subset of the Banach space X. For each $x\in B_r$, we have

$$\begin{aligned}
|(Fx)(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\\
&\times\Big(m(s)+dr^{\rho}+d'\gamma_0^{\rho'}r^{\rho'}+d''\lambda_0^{\rho''}r^{\rho''}+d_1r^{\rho_1}+...+d_nr^{\rho_n}\Big)ds\\
&+\frac{|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2})}\int_0^1(1-s)^{\frac{\alpha+\beta}{2}-1}\\
&\times\Big(m(s)+dr^{\rho}+d'\gamma_0^{\rho'}r^{\rho'}+d''\lambda_0^{\rho''}r^{\rho''}+d_1r^{\rho_1}+...+d_nr^{\rho_n}\Big)ds\\
&+\frac{|\lambda_1|}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\\
&\times\Big(m(s)+dr^{\rho}+d'\gamma_0^{\rho'}r^{\rho'}+d''\lambda_0^{\rho''}r^{\rho''}+d_1r^{\rho_1}+...+d_nr^{\rho_n}\Big)ds\\
&+\frac{|k|r}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}ds+\frac{|\lambda_1 ak|r}{\Gamma(\frac{\alpha-\beta}{2})}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}ds\\
&+\frac{|\lambda_1 k|r}{\Gamma(\alpha-\beta)}\int_0^{\eta}(\eta-s)^{\alpha-\beta-1}ds\\
\leq\ & \frac{|1+\lambda_1|\|m\|}{\Gamma(\alpha)}\left(\frac{1-l}{\alpha-l}\right)^{1-l}+\frac{|\lambda_1 a|\|m\|}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l}\\
&+|k|r\left(\frac{|1+\lambda_1|}{\Gamma(\alpha-\beta+1}+\frac{|\lambda_1 a|}{\Gamma(\frac{\alpha-\beta}{2}+1)}\right)
\end{aligned}$$

$$+(dr^{\rho}+d'\gamma_0^{\rho'}r^{\rho'}+d''\lambda_0^{\rho''}r^{\rho''}+d_1r^{\rho_1}+...+d_nr^{\rho_n})$$
$$\times\left[\frac{|1+\lambda_1|}{\Gamma(\alpha+1)}+\frac{|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2})}\right]$$

Also we have

$$\begin{aligned}
\left|D^{\beta_i}(Fx)(t)\right| \leq\ & \frac{1}{\Gamma(\alpha-\beta_i)}\int_0^t(t-s)^{\alpha-\beta_i-1}\\
&\times\Big(m(s)+dr^{\rho}+d'\gamma_0^{\rho'}r^{\rho'}+d''\lambda_0^{\rho''}r^{\rho''}+d_1r^{\rho_1}+...+d_nr^{\rho_n}\Big)ds\\
&+\frac{|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)\Gamma(\frac{\alpha+\beta}{2})}\int_0^1(1-s)^{\frac{\alpha+\beta}{2}-1}\\
&\times\Big(m(s)+dr^{\rho}+d'\gamma_0^{\rho'}r^{\rho'}+d''\lambda_0^{\rho''}r^{\rho''}+d_1r^{\rho_1}+...+d_nr^{\rho_n}\Big)ds\\
&+\frac{|\lambda_1|}{\Gamma(\alpha-\beta_i)}\int_0^{\eta}(\eta-s)^{\alpha-1}\\
&\times\Big(m(s)+dr^{\rho}+d'\gamma_0^{\rho'}r^{\rho'}+d''\lambda_0^{\rho''}r^{\rho''}+d_1r^{\rho_1}+...+d_nr^{\rho_n}\Big)ds\\
&+\frac{|k|r}{\Gamma(\alpha-\beta-\beta_i)}\int_0^t(t-s)^{\alpha-\beta-\beta_i-1}ds\\
&+\frac{|\lambda_1 ak|r\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2})\Gamma(\alpha-\beta_i)}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}ds\\
&+\frac{|\lambda_1 k|r\Gamma(\alpha)}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta_i)}\int_0^{\eta}(\eta-s)^{\alpha-\beta-1}ds\\
\leq\ & \frac{\|m\|}{\Gamma(\alpha-\beta_i)}\left(\left(\frac{1-l}{\alpha-\beta_i-l}\right)^{1-l}+|\lambda_1|\left(\frac{1-l}{\alpha-\beta_i-l}\right)^{1-l}\right.\\
&\left.+\frac{|\lambda_1|}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l}\right)
\end{aligned}$$

$$\begin{aligned}
&+(dr^{\rho}+d'\gamma_0^{\rho'}r^{\rho'}+d''\lambda_0^{\rho''}r^{\rho''}+d_1r^{\rho_1}+...+d_nr^{\rho_n})\\
&\times\left(\frac{1}{\Gamma(\alpha-\beta_i+1)}+\frac{|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)\Gamma(\frac{\alpha+\beta}{2}+1)}+\frac{|\lambda_1|}{\alpha\Gamma(\alpha-\beta_i)}\right)\\
&+|k|r(\frac{1}{\Gamma(\alpha-\beta-\beta_i+1)}+\frac{|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)}\\
&+\frac{|\lambda_1|\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\beta_i)}\Big),
\end{aligned}$$

for all $i = 1; 2, ..., n$. Thus,

$$
\begin{aligned}
\|Fx\| \leq\ & \frac{|1+\lambda_1|\|m\|}{\Gamma(\alpha)}\left(\frac{1-l}{\alpha-l}\right)^{1-l} + \frac{|\lambda_1 a|\|m\|}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l} \\
& + \sum_{i=1}^{n}\frac{\|m\|}{\Gamma(\alpha-\beta_i)}\left(\left(\frac{1-l}{\alpha-\beta_i-l}\right)^{1-l} + |\lambda_1|\left(\frac{1-l}{\alpha-l}\right)^{1-l}\right. \\
& \left. + \frac{|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2})}\left(\frac{1-l}{\frac{\alpha+\beta}{2}-l}\right)^{1-l}\right) \\
& + (dr^{\rho} + d'\gamma_0^{\rho'} r^{\rho'} + d"\lambda_0^{\rho"} r^{\rho"} + d_1 r^{\rho_1} + ... + d_n r^{\rho_n}) \\
& \times\left[\frac{|1+\lambda_1|}{\Gamma(\alpha+1)} + \frac{|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2})} + \sum_{i=1}^{n}\frac{1}{\Gamma(\alpha-\beta_i)}\left(\frac{1}{\alpha-\beta_i} + \frac{|\lambda_1 a|\Gamma(\alpha)}{\frac{\alpha+\beta}{2}+1} + \frac{|\lambda_1|}{\alpha}\right)\right] \\
& + |k|r\left(\frac{|1+\lambda_1|}{\Gamma(\alpha-\beta+1)} + \frac{|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2}+1)} + \sum_{i=1}^{n}\frac{1}{\Gamma(\alpha-\beta-\beta_i+1)}\right. \\
& \left. + \frac{|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)} + \frac{|\lambda_1|\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\beta_i)}\right) \\
\leq\ & \frac{r(1-C)}{n+4}(n+4) + Cr = r.
\end{aligned}
$$

Hence, F maps B_r into B_r. Since f is continuous, it is easy to get that F is also continuous. Now we show that F is completely continuous operator on B_r. For each $x \in B_r$, put

$$
N = \max_{t \in I} f(t, x(t), (\phi x)(t), (\psi x)(t), D^{\beta_1}x(t), D^{\beta_2}x(t), ..., D^{\beta_n}x(t)).
$$

Let $t_1, t_2 \in I$, with $t_1 < t_2$, we have
$|(Fx)(t_2) - (Fx)(t_1)|$

$$
\begin{aligned}
=\ & \left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), (\phi x)(s), (\psi x)(s), D^{\beta_1}x(s), ..., D^{\beta_n}x(s))ds\right. \\
& - \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
& \times f(s, x(s), (\phi x)(s), (\psi x)(s), D^{\beta_1}x(s), ..., D^{\beta_n}x(s))ds \\
& - \frac{k}{\Gamma(\alpha-\beta)}\left(\int_0^{t_2}(t_2-s)^{\alpha-\beta-1}x(s)ds - \int_0^{t_1}(t_1-s)^{\alpha-\beta-1}x(s)\right) \\
& + \frac{\lambda_1 a k(t_2^{\alpha-1} - t_1^{\alpha-1})}{\Gamma(\frac{\alpha-\beta}{2})}\int_0^1 (1-s)^{\frac{\alpha-\beta}{2}-1}x(s)ds
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{\lambda_1 a(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\frac{\alpha+\beta}{2})}\int_0^1 (1-s)^{\frac{\alpha+\beta}{2}-1} \\
& \times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))ds \\
& -\frac{\lambda_1(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1} \\
& \times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))ds \\
& +\frac{\lambda_1 k(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\alpha-\beta)}\int_0^\eta (\eta-s)^{\alpha-\beta-1}x(s)ds \\
\leq\ & \frac{2N(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}+\frac{2|k|\|x\|(t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
& +\frac{N(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+1)}+\frac{|k|\|x\||t_2^{\alpha-\beta}-t_1^{\alpha-\beta}|}{\Gamma(\alpha-\beta+1)} \\
& +\left(\frac{N|\lambda_1|}{\Gamma(\alpha)+1}+\frac{N|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2}+1)}+\frac{|\lambda_1 ak|\|x\|}{\Gamma(\frac{\alpha-\beta}{2}+1)}\right. \\
& \left.+\frac{|\lambda_1 k|\|x\|}{\Gamma(\alpha-\beta+1)}\right)(t_2^{\alpha-1}-t_1^{\alpha-1}).
\end{aligned}
$$

On the other hand, for each $i\in 1,2,...,n$, we have
$|D^{\beta_i}(Fx)(t_2)-D^{\beta_i}(Fx)(t_1)|$

$$
\begin{aligned}
=\ & \left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)}\right. \\
& \times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))ds \\
& -\int_0^{t_1}\frac{(t_1-s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} \\
& \times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))ds \\
& -\frac{k}{\Gamma(\alpha-\beta-\beta_i)}\left(\int_0^{t_2}(t_2-s)^{\alpha-\beta-\beta_i-1}x(s)ds\right. \\
& \left.-\int_0^{t_1}(t_1-s)^{\alpha-\beta-\beta_i-1}x(s)ds\right) \\
& +\frac{\lambda_1 ak\Gamma(\alpha)(t_2^{\alpha-\beta_i-1}-t_1^{\alpha-\beta_i-1})}{\Gamma(\frac{\alpha-\beta}{2})\Gamma(\alpha-\beta_i)}\int_0^1(1-s)^{\frac{\alpha-\beta}{2}-1}x(s)ds \\
& -\frac{\lambda_1 a(t_2^{\alpha-\beta_i-1}-t_1^{\alpha-\beta_i-1})}{\Gamma(\frac{\alpha+\beta}{2})\Gamma(\alpha-\beta_i)}\int_0^1(1-s)^{\frac{\alpha+\beta}{2}-1} \\
& \times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))ds \\
& -\frac{\lambda_1(t_2^{\alpha-\beta_i-1}-t_1^{\alpha-\beta_i-1})}{\Gamma(\alpha-\beta_i)}\int_0^\eta(\eta-s)^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times f(s,x(s),(\phi x)(s),(\psi x)(s),D^{\beta_1}x(s),...,D^{\beta_n}x(s))ds \\
&+\frac{\lambda_1 k\Gamma(\alpha)(t_2^{\alpha-\beta_i-1}-t_1^{\alpha-\beta_i-1})}{\Gamma(\alpha-\beta)\Gamma(\alpha-\beta_i)}\int_0^\eta(\eta-s)^{\alpha-\beta-1}x(s)ds\Bigg| \\
\leq\ &\frac{2N(t_2-t_1)^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}+\frac{2|k|\|x\|(t_2-t_1)^{\alpha-\beta-\beta_i}}{\Gamma(\alpha-\beta-\beta_i+1)}+\frac{N(t_2^{\alpha-\beta_i}-t_1^{\alpha-\beta_i})}{\Gamma(\alpha-\beta_i+1)} \\
&+\frac{|k|\|x\||t_2^{\alpha-\beta-\beta_i}-t_1^{\alpha-\beta-\beta_i}|}{\Gamma(\alpha-\beta-\beta_i+1)}+\Bigg(\frac{N|\lambda_1|}{\Gamma(\alpha-\beta_i)}+\frac{N|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\frac{\alpha+\beta}{2}+1)\Gamma(\alpha-\beta_i)} \\
&+\frac{|\lambda_1 ak|\|x\|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)}+\frac{|\lambda_1 k|\|x\|\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\beta_i)}\Bigg)(t_2^{\alpha-1}-t_1^{\alpha-1}).
\end{aligned}
$$

Hence,

$\|Fx(t_2)-Fx(t_1)\|$

$$
\begin{aligned}
\leq\ &\frac{2N(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}+\frac{2|k|\|x\|(t_2-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{N(t_2^{\alpha}-t_1^{\alpha})}{\Gamma(\alpha+1)} \\
&+\frac{|k|\|x\||t_2^{\alpha-\beta}-t_1^{\alpha-\beta}|}{\Gamma(\alpha-\beta+1)}+\Bigg(\frac{N|\lambda_1|}{\Gamma(\alpha)+1}+\frac{N|\lambda_1 a|}{\Gamma(\frac{\alpha+\beta}{2}+1)} \\
&+\frac{\lambda_1|ak|\|x\|}{\Gamma(\frac{\alpha-\beta}{2}+1)}+\frac{|\lambda_1 k|\|x\|}{\Gamma(\alpha-\beta+1)}\Bigg)(t_2^{\alpha-1}-t_1^{\alpha-1}) \\
&+\sum_{i=1}^{n}\Bigg(\frac{2N(t_2-t_1)^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}+\frac{2|k|\|x\|(t_2-t_1)^{\alpha-\beta-\beta_i}}{\Gamma(\alpha-\beta-\beta_i+1)} \\
&+\frac{N(t_2^{\alpha-\beta_i}-t_1^{\alpha-\beta_i})}{\Gamma(\alpha-\beta_i+1)}+\frac{|k|\|x\||t_2^{\alpha-\beta-\beta_i}-t_1^{\alpha-\beta-\beta_i}|}{\Gamma(\alpha-\beta-\beta_i+1)}\Bigg) \\
&+\Bigg(\frac{N|\lambda_1|}{\Gamma(\alpha-\beta_i)}+\frac{N|\lambda_1 a|\Gamma(\alpha)}{\Gamma(\frac{\alpha+\beta}{2}+1)\Gamma(\alpha-\beta_i)} \\
&+\frac{|\lambda_1 ak|\|x\|\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)}+\frac{|\lambda_1 k|\|x\|\Gamma(\alpha)}{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\beta_i)}\Bigg)(t_2^{\alpha-1}-t_1^{\alpha-1}),
\end{aligned}
$$

which implies that $\|Fx(t_2)-Fx(t_1)\|\to 0$, as $t_1\to t_2$. Thus F is equicontinuous and uniformly bounded, by using the Arzela-Ascoli theorem, one can get that F is completely continuous, Now, by using Theorem (3.3), F has a fixed point in B_r. Therefore, problem (1) has a solution. □

Example 3.4. Let as consider the following boundary value problem

$$\begin{cases} \left(D^{\frac{7}{4}} + \frac{1}{17}D^{\frac{3}{4}}\right)x(t) = f\left(t, x(t), (\phi x)(t), (\psi x)(t), D^{\frac{2}{3}}x(t), D^{\frac{1}{3}}x(t)\right), 0 \le t \le 1 \\ D^{\frac{1}{2}}x(0) = 0, \quad \frac{1}{5}D^{\frac{1}{2}}x(1) + x(\frac{1}{3}) = 0, \end{cases} \tag{6}$$

where

$$\begin{aligned} & f\left(t, x(t), (\phi x)(t), (\psi x)(t), D^{\frac{2}{3}}x(t), D^{\frac{1}{3}}x(t)\right) \\ = & \frac{t}{21}\left(\frac{|x(t)|}{1+|x(t)|} + \tan^{-1}(D^{\frac{2}{3}}x(t)\right) + \frac{|D^{\frac{1}{3}}x(t)|}{\sqrt{t+441}(1+|D^{\frac{1}{3}}x(t)|)} \\ & + \frac{1}{21}\int_0^t \frac{e^{-(s-t)}}{21}x(s)ds + \frac{1}{21}\int_0^t \frac{e^{\frac{-(s-t)}{2}}}{21}x(s)ds + sin\pi t, \end{aligned}$$

and $\alpha = \frac{7}{4}$, $k = \frac{1}{17}$, $\beta = \frac{3}{4}, \eta = \frac{1}{3}$, $a = \frac{1}{5}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{1}{3}$, $\gamma(t,s) = \frac{e^{-(s-t)}}{21}$, $\lambda(t,s) = \frac{e^{\frac{-(s-t)}{2}}}{21}$, with

$$\gamma_0 = \frac{e-1}{21}, \quad \lambda_0 = \frac{2(e-1)}{21}, \quad \Gamma(\frac{10}{8}) \approx 0.906, \quad \Gamma(\frac{7}{4}) \approx 0.919, \quad \Gamma(\frac{17}{12}) \approx 0.886,$$

$$\Gamma(\frac{13}{12}) \approx 0.958, \quad \Gamma(\frac{5}{3}) \approx 0.902, \quad \Gamma(\frac{4}{3}) \approx 0.892.$$

We obtain

$$\begin{aligned} & \Big| f\left(t, x(t), (\phi x)(t), (\psi x)(t), D^{\frac{2}{3}}x(t), D^{\frac{1}{3}}y(t)\right) \\ & - f\left(t, y(t), (\phi y)(t), (\psi y)(t), D^{\frac{2}{3}}y(t), D^{\frac{1}{3}}y(t)\right)\Big| \\ \le & \frac{1}{21}\left(|x-y| + |\phi x - \phi y| + |\psi x - \psi y| + |D^{\frac{2}{3}}x - D^{\frac{2}{3}}y| + |D^{\frac{1}{3}}y - D^{\frac{1}{3}}y|\right), \end{aligned}$$

with $L = \frac{1}{21}$. It follows that

$$\frac{(1+\lambda_1)k}{\Gamma(\alpha-\beta+1)} + \frac{\lambda_1 ak}{\Gamma(\frac{\alpha-\beta}{2}+1)} + \sum_{i=1}^{2}\Big(\frac{k}{\Gamma(\alpha-\beta-\beta_i+1)} + \frac{\lambda_1 ak\Gamma(\alpha)}{\Gamma(\frac{\alpha-\beta}{2}+1)\Gamma(\alpha-\beta_i)}$$
$$+ \frac{\lambda_1 k\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)\Gamma(\alpha-\beta+1)}\Big) \approx 0.508,$$

$$\frac{(1+\lambda_1)}{\Gamma(\alpha+1)} + \frac{\lambda_1 a}{\Gamma(\frac{\alpha+\beta}{2}+1)} + \sum_{i=1}^{2}\frac{1}{\Gamma(\alpha-\beta_i)}\left(\frac{1}{\alpha-\beta_i} + \frac{\lambda_1 a\Gamma(\alpha)}{\Gamma(\frac{\alpha+\beta}{2}+1)} + \frac{\lambda_1}{\alpha}\right) \approx$$
6.114.

Finally we have that

$$\Delta' \approx 0,868 < 1.$$

therefore, by Corollary(3.2) the problem (6) has unique solution on $[0,1]$.

REFERENCES

[1] Ahmed Alsaedi, Bashir Ahmad, Existence of solutions for nonlinear fractional integro-differential equations with three-point nonlocal fractional boundary conditions, *Advances in Difference Equations* (2010), 1-10.

[2] N. Adjeroud, Existence of positive solutions for nonlinear fractional differential equations with multi-point boundary conditions, *The Australian Journal of Mathematical Analysis and Applications*, **14**, No. 2 (2017), 1-14.

[3] N. Adjeroud, The existence and uniqueness of solutions for a nonlinear fractional Caputo-Langevin equation with multi-point boundary conditions, *International Journal of Pure and Applied Mathematics*, **116**, No. 1 (2017), 15-30.

[4] Bashir Ahmad, K. Sotiris, Existence of solutions for impulsive anti-periodic boundary value problems of fractional order, *Taiwanese Journal of Mathematics*, No. 3 (2011), 981-993.

[5] Bashir Ahmad, K. Sotiris, Ntouyas and Ahmed Alsaedi, New existence results for nonlinear fractional differential equations with three-point integral Boundary Conditions, *Advances in Difference Equations*, (2011), 1-11.

[6] Bashir Ahmad, Sotiris Ntouyas, Ravi Agarwal, Ahmed Alsaedi, Existence results for sequential fractional integro-differential equations with nonlocal multi-point and strip conditions, *Boundary Value Problems*, (2016).

[7] Bashir Ahmed, Juan J. Nieto, Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, *Boundary Value Problems*, (2011).

[8] Dumitru Baleanu, Sayyedeh Zahra Nazemi, Shahram Rezapour, Existence and uniqueness of solutions for multi-term nonlinear fractional integro-differential equations, *Advances in Difference Equations*, No. 368 (2013), 1-17.

[9] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[10] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, B.V., Amsterdam, 2006.

[11] A.A. Kilbas, O.I. Marichev, S.G. Samko, *Fractional Integral and Derivatives (Theory and Applications)*, Gorodon and Breach.

[12] Eric R. Kaufmann, Kouadio David Yao, Existence of solutions for a nonlinear fractional order differential equation, *Electronic Journal of Qualitative Theory of Differential Equations* (2009), 1-9.

[13] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, Academic Press, San Diego, 1999.

[14] Xiaoyou Liu, Zhenhai Liu, Separated boundary value problem fractional differential equations depending on lower-order derivative, *Advances in Difference Equations*, No. 78 (2013).

[15] Xiaoyou Liu, Yiliang Liu, Fractional differential equations with fractional non-separated boundary conditions, *Electronic Journal of Differential Equations*, No. 25 (2013), 1-13.

[16] Ying Wang, Lishan Liu, Uniqueness and existence of positive solutions for the fractional integro-differential equation, *Boundary Value Problems* (2017).

[17] Zhanbing Bai, Haishen Lu, Positive solutions for boundary value problem of nonlinear fractional differential equation, *Mathematical Analysis and Applications*, **311** (2005), 495-505.