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# THEORY AND ANALYSIS OF $\psi$-FRACTIONAL DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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**ABSTRACT:** In this note, we study the boundary value problems (BVPs for short) for the differential equations with $\psi$-fractional derivative. Some new existence and uniqueness results are derived by means of the contraction mapping principle and Schaefer's fixed point theorem. Further, we discuss the Ulam-Hyers stability.

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## 1. INTRODUCTION

In this paper, we investigate the existence and Ulam-Hyers stability results for $\psi$-fractional boundary vaule problem of the form

$$^{c}\mathfrak{D}^{\alpha;\psi}\mathfrak{u}(t) = \mathfrak{F}(t, \mathfrak{u}(t)), \quad \text{for each } t \in \mathfrak{I} := [0, T],\ \alpha \in (0, 1), \tag{1.1}$$

$$a\mathfrak{u}(0) + b\mathfrak{u}(T) = c, \tag{1.2}$$

where $^{c}\mathfrak{D}^{\alpha;\psi}$ is the $\psi$-Caputo fractional derivative of order $\alpha$. Let $\mathfrak{F}, \psi$ : $\mathfrak{I}\times\mathfrak{R} \to \mathfrak{R}$ is a continuous function and $a, b, c$ are real constants with $a+b \neq 0$. Fractional differential equations (FDEs) have recently confirmed to be important tools in the modelling of many phenomena in different fields of science and engineering. There are various applications to problems in viscoelasticity, electrochemistry, control, porous media, electromagnetics, etc. (see [2, 4] and references therein). There has been a significant growth in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives in modern years; see the monographs of Hilfer[9], Podlubny[16] and Samko et al. [20]. The theoretical study of these kinds of differential equations is significant for the applicability on the reality. For that reason, as a part of theoretical study, the pre-knowledge of the existence of a solution to FDEs is the first action for finding the analytic solution. Many natural phenomena can be formulated by BVPs of FDEs. We mention here some works on FDEs with boundary conditions (see [3, 5, 6, 7, 8, 15] and references therein). Very recently, Ricardo Almeida [18] introduced the so-called $\psi$ -fractional derivative with respect to another function.

The rest of the paper is arranged as follows. In Section 2, we recall some useful preliminaries. In Section 3, we give some sufficient conditions of the existence of the solutions and Ulam-Hyers stability is considered in Section 4.

## 2. FUNDAMENTAL CONCEPTS

By $C(\mathfrak{I}, \mathfrak{R})$ we denote the Banach space of all continuous functions from $\mathfrak{I}$ into $\mathfrak{R}$ with the norm

$$\|\mathfrak{u}\|_{\infty} := \sup\left\{|\mathfrak{u}(t)| : \ t \in \mathfrak{I}\right\}.$$

For detailed study on $\psi$-fractional derivative, we refer to [18, 19].

**Definition 2.1.** Let $\alpha > 0$, $\mathfrak{I} = [0, T]$ be a finite or infinite interval, $\mathfrak{F}$ an integrable function defined on $\mathfrak{I}$ and $\psi \in C^1(\mathfrak{I}, \mathfrak{R})$ an increasing function such that $\psi^{'}(t) \neq 0$, for all $t \in \mathfrak{I}$. Factional integrals and fractional derivatives of a fuction $\mathfrak{F}$ with respect to another function $\psi$ are defined as follows:

$$I^{\alpha;\psi}\mathfrak{F}(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\mathfrak{F}(s)ds,$$

and

$$\begin{aligned} \mathfrak{D}^{\alpha;\psi}\mathfrak{F}(t) :=& \left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n I^{n-\alpha;\psi}\mathfrak{F}(t) \\ =& \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{n-\alpha-1}\mathfrak{F}(s)ds, \end{aligned}$$

respectively, where $n = [\alpha] + 1$.

We declare the following generalization of Gronwall's lemma for $\psi$-fractional derivative.It plays a vital role in the proof of Ulam-Hyers stability.

**Lemma 2.2.** *(see Theorem 3,([19])) Let $\mathfrak{Z}, \mathfrak{W} : [0, T] \to [0, \infty)$ be continuous functions where $T \leq \infty$. If $\mathfrak{W}$ is nondecreasing and there are constants $k \geq 0$ and $0 < \alpha < 1$ such that*

$$\mathfrak{Z}(t) \leq \mathfrak{W}(t) + k\int_0^t \psi^{'}(t)(\psi(t) - \psi(s))^{\alpha-1}\mathfrak{Z}(s)ds, \quad t \in [0, T],$$

*then*

$$\mathfrak{Z}(t) \leq \mathfrak{W}(t) + \int_0^t \left(\sum_{n=1}^{\infty} \frac{(k\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\psi(t) - \psi(s))^{n\alpha-1}\mathfrak{W}(s)\right) ds, \quad t \in [0, T].$$

**Remark 2.3.** Under the hypothesis of Lemma 2.2, let $\mathfrak{W}(t)$ be a nondecreasing function on $[0, T]$. Then we have $\mathfrak{Z}(t) \leq \mathfrak{W}(t)E_{\alpha;\psi}(k\Gamma(\alpha)(\psi(t))^{\alpha})$.

**Theorem 2.4.** *(Banach's fixed point theorem) Let $C$ be a non-empty closed subset of a Banach space $X$, then any contraction mapping $\mathfrak{P}$ of $C$ into itself has a unique fixed point.*

**Theorem 2.5.** *(Schaefer's fixed point theorem) Let $\mathfrak{P} : C(\mathfrak{I}, \mathfrak{R}) \to C(\mathfrak{I}, \mathfrak{R})$ completely continuous operator. If the set*

$$\kappa = \{u \in C(\mathfrak{I}, \mathfrak{R}) : \mathfrak{u} = \lambda\mathfrak{P}(\mathfrak{u}) \quad \textit{for some} \quad \lambda \in (0, T)\}$$

*is bounded, then* $\mathfrak{P}$ *has at least a fixed point.*

## 3. EXISTENCE RESULTS

Let us begin by defining what we point out by a solution of the problem (1.1)-(1.2).

**Definition 3.1.** A function $\mathfrak{u} \in C^1(\mathfrak{I}, \mathfrak{R})$ is said to be a solution of (1.1)-(1.2) if $\mathfrak{u}$ satisfied the equation ${}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{u}(t) = \mathfrak{F}(t, \mathfrak{u}(t))$ on $\mathfrak{I}$, and the condition $a\mathfrak{u}(0) + b\mathfrak{u}(T) = c$.

We need the following lemma to derive the existence of solutions for the problem (1.1)-(1.2).

**Lemma 3.2.** *Let* $\alpha \in (0,1)$ *and let* $\mathfrak{F}, \psi : \mathfrak{I} \to \mathfrak{R}$ *be continuous. A function* $\mathfrak{u}$ *is a solution of the* $\psi$*-fractional integral equation*

$$\mathfrak{u}(t) = \mathfrak{u}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\mathfrak{F}(s, \mathfrak{u}(s))ds \tag{3.1}$$

*if and only if* $\mathfrak{u}$ *is a solution of the initial value problem for the* $\psi$*-fractional differential equation*

$$ {}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{u}(t) = \mathfrak{F}(t, \mathfrak{u}(t)), \quad \textit{for each } t \in \mathfrak{I} := [0,T], \ \alpha \in (0,1), \tag{3.2}$$

$$\mathfrak{u}(0) = \mathfrak{u}_0. \tag{3.3}$$

Since a result of Lemma 3.2 we have the following result which is helpful in what follows.

**Lemma 3.3.** *Let* $\alpha \in (0,1)$ *and let* $\mathfrak{F}, \psi : \mathfrak{I} \to \mathfrak{R}$ *be continuous. A function* $\mathfrak{u}$ *is a solution of the* $\psi$*-fractional integral equation*

$$\begin{aligned} \mathfrak{u}(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\mathfrak{F}(s, \mathfrak{u}(s))ds \\ & - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)} \int_0^T \psi^{'}(s)(\psi(T) - \psi(s))^{\alpha-1}\mathfrak{F}(s, \mathfrak{u}(s))ds - c\right] \end{aligned} \tag{3.4}$$

*if and only if* $\mathfrak{u}$ *is a solution of the* $\psi$*-fractional BVP*

$$ {}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{u}(t) = \mathfrak{F}(t, \mathfrak{u}(t)), \quad \textit{for each } t \in \mathfrak{I} := [0,T], \ \alpha \in (0,1), \tag{3.5}$$

$$a\mathfrak{u}(0) + b\mathfrak{u}(T) = c. \tag{3.6}$$

We impose the following assumptions:

(A1) The function $\mathfrak{F} : \mathfrak{I} \times \mathfrak{R} \to \mathfrak{R}$ is continuous.

(A2) There exists a constants $\mathfrak{K} > 0$ such that

$$\left|\mathfrak{F}(t,\mathfrak{X}) - \mathfrak{F}(t,\overline{\mathfrak{X}})\right| \leq \mathfrak{K}\left|\mathfrak{X} - \overline{\mathfrak{X}}\right|, \text{ for each } t \in \mathfrak{I},\ \forall\ \mathfrak{X},\overline{\mathfrak{X}} \in \mathfrak{R}.$$

(A3) There exists a constant $\mathfrak{M}$ such that

$$|\mathfrak{F}(t,\mathfrak{X})| \leq \mathfrak{M} \quad \text{for each } t \in \mathfrak{I} \text{ and } \forall\ \mathfrak{X} \in \mathfrak{R}.$$

Our first result is derived from Banach fixed point theorem.

**Theorem 3.4.** *Suppose (A1)-(A2) hold. If*

$$\frac{\mathfrak{K}(\psi(T))^{\alpha}}{\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right) < 1, \tag{3.7}$$

*then the BVP* (1.1)-(1.2) *has only one solution on* $\mathfrak{I}$.

*Proof.* Convert the problem (1.1)-(1.2) into a fixed point problem. Let $\Phi = C(\mathfrak{I},\mathfrak{R})$. Consider the operator $\mathfrak{P} : \Phi \to \Phi$ defined by

$$\begin{aligned}\mathfrak{P}\mathfrak{u}(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,\mathfrak{u}(s))ds\\ &\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1} f(s,\mathfrak{u}(s))ds - c\right].\end{aligned} \tag{3.8}$$

Noticeably, the fixed points of the operator $\mathfrak{P}$ are solution of the problem (1.1)-(1.2). We shall employ the Banach contraction principle to verify that $\mathfrak{P}$ defined by (3.8) has a fixed point. We shall demonstrate that $\mathfrak{P}$ is a contraction.

Let $\mathfrak{u},\mathfrak{v} \in \Phi$. Then, for each $t \in \mathfrak{I}$ we have

$$\begin{aligned}|\mathfrak{P}(\mathfrak{u})(t) - \mathfrak{P}(\mathfrak{v})(t)| \leq &\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{F}(s,\mathfrak{u}(t)) - \mathfrak{F}(s,\mathfrak{v}(t))\right| ds\\ &- \frac{|b|}{|a+b|\,\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}\left|\mathfrak{F}(s,\mathfrak{u}(t)) - \mathfrak{F}(s,\mathfrak{v}(t))\right| ds\\ \leq &\frac{\mathfrak{K}\left\|\mathfrak{u}-\mathfrak{v}\right\|_{\infty}}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}ds\end{aligned}$$

$$- \frac{|b| \mathfrak{K} \|\mathfrak{u} - \mathfrak{v}\|_\infty}{|a+b| \Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} ds$$
$$\leq \left[ \frac{\mathfrak{K}(\psi(T))^\alpha}{\Gamma(\alpha+1)} \left( 1 + \frac{|b|}{|a+b|} \right) \right] \|\mathfrak{u} - \mathfrak{v}\|_\infty .$$

Therefore,

$$\|\mathfrak{P}(\mathfrak{u}) - \mathfrak{P}(\mathfrak{v})\|_\infty \leq \left[ \frac{\mathfrak{K}(\psi(T))^\alpha}{\Gamma(\alpha+1)} \left( 1 + \frac{|b|}{|a+b|} \right) \right] \|\mathfrak{u} - \mathfrak{v}\|_\infty .$$

As a result by (3.7), $\mathfrak{P}$ is a contraction. Since a result of Banach fixed point theorem, we deduce that $\mathfrak{P}$ has a fixed point which is a solution of the problem (1.1)-(1.2). □

The next result is derived from Schaefer's fixed point theorem.

**Theorem 3.5.** *Suppose (A1)-(A3) hold. Then the BVP* (1.1)-(1.2) *has at least one solution on* $\mathfrak{I}$.

*Proof.* We shall make use of Schaefer's fixed point theorem to verify that $\mathfrak{P}$ defined by (3.8) has a fixed point. The proof will be given in some stages.
**Stage 1:** The operator $\mathfrak{P}$ is continuous.

Let $\{\mathfrak{u}_n\}$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in $\Phi$. Then for each $t \in \mathfrak{I}$

$$\begin{aligned}
&|\mathfrak{P}(\mathfrak{u}_n)(t) - \mathfrak{P}(\mathfrak{u})(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathfrak{u}_n(s)) - \mathfrak{F}(s, \mathfrak{u}(s))| ds \\
&\quad + \frac{|b|}{|a+b| \Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathfrak{u}_n(s)) - \mathfrak{F}(s, \mathfrak{u}(s))| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \sup_{s \in \mathfrak{I}} |\mathfrak{F}(s, \mathfrak{u}_n(s)) - \mathfrak{F}(s, \mathfrak{u}(s))| ds \\
&\quad + \frac{|b|}{|a+b| \Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} \sup_{s \in \mathfrak{I}} |\mathfrak{F}(s, \mathfrak{u}_n(s)) - \mathfrak{F}(s, \mathfrak{u}(s))| ds \\
&\leq \frac{\|\mathfrak{F}(\cdot, \mathfrak{u}_n(\cdot)) - \mathfrak{F}(\cdot, \mathfrak{u}(\cdot))\|_\infty}{\Gamma(\alpha)} \left[ \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \right. \\
&\quad \left. + \frac{|b|}{|a+b|} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} ds \right] \\
&\leq \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} \left( 1 + \frac{|b|}{|a+b|} \right) \|\mathfrak{F}(\cdot, \mathfrak{u}_n(\cdot)) - \mathfrak{F}(\cdot, \mathfrak{u}(\cdot))\|_\infty .
\end{aligned}$$

Since $\mathfrak{F}$ is a continuous function, we have

$$\|\mathfrak{P}(\mathfrak{u}_n) - \mathfrak{P}(\mathfrak{u})\|_\infty \leq \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \|\mathfrak{F}(\cdot, \mathfrak{u}_n(\cdot)) - \mathfrak{F}(\cdot, \mathfrak{u}(\cdot))\|_\infty \quad \text{as } n \to \infty.$$

**Stage 2:** The operator $\mathfrak{P}$ maps bounded sets into bounded sets in $\Phi$.

In fact, it is sufficient to prove that for any $q > 0$, there exists a poitive constant $\zeta$ such that for each $\mathfrak{u} \in \mathfrak{D}_q = \{\mathfrak{u} \in \Phi : \ \|\mathfrak{u}\|_\infty \leq q\}$, we have $\|\mathfrak{P}(\mathfrak{u})\|_\infty \leq \zeta$.

By (A3) we have for each $t \in \mathfrak{I}$

$$\begin{aligned}
|\mathfrak{P}(\mathfrak{u})(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathfrak{u}(s))| \, ds \\
&\quad + \frac{|b|}{|a+b|} \frac{1}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathfrak{u}(s))| \, ds + \frac{|c|}{|a+b|} \\
&\leq \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \\
&\quad + \frac{|b|}{|a+b|} \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} ds + \frac{|c|}{|a+b|} \\
&\leq \frac{\mathfrak{M}}{\Gamma(\alpha+1)} (\psi(T))^\alpha + \frac{\mathfrak{M}\,|b|}{|a+b|\,\Gamma(\alpha+1)} (\psi(T))^\alpha + \frac{|c|}{|a+b|}.
\end{aligned}$$

Therefore,

$$\|\mathfrak{P}(\mathfrak{u})\|_\infty \leq \frac{\mathfrak{M}}{\Gamma(\alpha+1)} (\psi(T))^\alpha + \frac{\mathfrak{M}\,|b|}{|a+b|\,\Gamma(\alpha+1)} (\psi(T))^\alpha + \frac{|c|}{|a+b|} := \zeta.$$

**Stage 3:** The operator $\mathfrak{P}$ maps bounded sets into equicontinuous sets of $\Phi$.

Let $t_1, t_2 \in \mathfrak{I}$, $t_1 < t_2$, $\mathfrak{D}_q$ be a bounded set of $\Phi$ as in Stage 2, and let $\mathfrak{u} \in \mathfrak{D}_q$. Then

$$\begin{aligned}
&|\mathfrak{P}(\mathfrak{u})(t_2) - \mathfrak{P}(\mathfrak{u})(t_1)| \\
&\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s) \left[ (\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1} \right] \mathfrak{F}(s, \mathfrak{u}(s)) ds \right. \\
&\qquad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathfrak{u}(s)) ds \right| \\
&\leq \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s) \left[ (\psi(t_2) - \psi(s)))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1} \right] ds
\end{aligned}$$

$$+\frac{\mathfrak{M}}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi^{'}(s)(\psi(t_2)-\psi(s))^{\alpha-1}ds$$
$$\leq\frac{\mathfrak{M}}{\Gamma(\alpha+1)}(\psi(t_2)-\psi(t_1))^{\alpha}+\frac{\mathfrak{M}}{\Gamma(\alpha+1)}((\psi(t_1))^{\alpha}-(\psi t_2))^{\alpha}).$$

Since $t_1 \to t_2$, the right-hand side of above inequality tends to 0. Because, a result of Stage1 to 3 together with Arzela-Ascoli theorem, we can finish that $\mathfrak{F}:\Phi\to\Phi$ is continuous and completely continuous.

**Stage 4:** A priori bounds.

Now it remains to prove that the set

$$\kappa=\{\mathfrak{u}\in\Phi:\ \mathfrak{u}=\lambda\mathfrak{P}(\mathfrak{u})\quad\text{for some }\lambda\in(0,1)\}$$

is bounded.

Let $\mathfrak{u}\in\kappa$, then $\lambda\mathfrak{P}(\mathfrak{u})$ for some $\lambda\in(0,1)$. Hence, for each $t\in\mathfrak{I}$ we have

$$\mathfrak{u}(t)=\lambda\left[\frac{1}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{F}(s,\mathfrak{u}(s))ds\right.$$
$$\left.-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T\psi^{'}(s)(\psi(T)-\psi(s))^{\alpha-1}\mathfrak{F}(s,\mathfrak{u}(s))ds-c\right]\right].$$

We complete this stage by considering the estimation in Stage 2. The same as a result of Schaefer's fixed point theorem, we finish the proof that $\mathfrak{P}$ has fixed point which is solution of the problem (1.1)-(1.2). □

## 4. ULAM-HYERS-RASSIAS STABILITY

In this part, we study the Ulam stability of BVP for $\psi$-fractional differential equations (1.1)-(1.2). There are many works on the Ulam-stability of solutions for FDEs. We mention here some works [1, 10, 11, 12, 13, 14, 17, 21]; also see the references cited therein.A similar idea can be found in [7]; but there is no work on Ulam stability results for $\psi$- fractional differential equations with boundary conditions. Now we consider the Ulam stability for the following problem

$$^{c}\mathfrak{D}^{\alpha;\psi}\mathfrak{u}(t)=\mathfrak{F}(t,\mathfrak{u}(t)),\quad t\in\mathfrak{I}:=[0,T],\tag{4.1}$$

and the following inequalities:

$$\left|{}^{c}\mathfrak{D}^{\alpha;\psi}\mathfrak{z}(t) - \mathfrak{F}(t,\mathfrak{z}(t))\right| \le \epsilon, \quad t \in \mathfrak{I}, \tag{4.2}$$

$$\left|{}^{c}\mathfrak{D}^{\alpha;\psi}\mathfrak{z}(t) - \mathfrak{F}(t,\mathfrak{z}(t))\right| \le \epsilon\varphi(t), \quad t \in \mathfrak{I}, \tag{4.3}$$

$$\left|{}^{c}\mathfrak{D}^{\alpha;\psi}\mathfrak{z}(t) - \mathfrak{F}(t,\mathfrak{z}(t))\right| \le \varphi(t), \quad t \in \mathfrak{I}. \tag{4.4}$$

**Definition 4.1.** The equation (4.1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{z} \in \Phi$ of the inequality (4.2) there exists a solution $\mathfrak{u} \in \Phi$ of equation (4.1) with

$$|\mathfrak{z}(t) - \mathfrak{u}(t)| \le C_f\epsilon, \; t \in J.$$

**Definition 4.2.** The equation (4.1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C\left([0,\infty),[0,\infty)\right), \psi_f(0) = 0$ such that for each solution $\mathfrak{z} \in \Phi$ of the inequality (4.2) there exists a solution $\mathfrak{u} \in \Phi$ of equation (4.1) with

$$|\mathfrak{z}(t) - \mathfrak{u}(t)| \le \psi_f\epsilon, \quad t \in J.$$

**Definition 4.3.** The equation (4.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in \Phi$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{z} \in \Phi$ of the inequality (4.3) there exists a solution $\mathfrak{u} \in \Phi$ of equation (4.1) with

$$|\mathfrak{z}(t) - \mathfrak{u}(t)| \le C_f\epsilon\varphi(t), \quad t \in J.$$

**Definition 4.4.** The equation (4.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in \Phi$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{z} \in \Phi$ of the inequality (4.4) there exists a solution $\mathfrak{u} \in \Phi$ of equation (4.1) with

$$|\mathfrak{z}(t) - \mathfrak{u}(t)| \le C_{f,\varphi}\varphi(t), \quad t \in J.$$

**Remark 4.5.** A function $\mathfrak{z} \in \Phi$ is a solution of (4.2) if and only if there exists a function $g \in \Phi$ (which depend on $\mathfrak{z}$) such that

1. $|g(t)| \le \epsilon$, $t \in \mathfrak{I}$;

2. ${}^{c}\mathfrak{D}^{\alpha;\psi}\mathfrak{z}(t) = \mathfrak{F}(t,\mathfrak{z}(t)) + g(t)$, $t \in \mathfrak{I}$.

**Remark 4.6.** Let $\alpha \in (0,1)$, if $\mathfrak{Z} \in \Phi$ is a solution of the inequality (4.2), then $\mathfrak{Z}$ is a solution of the following inequality

$$\left|\mathfrak{Z}(t) - \mathfrak{A}_{\mathfrak{z}} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{F}(s,\mathfrak{Z}(s))ds\right| \leq \epsilon \frac{(\psi(T))^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right).$$

In fact, by Remark 4.5, we have that

$${}^{c}\mathfrak{D}^{\alpha;\psi}\mathfrak{Z}(t) = \mathfrak{F}(t,\mathfrak{Z}(t)) + g(t), \quad t \in \mathfrak{I}.$$

Then

$$\begin{aligned}\mathfrak{Z}(t) =& \mathfrak{A}_{\mathfrak{z}} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{F}(s,\mathfrak{Z}(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s)ds \\ &- \left(\frac{b}{a+b}\right)\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s)ds, \quad t \in \mathfrak{I},\end{aligned}$$

with

$$\mathfrak{A}_{\mathfrak{Z}} = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{F}(s,\mathfrak{Z}(s))ds\right].$$

From this it follows that

$$\begin{aligned}&\left|\mathfrak{Z}(t) - \mathfrak{A}_{\mathfrak{z}} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{F}(s,\mathfrak{Z}(s))ds\right| \\ =&\left|\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s)ds \right.\\ &\left. - \left(\frac{b}{a+b}\right)\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s)ds\right| \\ \leq&\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s)|\,ds \\ &- \left(\frac{b}{a+b}\right)\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s)|\,ds \\ \leq&\epsilon\frac{(\psi(T))^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right).\end{aligned}$$

**Remark 4.7.** Undoubtedly,

- Definition 4.1 $\Rightarrow$ Definition 4.2.
- Definition 4.3 $\Rightarrow$ Definition 4.4.

**Remark 4.8.** A solution of the $\psi$-fractional diferential equations with boundary condition inequality (4.2) is called an $\epsilon$-solution of the problem (4.1).

**Theorem 4.9.** *Suppose (A1)-(A2) and* (3.7) *hold. Then, the problem* (1.1)-(1.2) *is Ulam-Hyers stable.*

*Proof.* Let $\epsilon > 0$ and let $\mathfrak{Z} \in \Phi$ be a function which satisfies inequality (4.2) and let $\mathfrak{u} \in \Phi$ be the unique solution of the following problem

$$\begin{aligned} &{}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{u}(t) = \mathfrak{F}(t, \mathfrak{u}(t)), \quad t \in \mathfrak{I},\ \alpha \in (0,1), \\ &\mathfrak{u}(0) = \mathfrak{Z}(0),\ \mathfrak{u}(T) = \mathfrak{Z}(T). \end{aligned}$$

Using Lemma 3.3, we obtain

$$\mathfrak{u}(t) = \mathfrak{A}_{\mathfrak{u}} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\mathfrak{F}(s, \mathfrak{u}(s))ds.$$

Alternatively , if $\mathfrak{u}(0) = \mathfrak{Z}(0),\ \mathfrak{u}(T) = \mathfrak{Z}(T)$, then $\mathfrak{A}_{\mathfrak{u}} = \mathfrak{A}_{\mathfrak{Z}}$.
In fact,

$$\begin{aligned} |\mathfrak{A}_{\mathfrak{u}} - \mathfrak{A}_{\mathfrak{Z}}| &\le \frac{|b|}{|a+b|\,\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \left|\mathfrak{F}(s, \mathfrak{u}(s)) - \mathfrak{F}(s, \mathfrak{Z}(s))\right| ds \\ &\le \frac{\mathfrak{K}\,|b|}{|a+b|} I^{\alpha;\psi} \left|\mathfrak{u}(T) - \mathfrak{Z}(T)\right| \\ &= 0. \end{aligned}$$

Therefore, $\mathfrak{A}_u = \mathfrak{A}_{\mathfrak{Z}}$. We have

$$\mathfrak{u}(t) = \mathfrak{A}_{\mathfrak{Z}} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\mathfrak{F}(s, \mathfrak{u}(s))ds.$$

By integration of the inequality (4.2) and using Remark 4.6, we obtain

$$\begin{aligned} &\left|\mathfrak{Z}(t) - \mathfrak{A}_{\mathfrak{z}} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\mathfrak{F}(s, \mathfrak{Z}(s))ds\right| \\ &\qquad\qquad \le \epsilon \frac{(\psi(T))^{\alpha}}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right). \end{aligned}$$

We have for any $t \in \mathfrak{I}$

$$\begin{aligned}|\mathfrak{Z}(t)-\mathfrak{u}(t)| \leq & \left|\mathfrak{Z}(t)-\mathfrak{A}_{\mathfrak{z}}-\frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{F}(s,\mathfrak{Z}(s))ds\right| \\ & +\frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{F}(s,\mathfrak{u}(s))-\mathfrak{F}(s,\mathfrak{Z}(s))\right| ds \\ \leq & \epsilon\frac{(\psi(T))^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right) \\ & +\frac{\mathfrak{K}}{\Gamma(\alpha)}\int_0^t \psi^{'}(t)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{Z}(s)-\mathfrak{u}(s)\right| ds.\end{aligned}$$

Using Gronwall inequality, Lemma 2.2 and Remark 2.3, we obtain

$$|\mathfrak{Z}(t)-\mathfrak{u}(t)| \leq \left(1+\frac{|b|}{|a+b|}\right)\frac{\epsilon(\psi(T))^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha;\psi}\left(\mathfrak{K}(\psi(T))^{\alpha}\right).$$

Thus, the problem (1.1)-(1.2) is Ulam-Hyers stable. □

**Theorem 4.10.** *Suppose (A1)-(A2), inequality* (3.7) *and*

*(A4) there exists an increasing function* $\varphi \in \Phi$ *and* $\lambda_{\varphi} > 0$ *such that*

$$I^{\alpha;\psi}\varphi(t) \leq \lambda_{\varphi}\varphi(t), \quad \text{for each} \quad t \in \mathfrak{I}$$

*hold. Then the problem* (1.1)-(1.2) *is Ulam-Hyers-Rassias stable.*

**Remark 4.11.** Under the assumptions of Theorem 4.9, we consider the problem (1.1)-(1.2) and the inequality (4.4). One can repeat the same process to verify that the problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable.

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