Communications in Applied Analysis, **22**, No. 3 (2018), 383-399 ISSN: 1083-2564

EXISTENCE OF SOLUTIONS FOR A PERTURBED SECOND ORDER PROBLEM ON THE HALF-LINE VIA EKELAND'S VARIATIONAL PRINCIPLE

DAHMANE BOUAFIA[1], JOHN R. GRAEF[2],
AND TOUFIK MOUSSAOUI[3]

[1]Laboratory of Fixed Point Theory and Applications
École Normale Supérieure
Kouba, Algiers, ALGERIA

[2]Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403, USA

[3]Laboratory of Fixed Point Theory and Applications
Department of Mathematics E.N.S.
Kouba, Algiers, ALGERIA

ABSTRACT: In this paper, the authors discuss the existence of nontrivial solutions to a perturbed second order problem on the half-line. The Ekeland variational principle plays an essential role in the proof.

AMS Subject Classification: 34B15, 34B18, 34B40

Received: January 7, 2018; **Accepted:** May 29, 2018;
Published: June 1, 2018. **doi:** 10.12732/caa.v22i3.4
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION

Variational methods and critical point theory have proved to be very useful and popular techniques in proving existence of solutions to boundary value problems in a variety of settings including ordinary differential equations, partial differential equations, impulsive problems, etc. As some recent examples of such results we refer the reader to the papers [2, 4, 6, 7, 8, 9, 10, 15, 17, 18]. By comparison to problems on a finite interval, somewhat fewer results for boundary value problems on the half-line have appeared in the literature; recent examples include [9, 10, 13, 17]. A great deal of work has been done on second order differential equations, and due to their significance in physics, many researchers have studied extensively the existence and multiplicity of solutions to boundary value problems (BVPs) on the half-line. Many results for the existence of solutions, positive solutions, multiple solutions have been obtained by using fixed point theory and topological degree; see for example [8, 9, 21] and the references contained therein. There are some results on boundary value problems using variational methods to investigate the existence and multiplicity of solutions for perturbed second-order differential equation on the half-line (see [21, 14, 16]).

One approach that perhaps is not as well-known as some others, but has been shown to be effective for problems of this type, is Ekeland's variational principle. In this paper we will study the existence of solutions to the perturbed Dirichlet problem

$$\begin{cases} -u''(x) + \eta u(x) = h(x)u^r(x) + \lambda q(x) f(x, u(x)), & x \in [0, +\infty), \\ u(0) = u(+\infty) = 0, \end{cases} \tag{1.1}$$

where $f : [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is a continuous function, η and λ are positive real parameters, and $0 < r < 1$.

In the case of $\eta = 1$ and the absence of a perturbation term, i.e., $h(x) \equiv 0$, this problem was considered by Bouafia, Moussaoui, and O'Regan [2]. They studied the existence of positive solutions to this special case of problem (1.1) by applying Ekeland's variational principle. As we will point out below, their main result, namely, [2, Theorem 2.1] is included as a special case of the results obtained in this paper. Here, we shall prove that (1.1) possesses at least one solution. Our nonlinearity $f \in C([0, +\infty) \times \mathbb{R}; \mathbb{R})$ satisfies less restrictive assumptions than in [2]. In particular, our results are obtained without using a condition at infinity as in paper [2]. Moreover, near zero we compare the nonlinearity f to u^r $(0 < r < 1)$ rather than u. (See condition (H_4) below compared to condition (H_2) in [2].)

Throughout this paper we assume that the following assumptions are satisfied:

(H_0) There are constants a, $b \in \mathbb{R}^+ \setminus \{0\}$ such that

$$|f(x,u)| \leq a|u|^r + b, \quad \text{for all } x \in \mathbb{R}^+ \text{ and } u \in \mathbb{R};$$

(H_1) h, q, and $\frac{q}{h} : [0,+\infty) \longrightarrow (0,+\infty)$ belong to $L^1[0,+\infty) \cap L^\infty[0,+\infty)$;

(H_2) There is a continuously differentiable and bounded function $p : [0,+\infty) \longrightarrow (0,+\infty)$ such that the functions $\frac{h}{p^r}$, $\frac{h}{p^{r+1}}$, $\frac{q}{p^r}$, $\frac{q}{p^2}$, and $\frac{h^{\frac{2}{1-r}}}{q^{\frac{r+1}{1-r}}}$ all belong to $L^1[0,+\infty)$;

(H_3) $M = \max\{\|p\|_{L^2}, \|p'\|_{L^2}\} < +\infty$,

$$M_0 = \int_0^{+\infty} q(x) \left(\int_0^x \frac{ds}{p(s)} \right) dx < +\infty,$$

$$M^\star = \left[\int_0^{+\infty} h(x) \left(\int_0^x \frac{ds}{p(s)} \right)^{\frac{r+1}{2}} dx \right]^{\frac{1}{r+1}} < +\infty,$$

and

$$M_1 = \left[\sup_{x \geq 0} p(x) \right]^{\frac{1}{2}} M^\star;$$

(H_4) $f(x,0) = 0$, $\displaystyle\lim_{u \to 0^+} \frac{f(x,u)}{u^r} = +\infty$ uniformly for $x \in [0,+\infty)$.

We now introduce a Hilbert space that will be an appropriate setting for our problem. Let

$$H_0^1(0,+\infty) = \left\{ u \text{ measurable} : \; u, \; u' \in \; L^2(0,+\infty), \; u(0) = u(+\infty) = 0 \right\}$$

with the natural norm

$$\|u\| = \left(\int_0^{+\infty} |u'(x)|^2 dx + \eta \int_0^{+\infty} |u(x)|^2 dx \right)^{\frac{1}{2}}$$

and endowed with the inner product

$$(u,v)=\int_0^{+\infty} u'(x)\cdot v'(x)dx+\eta\int_0^{+\infty} u(x)\cdot v(x)dx.$$

Let the space $C_{l,p}[0,+\infty)$ be defined by

$$C_{l,p}[0,+\infty)=\{u\in C([0,+\infty),\mathbb{R}):\lim_{x\to+\infty} p(x)u(x)\ \text{ exists }\}$$

with the corresponding norm given by

$$\|u\|_{\infty,p}=\sup_{x\in[0,+\infty)} p(x)|u(x)|.$$

We also consider the spaces $L_q^2(0,+\infty)$ and $L_h^{r+1}(0,+\infty)$ defined by

$$L_q^2(0,+\infty)=\{u:(0,+\infty)\to\mathbb{R} \text{ is measurable and } \sqrt{q}u\in L^2(0,+\infty)\}$$

and

$$L_h^{r+1}(0,+\infty)=\{u:(0,+\infty)\to\mathbb{R} \text{ is measurable and } \int_0^{+\infty} h(x)|u|^{r+1}dx<+\infty\},$$

equipped with their respective norms

$$\|u\|_{L_q^2}=\left(\int_0^{+\infty} q(x)u^2(x)dx\right)^{\frac{1}{2}} \text{ and } \quad \|u\|_{L_h^{r+1}}=\left(\int_0^{+\infty} h(x)|u|^{r+1}dx\right)^{\frac{1}{r+1}}. \tag{1.2}$$

Next we give some needed lemmas.

Lemma 1.1. ([12]) *$H_0^1(0,+\infty)$ is embedded continuously and compactly in $C_{l,p}[0,+\infty)$.*

Lemma 1.2. ([2]) *$C_{l,p}[0,+\infty)$ is continuously embedded in $L_q^2(0,+\infty)$.*

An immediate consequence of these two lemmas is the following result.

Corollary 1.3. ([2]) *$H_0^1(0,+\infty)$ is continuously and compactly embedded in $L_q^2(0,+\infty)$.*

We will also need the following results in the remainder of our paper.

Lemma 1.4. *$C_{l,p}[0,+\infty)$ is continuously embedded in $L_h^{r+1}(0,+\infty)$.*

Proof. For all $u \in C_{l,p}[0,+\infty)$, we have

$$\begin{aligned}
\|u\|_{L_h^{r+1}}^{r+1} &= \int_0^{+\infty} h(x)|u|^{r+1}(x)dx \\
&= \int_0^{+\infty} \frac{h(x)}{p^{r+1}(x)} p^{r+1}(x)|u|^{r+1}(x)dx \\
&\leq \|u\|_{\infty,p}^{r+1} \int_0^{+\infty} \frac{h(x)}{p^{r+1}(x)}dx = \left\|\frac{h}{p^{r+1}}\right\|_{L^1} \|u\|_{\infty,p}^{r+1}.
\end{aligned}$$

Thus, $\|u\|_{L_h^{r+1}} \leq C_1\|u\|_{\infty,p}$, where $C_1 = \left(\left\|\frac{h}{p^{r+1}}\right\|_{L^1}\right)^{\frac{1}{r+1}}$, and this proves the lemma. □

Lemma 1.5. *$L_q^2(0,+\infty)$ is continuously embedded in $L_h^{r+1}(0,+\infty)$.*

Proof. For all $u \in L_q^2(0,+\infty)$, we have

$$\begin{aligned}
\|u\|_{L_h^{r+1}}^{r+1} &= \int_0^{+\infty} h(x)|u|^{r+1}(x)dx \\
&\leq \int_0^{+\infty} \frac{h(x)}{q^{\frac{r+1}{2}}(x)} q^{\frac{r+1}{2}}(x)|u|^{r+1}(x)dx \\
&\leq \left(\int_0^{+\infty} q(x)u^2(x)dx\right)^{\frac{r+1}{2}} \left(\int_0^{+\infty} \left(\frac{h(x)}{q^{\frac{r+1}{2}}(x)}\right)^{\frac{2}{1-r}} dx\right)^{\frac{1-r}{2}} \\
&= \left(\int_0^{+\infty} \frac{h^{\frac{2}{1-r}}(x)}{q^{\frac{r+1}{1-r}}(x)}dx\right)^{\frac{1-r}{2}} \|u\|_{L_q^2}^{r+1}.
\end{aligned}$$

Hence,

$$\|u\|_{L_h^{r+1}} \leq C_2\|u\|_{L_q^2}, \text{ where } C_2 = \left(\int_0^{+\infty} \frac{h^{\frac{2}{1-r}}(x)}{q^{\frac{r+1}{1-r}}(x)}dx\right)^{\frac{1-r}{2(1+r)}}.$$

This proves the lemma. □

The following corollary is immediate.

Corollary 1.6. *$H_0^1(0,+\infty)$ is compactly embedded in $L_h^{r+1}(0,+\infty)$.*

We denote by μ_1 the first eigenvalue of the problem

$$\begin{cases} -u''(x) + \eta u(x) = \mu h(x) u^r(x), \quad x \geq 0, \\ u(0) = u(+\infty) = 0, \end{cases} \tag{1.3}$$

namely,

$$\mu_1 = \inf_{u \in H_0^1 \setminus \{0\}} \frac{\|u\|}{\|u\|_{L_h^{r+1}}}.$$

Lemma 1.7. *The first eigenvalue μ_1 is positive and corresponds to a positive eigenfunction $\psi_1 \in H_0^1(0,+\infty) \setminus \{0\}$.*

Proof. Using the same idea as in [1], for $u \in H_0^1(0,+\infty)$, let

$$N_1(u) = \|u\| \quad \text{and} \quad N_2(u) = \|u\|_{L_h^{r+1}},$$

and define the functional $P : H_0^1(0,+\infty) \setminus \{0\} \to \mathbb{R}$ by

$$P(u) = \frac{N_1(u)}{N_2(u)}.$$

Then,

$$\mu_1 = \inf_{u \in H_0^1 \setminus \{0\}} P(u).$$

Let $u \in H_0^1(0,+\infty)$; then for $x > 0$,

$$\begin{aligned} |u(x)|^{r+1} &= \left| \int_0^x u'(s)ds \right|^{r+1} = \left| \int_0^x \sqrt{p(s)} u'(s) \frac{1}{\sqrt{p(s)}} ds \right|^{r+1} \\ &\leq \left(\int_0^x p(s) u'^2(s) ds \right)^{\frac{r+1}{2}} \left(\int_0^x \frac{ds}{p(s)} \right)^{\frac{r+1}{2}} \\ &\leq \left[\sup_{x \geq 0} p(x) \right]^{\frac{r+1}{2}} \left(\int_0^{+\infty} u'^2(s) ds \right)^{\frac{r+1}{2}} \left(\int_0^x \frac{ds}{p(s)} \right)^{\frac{r+1}{2}}. \end{aligned}$$

Hence, for $x > 0$,

$$h(x)|u(x)|^{r+1} \leq \left[\sup_{x \geq 0} p(x) \right]^{\frac{r+1}{2}} h(x) \left(\int_0^x \frac{ds}{p(s)} \right)^{\frac{r+1}{2}} \left(\int_0^{+\infty} u'^2(s) ds \right)^{\frac{r+1}{2}},$$

so

$$\|u\|_{L_h^{r+1}} \leq M_1 \|u\|, \tag{1.4}$$

which gives

$$\mu_1 = \inf_{u \in H_0^1 \setminus \{0\}} \frac{\|u\|}{\|u\|_{L_h^{r+1}}} \geq \frac{1}{M_1} > 0\cdot$$

Let (u_n) be a minimizing sequence for this quantity. Then we know that $(|u_n|)$ is a minimizing sequence for P, so we can assume that $u_n(x) \geq 0$ for $x \in [0, +\infty)$. The functional P is homogeneous of degree zero, i.e., $P(\alpha u) = P(u)$, for every $\alpha \in \mathbb{R}$. Setting $\widetilde{u_n} = \frac{u_n}{\|u_n\|_{L_h^{r+1}}}$, for every n, we can assume that $\|u_n\|_{L_h^{r+1}} = 1$. Note that $\lim_{n \to +\infty} P(u_n) = \inf_{u \in H_0^1 \setminus \{0\}} P(u) = \mu_1$, so the sequence $(P(u_n))$ is bounded. From this and the fact that $P(u_n) = \|u_n\|$, we see that (u_n) is bounded in H_0^1. From Lemma 1.1 and the reflexivity and separability of H_0^1, there exists a subsequence (u_{n_k}) of (u_n) such that, as $k \to +\infty$,

$$\begin{cases} u_{n_k} \rightharpoonup \overline{u}, & \text{in } H_0^1, \\ u_{n_k} \to \overline{u}, & \text{in } C_{l,p}. \end{cases}$$

Therefore, $u_{n_k}(x) \to \overline{u}(x)$ for all $x \in [0, +\infty)$. By Lemma 1.4, (u_{n_k}) converges in norm to $\overline{u}$ in L_h^{r+1}. Thus, $\|\overline{u}\|_{L_h^{r+1}} = 1$ and $\overline{u}(x) \geq 0$, for $x \in [0, +\infty)$. Finally, by the weak lower semicontinuity of the norm, we obtain

$$P(\overline{u}) = N_1(\overline{u}) \leq \liminf_k N_1(u_{n_k}) = \liminf_k P(u_{n_k}) = \mu_1.$$

Thus, $\overline{u} \in H_0^1 \setminus \{0\}$ and $P(\overline{u}) = \mu_1$. □

We also need the following weak Ekeland variational principle.

Theorem 1.8. ([11]) *(Weak Ekeland variational principle) Let (E, d) be a complete metric space and let $J : E \to \mathbb{R}$ be a lower semi-continuous functional that is bounded from below. Then, for each $\varepsilon > 0$, there exists $u_\varepsilon \in E$ with*

$$J(u_\varepsilon) \leq \inf_E J + \varepsilon,$$

and if $w \in E$ with $w \neq u_\varepsilon$, then

$$J(u_\varepsilon) < J(w) + \varepsilon d(u_\varepsilon, w).$$

2. MAIN RESULTS

We denote by F the primitive of f with respect to its second variable, i.e., $F(x,u) = \int\limits_0^u f(x,s)ds$. To begin, we define the Euler-Lagrange functional associated with problem (1.1). Let $J : H_0^1(0,+\infty) \to \mathbb{R}$ be defined by

$$J(u) = \frac{1}{2}\int\limits_0^{+\infty}\Big(u'^2(x) + \eta u^2(x)\Big)dx - \frac{1}{r+1}\int\limits_0^{+\infty} h(x)|u(x)|^{r+1}dx - \lambda \int\limits_0^{+\infty} q(x)F(x,u(x))dx.$$

Proposition 2.1. Assume that conditions (H_0)–(H_2) hold. Then the functional J is continuously differentiable and its Fréchet derivative is given by

$$\langle J'(u), v\rangle = \int\limits_0^{+\infty}\Big(u'(x)v'(x) + \eta u(x)v(x)\Big)dx - \int\limits_0^{+\infty} h(x)u^r(x)v(x)dx - \lambda\int\limits_0^{+\infty} q(x)f(x,u)v(x)dx$$

for all $v \in H_0^1(0,+\infty)$.

Proof. First we will show that J is Gâteaux-differentiable. For all $x \geq 0$, $v \in H_0^1(0,+\infty)$, and any $t > 0$, we have

$$\begin{aligned}
J(u+tv) - J(u) = {} & \frac{1}{2}\int\limits_0^{+\infty}\Big(|(u+tv)'(x)|^2 + \eta|(u+tv)(x)|^2\Big)dx \\
& - \frac{1}{r+1}\int\limits_0^{+\infty} h(x)|(u+tv)(x)|^{r+1}dx \\
& - \lambda\int\limits_0^{+\infty} q(x)F(x,(u+tv)(x))dx \\
& - \frac{1}{2}\int\limits_0^{+\infty}\Big(|u'(x)|^2 + \eta|u(x)|^2\Big)dx + \frac{1}{r+1}\int\limits_0^{+\infty} h(x)|u(x)|^{r+1}dx
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda \int_0^{+\infty} q(x)F(x, u(x))dx \\
&= \frac{t^2}{2} \int_0^{+\infty} |v(x)'|^2 dx + \frac{t^2}{2}\eta \int_0^{+\infty} |v(x)|^2 dx + t \int_0^{+\infty} u'(x)v'(x)dx \\
&+ \eta t \int_0^{+\infty} u(x)v(x)dx \\
&- \frac{1}{r+1} \int_0^{+\infty} h(x)\Big[|(u+tv)(x)|^{r+1} - |u(x)|^{r+1}\Big]dx \\
&- \lambda \int_0^{+\infty} q(x)\Big[F(x, (u+tv)(x)) - F(x, u(x))\Big]dx \\
&= \frac{t^2}{2} \int_0^{+\infty} |v'(x)|^2 dx + \frac{t^2}{2}\eta \int_0^{+\infty} |v(x)|^2 dx + t \int_0^{+\infty} u'(x)v'(x)dx \\
&+ t\eta \int_0^{+\infty} u(x)v(x)dx - t \int_0^{+\infty} h(x)(u(x) + t\theta v(x))^r v(x)dx \\
&- \lambda t \int_0^{+\infty} q(x)f(x, u(x) + t\theta v(x))v(x)dx,
\end{aligned}
$$

where $0 < \theta < 1$. Then,

$$
\begin{aligned}
\frac{J(u+tv) - J(u)}{t} &= \frac{t}{2} \int_0^{+\infty} |v'(x)|^2 dx + \frac{t}{2}\eta \int_0^{+\infty} |v(x)|^2 dx + \int_0^{+\infty} u'(x)v'(x)dx \\
&+ \eta \int_0^{+\infty} u(x)v(x)dx - \int_0^{+\infty} h(x)(u(x) + t\theta v(x))^r v(x)dx \\
&- \lambda \int_0^{+\infty} q(x)f(x, u(x) + t\theta v(x))v(x)dx.
\end{aligned}
$$

Letting $t \to 0$, we obtain

$$
\langle J'(u), v\rangle = \int_0^{+\infty} \Big(u'(x)v'(x) + \eta u(x)v(x)\Big)dx - \int_0^{+\infty} h(x)u^r(x)v(x)dx
$$

$$-\lambda \int_0^{+\infty} q(x)f(x,u(x))v(x)dx$$

Next we show that J' is continuous, so let $(u_n) \subset H_0^1(0,+\infty)$ with $u_n \to u$ as $n \to +\infty$. From (H_0) and (H_2), we have

$$\begin{aligned} q(x)|f(x,u_n(x))| &\leq aq(x)|u(x)|^r + bq(x) \\ &\leq a \sup_{x\in[0,+\infty)} |(pu)(x)|^r \frac{q(x)}{p^r(x)} + bq(x) \\ &= a\|u\|_{\infty,p}^r \frac{q(x)}{p^r(x)} + bq(x) \in L^1(0,+\infty). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} h(x)|u_n(x)|^r &= p^r(x)|u_n(x)|^r \frac{h(x)}{p^r(x)} \\ &\leq \sup_{x\in[0,+\infty)} |(pu)(x)|^r \frac{h(x)}{p^r(x)} \\ &= \|u\|_{\infty,p}^r \frac{h(x)}{p^r(x)} \in L^1(0,+\infty). \end{aligned}$$

Then by the Lebesgue dominated convergence theorem,

$$\lim_{n\to+\infty} \int_0^{+\infty} q(x)f(x,u_n(x))dx = \int_0^{+\infty} q(x)f(x,u(x))dx,$$

and

$$\lim_{n\to+\infty} \int_0^{+\infty} h(x)u_n^r(x)dx = \int_0^{+\infty} h(x)u^r(x)dx.$$

Hence,

$$\begin{aligned} \langle J'(u_n) - J'(u), v\rangle = &\int_0^{+\infty} \Big(u_n'v' + \eta u_n v\Big)dx - \int_0^{+\infty} h(x)u_n^r v dx \\ &- \lambda \int_0^{+\infty} q(x)f(x,u_n)v dx \\ &- \int_0^{+\infty} \Big(u'v' + \eta uv\Big)dx + \int_0^{+\infty} h(x)u^r v dx \end{aligned}$$

$$
\begin{aligned}
&+\lambda \int_0^{+\infty} q(x) f(x,u) v dx \\
&= \int_0^{+\infty} \Big((u_n' - u')v' + \eta (u_n - u) v \Big) dx \\
&\quad - \int_0^{+\infty} h(x) \Big(u_n^r - u^r \Big) v dx \\
&\quad - \lambda \int_0^{+\infty} q(x) \Big(f(x,u_n) - f(x,u) \Big) v dx.
\end{aligned}
$$

Passing to the limit as $n \to +\infty$ and using (H_0) and the Lebesgue dominated convergence theorem, we obtain that $J'(u_n) \to J'(u)$ as $n \to +\infty$. Thus, J' is continuous and this completes the proof of the proposition. $\square$

We now define what is meant by a weak solution of our problem.

Definition 2.2. We say that $u \in H_0^1(0,+\infty)$ is a weak solution of problem (1.1) if for any $v \in H_0^1(0,+\infty)$ we have

$$
\langle J'(u), v \rangle = \int_0^{+\infty} \Big(u'v' + \eta u v \Big) dx - \int_0^{+\infty} h(x) u^r v dx - \lambda \int_0^{+\infty} q(x) f(x,u) v dx = 0.
$$

Remark 2.3. Since the nonlinear term f is continuous, a weak solution of problem (1.1) is also a classical solution.

Our main existence result is contained in the following theorem.

Theorem 2.4. *Assume that $(H_0) - (H_4)$ hold. Then there exists $\overline{\lambda} > 0$ such that problem (1.1) has at least one solution u_λ for every $\lambda \in (0, \overline{\lambda})$.*

Proof. In view of (H_0), there exists $\delta_1 > 0$ such that

$$
|F(x,u)| \le a \frac{1}{r+1} |u|^{r+1} + b|u| \le K |u|^{r+1} \quad \text{for all} \;\; u > \delta_1.
$$

Also, from (H_0), there exists $K_1 > 0$ such that

$$
|F(x,u)| \le K_1 \quad \text{for all} \;\; u \in [-\delta_1, \delta_1] \quad \text{and all } x \in (0,+\infty).
$$

Therefore,

$$|F(x,u)| \le K_1 + K|u|^{r+1} \quad \text{for all } u \in \mathbb{R}^+ \text{ and all } x \in [0,+\infty). \tag{2.1}$$

From (1.2), (1.4), (2.1), (H_1), and Corollary 1.3, we obtain

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{r+1}\int_0^{+\infty} h(x)|u|^{r+1}dx - \lambda \int_0^{+\infty} q(x)F(x,u(x))dx \\
&\ge \frac{1}{2}\|u\|^2 - \frac{1}{r+1}\int_0^{+\infty} h(x)|u|^{r+1}dx - \lambda \int_0^{+\infty} q(x)\Big(K_1 + K|u|^{r+1}\Big)dx \\
&\ge \frac{1}{2}\|u\|^2 - \frac{1}{r+1}\int_0^{+\infty} h(x)|u|^{r+1}dx - \lambda K \int_0^{+\infty} q(x)|u|^{r+1}dx \\
&\quad - \lambda K_1 \int_0^{+\infty} q(x)dx \\
&\ge \frac{1}{2}\|u\|^2 - \frac{1}{r+1}\|u\|^{r+1}_{L_h^{r+1}} - \lambda K \int_0^{+\infty} \frac{q(x)h(x)}{h(x)}|u|^{r+1}dx - \lambda K_1 \|q\|_{L^1} \\
&\ge \frac{1}{2}\|u\|^2 - \frac{M_1^{r+1}}{r+1}\|u\|^{r+1} - \frac{\lambda K M_1^{r+1}}{r+1}\|\frac{q}{h}\|_\infty \|u\|^{r+1} - \lambda K_1 \|q\|_{L^1}
\end{aligned}$$

Choose $R > \max\left\{1, \left(\frac{2}{r+1}M_1^{r+1}\right)^{\frac{1}{1-r}}\right\}$; then for $u \in H_0^1(0,+\infty)$ with $\|u\| \le R$, we have

$$J(u) \ge \frac{1}{2}\|u\|^2 - \frac{M_1^{r+1}}{r+1}\|u\|^{r+1} - \frac{\lambda K M_1^{r+1}}{r+1}\|\frac{q}{h}\|_\infty \|u\|^{r+1} - \lambda K_1 \|q\|_{L^1}.$$

Now choose ρ so that

$$\max\left\{1, \left(\frac{2}{r+1}M_1^{r+1}\right)^{\frac{1}{1-r}}\right\} < \rho < R. \tag{2.2}$$

Note that $\overline{\lambda} := \dfrac{\frac{1}{2}\rho^2 - \frac{M_1^{r+1}}{r+1}\rho^{r+1}}{\frac{KM_1^{r+1}}{r+1}\|\frac{q}{h}\|_\infty \rho^{r+1} + K_1\|q\|_{L^1}} > 0$ by (2.2). Now if $\lambda < \overline{\lambda}$, then

$$J(u) > 0 \quad \text{if } \|u\| = \rho, \qquad \inf_{u \in \partial \overline{B_\rho(0)}} J(u) > 0, \tag{2.3}$$

and

$$J(u) \ge -C \ \text{ if } \|u\| \le \rho, \text{ for some } C > 0.$$

Hence, the functional J is bounded from below on $\overline{B_\rho(0)}$.

Let $\psi_1 \in H_0^1(0,+\infty)$ be the eigenfunction corresponding to the first eigenvalue of problem (1.3) as described in Lemma 1.7. Fix $\lambda \in \big(0, \overline{\lambda}\big)$. By (H_4), for any

$$D > \frac{\|\psi_1\|^2_{L^2_q}}{2\lambda\|\psi_1\|^2} \tag{2.4}$$

there exists $0 < \epsilon_D < 1$ such that

$$f(x,u) \geq Du^r \quad \text{for} \quad 0 < u < \epsilon_D.$$

Since $0 < u < 1$,

$$f(x,u) \geq Du^r \quad \text{implies} \quad F(x,u) \geq Du^{r+1} \geq Du^2. \tag{2.5}$$

Since the function ψ_1 is continuous on $[0,+\infty)$ and $\psi_1(0) = \psi_1(+\infty) = 0$, there exists $\widehat{c} > 0$, such that $\sup\limits_{x\in[0,+\infty)} \psi_1(x) \leq \widehat{c}$, Thus, for every $0 < t < \frac{1}{\widehat{c}}$, by (2.5), Lemma 1.5 and 1.7 we have

$$\begin{aligned}
J(t\psi_1) &= \frac{t^2}{2}\|\psi_1\|^2 - \frac{1}{r+1}\|t\psi_1\|^{r+1}_{L_h^{r+1}} - \lambda \int_0^{+\infty} q(x)F(x,t\psi_1(x))dx \\
&\leq \frac{t^2}{2}\|\psi_1\|^2 - \frac{t^{r+1}}{r+1}\|\psi_1\|^{r+1}_{L_h^{r+1}} - \lambda D t^2 \int_0^{+\infty} q(x)\psi_1^2(x)dx \\
&= \frac{t^2}{2}\|\psi_1\|^2 - \frac{t^{r+1}}{(r+1)\mu_1^{r+1}}\|\psi_1\|^{r+1} - \lambda D t^2 \|\psi_1\|^2_{L^2_q} < 0,
\end{aligned}$$

by (2.4). Then, in view of (2.3), we see that

$$\inf_{u\in\overline{B_\rho(0)}} J(u) < 0 < \inf_{u\in\partial\overline{B_\rho(0)}} J(u).$$

By applying Ekeland's variational principle (Theorem 1.8 above) in $\overline{B_\rho(0)}$, there is a minimizing sequence $(u_n)_{n\geq1} \subset \overline{B_\rho(0)}$ such that

$$J(u_n) \leq \inf_{u\in\overline{B_\rho(0)}} J(u) + \frac{1}{n}$$

and

$$J(u_n) \leq J(w) + \frac{1}{n}\|w - u_n\| \quad \text{for all } w \in \overline{B_\rho(0)} \text{ with } w \neq u_n. \tag{2.6}$$

If we set $w = u_n + tv$, for all $t > 0$ with $v \in H_0^1(0,+\infty)$, and $n \in \mathbb{N} \setminus \{0\}$, then we have

$$J(u_n) \leq J(u_n + tv) + \frac{1}{n} t\|v\|,$$

and so

$$\frac{J(u_n) - J(u_n + tv)}{t} < \frac{1}{n}\|v\| \quad \text{for all } n \in \mathbb{N} \setminus \{0\}.$$

Hence,

$$-\langle J'(u_n), v\rangle \leq \frac{1}{n}\|v\| \quad \text{for all } n \in \mathbb{N} \setminus \{0\}.$$

If we take $w = u_n - tv$ in (2.6), then we obtain $\langle J'(u_n), v\rangle \leq \frac{1}{n}\|v\|$ for all $n \in \mathbb{N} \setminus \{0\}$. Thus,

$$\sup_{\|v\|\leq 1} |\langle J'(u_n), v\rangle| \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N} \setminus \{0\}.$$

Therefore, we have

$$\|J'(u_n)\| \to 0, \text{and } \ J(u_n) \to c_\lambda \quad \text{as } n \to +\infty,$$

where c_λ stands for the infimum of $J(u)$ on $\overline{B_\rho(0)}$. Since $(u_n)_{n\geq 1}$ is bounded and $\overline{B_\rho(0)}$ is a closed convex set, there exist $u_\lambda \in \overline{B_\rho(0)} \subset H_0^1(0,+\infty)$ and a subsequence, still denoted by $(u_n)_{n\geq 1}$, such that

$$\begin{cases} u_n \to u_\lambda, & \text{weakly in } H_0^1(0,+\infty); \\ u_n \to u_\lambda, & \text{strongly in } L_h^{r+1}[0,+\infty); \\ u_n(x) \to u_\lambda(x), & \text{for } x \in (0,+\infty). \end{cases}$$

By the Lebesgue dominated convergence theorem and passing to the limit in $\langle J'(u_n), v\rangle$, as $n \to +\infty$, we obtain

$$\int_0^{+\infty} \Big(u_\lambda' v' + \eta u_\lambda v\Big) dx - \int_0^{+\infty} h(x) u_\lambda^r v dx - \lambda \int_0^{+\infty} q(x) f(x, u_\lambda) v dx = 0$$

for all $v \in H_0^1(0,+\infty)$. That is, $\langle J'(u_\lambda), v\rangle = 0$ for all $v \in H_0^1(0,+\infty)$. Hence, u_λ is a critical point of the functional J, which in turn is a solution of our problem. This proves the theorem. $\square$

Remark 2.5. If $\eta = 1$ and $h(x) \equiv 0$, then our result reduces to [2, Theorem 2.1].

3. EXAMPLES

In this section we give an example to illustrate our results.

Example 3.1. Consider the problem

$$\begin{cases} -u''(x)+\eta u = e^{-\frac{4}{3}x}u^r(x)+\lambda e^{-\frac{3}{2}x}(u^{\frac{1}{4}}+u^{\frac{1}{5}}), & x\in[0,+\infty),\\ u(0)=u(+\infty)=0, \end{cases} \tag{3.1}$$

Here we have $f(x,u)=u^{\frac{1}{4}}+u^{\frac{1}{5}}$, $q(x)=e^{-\frac{3}{2}x}$, $h(x)=e^{-\frac{4}{3}x}$, and $\dfrac{q}{h}(x)=e^{-\frac{1}{6}x}$. Then,

$$|f(x,u)|=|u^{\frac{1}{4}}+u^{\frac{1}{5}}|\le 2|u^r|+1$$
$$\text{for all } x\in\mathbb{R}^+ \text{ and all } u\in\mathbb{R}^+ \text{ where } r\in\left[\tfrac{1}{3},\tfrac{7}{9}\right).$$

Choosing $p(x)=2e^{-\frac{1}{2}x}$, we see that

$$\frac{h}{p^r}(x)=2^{-r}e^{-\frac{8-3r}{6}x},\quad \frac{h}{p^{r+1}}(x)=2^{-(r+1)}e^{-\frac{5-3r}{6}x},\quad \frac{q}{p^r}(x)=2^{-r}e^{-\frac{3-r}{2}x},$$

$$\frac{q}{p^2}(x)=\frac{1}{4}e^{-\frac{1}{2}x},\quad \frac{h^{\frac{2}{1-r}}}{q^{\frac{r+1}{1-r}}}(x)=e^{\frac{9r-7}{6(1-r)}x}$$

are in $L^1[0,+\infty)$. It is easy to see that conditions (H_0)–(H_3) hold. So by Theorem 2.4, (3.1) has at least one solution for each $\lambda\in(0,\overline{\lambda})$.

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