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SYSTEM OF INITIAL VALUE PROBLEMS INVOLVING RIEMANN-LIOUVILLE SEQUENTIAL FRACTIONAL DERIVATIVE

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ABSTRACT: In this paper, system of initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative is studied by using monotone iterative technique coupled with lower-upper solutions. Monotone iterative technique is successfully applied to obtain existence and uniqueness of solutions of system of initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative.

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1. INTRODUCTION

Fractional differential equations occur more frequently in different research areas and engineering such as physics, chemistry, control of dynamical systems etc [2, 3, 6, 13, 21]. During the last decade many researchers paid attention towards existence and uniqueness results for initial value problems [4, 11, 26], boundary value problems [1, 15, 17], periodic boundary value problem [12, 18, 22] and integral boundary value problem [19, 23]. Some recent results on the theory of fractional differential equations due to Lakshmikantham et. al. can be seen in [7, 8, 9, 10].

Recently, Wei et. al. [24, 25] developed monotone iterative technique for initial value problems and periodic boundary value problems involving Riemann- Liouville sequential fractional derivative and technique is successfully applied to study existence and uniqueness results for initial value problems and periodic boundary value problems. In the year 2012, Nanware and Dhaigude developed monotone method for system of Caputo fractional differential equations with periodic boundary conditions when the function is quasimonotone nondecreasing [5, 14], Riemann-Liouville fractional differential equations with integral boundary conditions when the function on the right is sum of nondecreasing and nonincreasing functions [14, 16] and system of Riemann-Liouville fractional differential equations with integral boundary conditions when the function is quasimonotone nondecreasing [14, 20]. Monotone method is successfully applied to obtain existence and uniqueness of solutions of the problems. In this paper, we shall consider the following system of initial value problems involving Riemann-Liouville sequential fractional derivative when the function on the right hand side is quasimonotone nondecreasing :

$$\begin{aligned} (\mathscr{D}^{2q}_{0+}u_i)(t) &= f_i(t, u_1, u_2, \mathscr{D}^q_{0+}u_1, \mathscr{D}^q_{0+}u_2), \quad t \in (0, T], \\ t^{1-q}u_i(t) &= u^i_0, \quad t^{1-q}(\mathscr{D}^q u_i)(t)|_{t=0} = u^i_1, \quad i = 1, 2. \end{aligned} \tag{1.1}$$

where $f_i \in C([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ is quasimonotone nondecreasing. Monotone method is developed for the initial value problem (1.1). Existence and uniqueness results are obtained for initial value problem (1.1) by using monotone method.

The paper is arranged in the following way:

In the second section definitions and basic results are considered. Comparison results and some important Lemmas are also given. In the last section monotone method is developed for system of initial value problems for fractional differential equations involving Riemann-Liouville fractional derivative and technique developed is successfully applied to obtain existence and uniqueness of solution of the problem (1.1).

2. DEFINITIONS AND BASIC RESULTS

Let $J = [0, T]$ be a compact interval on the real axis $\mathbb{R}$, and $u(t)$ be a measurable function, that is, $u \in L_1(0, T)$. Let $t \in J$ and $q \in \mathbb{R}$ $(0 < q \leq 1)$. The relation between the Riemann-Liouville sequential fractional derivatives and non-sequential Riemann-Liouville fractional derivative [6] is given by

$$\left(\mathscr{D}^{2q}u\right)(t) = \left(D^{2q}\left[u(x) - \left(I^{1-q}u\right)\frac{(x-0)^{q-1}}{\Gamma(q)}\right]\right)(t). \tag{2.1}$$

Definition 2.1. A function $u_i(t)$ is called a classical solution of initial value problem (1.1) if:

(i) $u_i(t)$ is continuous on $(0, T]$; $t^{1-q}u_i(t), t^{1-q}(\mathscr{D}^q u_i)(t)$ are continuous on $[0, T]$, and its fractional integrals $(I^{1-q}u_i)(t)$, $(I^{1-q}D^q u_i)(t)$ are continuously differentiable for $(0, T]$,

(ii) $u_i(t)$ satisfies initial value problem (1.1).

Definition 2.2. Define the following classes:

$$C([0,T]) = \left\{u_i : u_i(t) \text{ is continuous on } [0,T], ||u_i(t)||_C = \max_{t\in(0,T]} |u_i(t)|\right\},$$

$$C_{1-q}([0,T]) = \left\{u_i \in C([0,T]) : t^{1-q}u_i(t) \in C([0,T]), \quad ||u_i(t)||_{C_{1-q}} = ||t^{1-q}u_i(t)||_C\right\},$$

$$C^q_{1-q}([0,T]) = \left\{u_i \in C_{1-q}([0,T]) : t^{1-q}(\mathscr{D}^q u_i)(t) \in C([0,T])\right\}.$$

Definition 2.3. A function $v^0(t) = (v_1^0, v_2^0) \in C_{1-q}^q([0,T])$ is called a lower solution of initial value problem (1.1) if it satisfies

$$(\mathscr{D}^{2q} v_i^0)(t) \le f_i(t, v_1^0, v_2^0, \mathscr{D}^q v_1^0, \mathscr{D}^q v_2^0), \quad t \in (0,T]$$
$$t^{1-q} v_i^0(t) \le v_i^0, \quad t^{1-q}(\mathscr{D}^q v_i^0)(t)|_{t=0} \le v_i^1.$$

Definition 2.4. A function $w^0(t) = (w_1^0, w_2^0) \in C_{1-q}^q([0,T])$ is called a upper solution of initial value problem (1.1) if it satisfies

$$(\mathscr{D}^{2q} w_i^0)(t) \ge f_i(t, w_1^0, w_2^0, \mathscr{D}^q w_1^0, \mathscr{D}^q w_2^0), \quad t \in (0,T]$$
$$t^{1-q} w_i^0(t)|_{t=0} \ge w_i^0, \quad t^{1-q}(\mathscr{D}^q w_i^0)(t)|_{t=0} \ge w_i^1.$$

Assume that

$$v_i^0(t) \le w_i^0(t), t \in (0,T] : t^{1-q} v_i^0(t)|_{t=0} \le t^{1-q} w_i^0(t)|_{t=0},$$
$$t^{1-q}(\mathscr{D}^q v_i^0)(t)|_{t=0} \le t^{1-q}(\mathscr{D}^q w_i^0)(t)|_{t=0}$$

Definition 2.5. Define the sector in space $C_{1-q}^q([0,T])$:

$$[v^0, w^0] = \Big\{ u_i \in C_{1-q}^q([0,T]) \mid v_i^0 \le u_i \le w_i^0, t \in (0,T] :$$
$$t^{1-q} v_i^0|_{t=0} \le t^{1-q} u_i|_{t=0} \le t^{1-q} w_i^0|_{t=0},$$
$$t^{1-q}(\mathscr{D}^q v_i^0)|_{t=0} \le t^{1-q}(\mathscr{D}^q u_i)|_{t=0} \le t^{1-q}(\mathscr{D}^q w_i^0)|_{t=0} \Big\}$$

Following Lemma gives the existence of solution of the linear initial value problem for fractional differential equation

Lemma 2.6. *[24] Suppose that $u(t) \in C_{1-q}([0,T])$, then the linear initial value problem*

$$\begin{aligned} \mathscr{D}^q u(t) + M u(t) &= \sigma(t), \quad t \in (0,T], \\ t^{1-q} u(t)|_{t=0} &= u_0, \end{aligned} \tag{2.2}$$

where $M \in \mathbb{R}$ is a constant and $\sigma(t) \in C_{1-q}([0,T])$, has the following integral representation of solution

$$u(t) = \Gamma(q) u_0 e_q(-M, t) + \Big[e_q(-M, x) * \sigma(x) \Big](t), \tag{2.3}$$

where

$$(g * f)(t) = \int_0^t g(t-x) f(x) dx, \quad e_q(\lambda, z) = z^{q-1} E_{q,q}(\lambda z^q)$$

$$= z^{q-1}\sum_{k=0}^{\infty}\lambda^k\frac{z^{qk}}{\Gamma[(k+1)q]},$$

where

$$E_{q,q}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma[(k+1)q]}$$

is Mittag-Leffler function of the two parameter.

Lemma 2.7. *[24] Suppose that $u(t)\in C^q_{1-q}([0,T])$, then the linear initial value problem*

$$(\mathscr{D}^{2q}u)(t)+N\mathscr{D}^q u(t)+Mu(t)=\sigma(t),\quad t\in(0,T],$$
$$t^{1-q}u(t)|_{t=0}=u_0,\quad t^{1-q}(\mathscr{D}^q u)(t)|_{t=0}=u_1$$

where $N,M\in\mathbb{R}, N^2\geq 4M$ are constants and $\sigma(t)\in C_{1-q}([0,T])$, has the following representation of solution

$$u(t)=\Gamma(q)u_0e_q(\lambda_2,t)+\Gamma(q)(u_1-\lambda_2u_0)\Big[e_q(\lambda_2,x)*e_q(\lambda_1,x)\Big](t)+$$
$$\Big[e_q(\lambda_2,x)*e_q(\lambda_1,x)*\sigma(x)\Big](t),$$

where

$$\lambda_1=\frac{-N+\sqrt{N^2-4M}}{2},\quad \lambda_2=\frac{-N-\sqrt{N^2-4M}}{2}\leq 0.$$

Lemma 2.8. *[24] Prove that:*

$$\Big[e_q(\lambda_2,x)*e_q(\lambda_1,x)\Big](t)=\Big[e_q(\lambda_1,x)*e_q(\lambda_2,x)\Big](t)$$
$$=\frac{1}{\lambda_1-\lambda_2}\Big[e_q(\lambda_1,x)-e_q(\lambda_2,x)\Big](t),\ t\in\mathbb{R}.$$

Lemma 2.9. *[24] For $0\leq q\leq 1$, there exist positive constants $b^0_n>0, b^1_n>0, b^2_n>0, ..., b^n_n>0$, such that $\omega_n(kq)=\sum_{i=0}^n b^i_n C^{i+1}_{k+i}$.*
Hence, we have $(k-1)\omega_n(kq)=\sum_{i=0}^n(i+2)b^i_nC^{i+2}_{k+i}$.

$$(1+kq)(1+\frac{kq}{2})...(1+\frac{kq}{n})=\frac{1}{q}\sum_{i=0}^{n}\frac{1}{i+1}b^i_nC^i_{k+i}.$$

Following comparison results play a vital role in the later section.

Lemma 2.10. *[24] If $w(t) \in C_{1-q}([0,T])$ and satisfies the relations*

$$D^q w(t) + Mw(t) \geq 0, \quad t \in (0,T]$$
$$t^{1-q} w(t)|_{t=0} \geq 0,$$

where $M \in \mathbb{R}$ is constant. Then $w(t) \geq 0, \quad t \in (0,T]$.

3. MONOTONE METHOD AND APPLICATIONS

In this section monotone method is developed for the system of initial value problem for fractional differential equations involving Riemann-Liouville fractional derivative when the function on the right hand side is quasimonotone nondecreasing and monotone method is successfully applied to obtain existence of solution of the initial value problem (1.1). The uniqueness of the solution of the problem is also obtained.

Theorem 3.1. *Assume that*

(i) *$v_i^0, w_i^0 \in C_{1-q}^q([0,T])$ are ordered lower and upper solutions of IVP (1.1), $f = (f_1, f_2) \in C([0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ is quasimonotone nondecreasing*

(ii) *f_i satisfies one-sided Lipschitz condition*

$$f_i(t, w_1, w_2, \mathscr{D}^q w_1, \mathscr{D}^q w_2) - f_i(t, v_1, v_2, \mathscr{D}^q v_1, \mathscr{D}^q v_2) \geq -N_i(\mathscr{D}^q w_i - \mathscr{D}^q v_i) - M_i(w_i - v_i),$$

where $N_i, M_i \in \mathbb{R}, N_i^2 > 4M_i$.

(iii) *There exist constants $N_i, M_i \in \mathbb{R}, N_i^2 > 4M_i$ such that (ii) holds and for $t \in (0,T], v_i \leq y_i \leq y_i^* \leq w_i, D(t) \leq z_i \leq D^*(t), D(t) \leq z_i^* \leq D^*(t)$ such that*

$$f_i(t, y_1, y_2, z_1, z_2) - f_i(t, y_1^*, y_2^*, z_1^*, z_2^*) \leq N_i(z_i - z_i^*) + M_i(y_i - y_i^*),$$

where $D(t) = \mathscr{D}_{0+}^q v_i(t) + \lambda_2^i (w_i(t) - v_i(t)),$

$$D^*(t) = \mathscr{D}_{0+}^q w_i(t) - \lambda_2^i (w_i(t) - v_i(t)),$$

$$\lambda_1^i = \frac{-N_i + \sqrt{N_i^2 - 4M_i}}{2} \geq 0 > \lambda_2^i = \frac{-N_i - \sqrt{N_i^2 - 4M_i}}{2}. \qquad (3.1)$$

Then there exist sequences $\{v_i^n(t)\}, \{w_i^n(t)\} \subset C_{1-q}^q([0,T])$ *with* $v_i^0(t) = v_i(t), w_i^0(t) = w_i(t)$ *such that for* $t \in (0,T]$,

$$\lim_{n\to\infty} v_i^n(t) = v_i(t), \quad \lim_{n\to\infty} w_i^n(t) = w_i(t)$$

and $v_i(t), w_i(t)$ *are minimal and maximal solutions on the ordered interval* $[v^0, w^0]$ *for initial value problem* (1.1) *respectively and for any solution* $u_i(t)$ *of initial value problem* (1.1) *such that* $u_i(t) \in \Omega$, *we have*

$$v_i^0 \le v_i^1 \le v_i^2 \le ... \le v_i^n \le v_i \le u_i \le w_i \le w_i^n \le ... \le w_i^2 \le w_i^1 \le w_i^0.$$

Proof. Let

$$\sigma(\eta_i)(t) = f_i(t, \eta_1, \eta_2, \mathscr{D}^q\eta_1, \mathscr{D}^q\eta_2) + N_i\mathscr{D}^q\eta_i(t) + M_i\eta_i(t), \quad t \in (0,T].$$

For any $\eta(t) = (\eta_1(t), \eta_2(t)) \in \Omega$, consider the linear initial value problem

$$\begin{aligned} (\mathscr{D}^{2q}u_i)(t) + N_i\mathscr{D}^q u_i(t) + M_i u_i(t) = \sigma(\eta_i)(t), \quad t \in (0,T], \\ t^{1-q}u_i(t)|_{t=0} = u_i^0, \quad t^{1-q}(\mathscr{D}^q u_i)(t)|_{t=0} = u_i^1. \end{aligned} \tag{3.2}$$

By Lemma 2.1 and relation (2.1), linear initial value problem (3.2) has exactly one solution $u_i(t) \in C_{1-q}^q([0,T])$ and is given by

$$\begin{aligned} u_i(t) &= (A\eta_i)(t) \\ &= \Gamma(q)u_i^0 e_q(\lambda_2^i, t) + \Gamma(q)(u_i^1 - \lambda_2 u_i^0)\Big[e_q(\lambda_2^i, x) * e_q(\lambda_1^i, x)\Big](t) + \\ &\qquad \Big[e_q(\lambda_2^i, x) * e_q(\lambda_1^i, x) * \sigma\eta_i(x)\Big](t), \end{aligned}$$

where

$$\lambda_1^i = \frac{-N_i + \sqrt{N_i^2 - 4M_i}}{2}, \quad \lambda_2^i = \frac{-N_i - \sqrt{N_i^2 - 4M_i}}{2} \le 0.$$

$$\begin{aligned} (\mathscr{D}^q A\eta_i)(t) &= \Gamma(q)u_i^0\lambda_2^i e_q(\lambda_2^i, t) + \\ &\quad \Gamma(q)(u_i^1 - \lambda_2 u_i^0)\frac{1}{\lambda_1^i - \lambda_2^i}\Big[\lambda_1^i e_q(\lambda_1^i, x) - \lambda_2^i e_q(\lambda_1^i, x)\Big](t) + \\ &\quad \frac{1}{\lambda_1^i - \lambda_2^i}\Big\{\lambda_1^i e_q(\lambda_1^i, x) * \sigma(\eta_i)(x) - \lambda_2^i e_q(\lambda_2^i, x) * (\sigma\eta_i)(x)\Big\}(t). \end{aligned}$$

Then A is an operator from Ω into $C^q_{1-q}([0,T])$ and $\eta_i(t)$ is a solution of initial value problem (1.1) if and only if $\eta_i = A\eta_i$. Since $\lambda^i_1 \geq 0 \geq \lambda^i_2$ in (3.1), we have

$$\frac{1}{\lambda^i_1 - \lambda^i_2}\Big[\lambda^i_1 e_q(\lambda^i_1, x) - \lambda^i_2 e_q(\lambda^i_2, x)\Big](t) \geq 0, \quad t \in (0,T].$$

Using Lemma 2.3 and Lemma 2.4, definitions of lower and upper solutions, definition of $u_i(t)$ and $(\mathscr{D}^q A\eta_i)(t)$, we get

$$\begin{aligned} v_i(t) &= (A\eta_i)(t) \\ &= \Gamma(q)u^0_i e_q(\lambda^i_2, t) + \Gamma(q)(u^1_i - \lambda^i_2 u^0_i)\Big[e_q(\lambda^i_2, x) * e_q(\lambda^i_1, x)\Big](t) + \\ &\qquad \Big[e_q(\lambda^i_2, x) * e_q(\lambda^i_1, x) * (\sigma\eta_i)(x)\Big](t) \end{aligned}$$

$$\begin{aligned} v_i(t) &\leq Av_i(t) \\ &= \Gamma(q)u^0_i e_q(\lambda^i_2, t) + \Gamma(q)(u^1_i - \lambda^i_2 u^0_i)\Big[e_q(\lambda^i_2, x) * e_q(\lambda^i_1, x)\Big](t) + \\ &\qquad \Big[e_q(\lambda^i_2, x) * e_q(\lambda^i_1, x) * \sigma v_i(x)\Big](t) \\ &\leq (A\eta_i)(t) \leq (Aw_i)(t) \leq w_i(t), \quad \forall \quad \eta_i(t) \in \Omega. \end{aligned}$$

Thus, we have

$$v_i \leq Av_i \leq A\eta_i \leq Aw_i \leq w_i, \quad \forall \quad \eta_i \in \Omega \tag{3.3}$$

and if

$$\begin{gathered} v_i \leq \theta_i \leq \phi_i \leq w_i \ \text{ then } (\sigma\theta_i) \leq (\sigma\phi_i), A\theta_i \leq A\phi_i, \\ \text{and } \mathscr{D}^q A\theta_i \leq \mathscr{D}^q A\phi_i. \end{gathered} \tag{3.4}$$

By Lemma 2.5, we know that, for $i = 1, 2$
$z^1_i(t) = \mathscr{D}^q(A\eta_i - v_i)(t) - \lambda^i_2(A\eta_i - v_i)(t) \geq 0, \ \ t \in (0,T], \quad \forall \quad \eta_i \in \Omega.$
Hence

$$\begin{aligned} \mathscr{D}^q(A\eta_i)(t) &= \mathscr{D}^q v_i(t) + \lambda^i_2(A\eta_i - v_i)(t) \\ &\geq \mathscr{D}^q v_i(t) + \lambda^i_2(w_i - v_i)(t) \\ &\geq D(t), \ \ t \in (0,T], \quad \forall \quad \eta_i \in \Omega. \end{aligned}$$

Similarly, we can show $\mathscr{D}^q(A\eta_i)(t) \le D^*(t), \ \ t \in (0,T], \quad \forall \quad \eta_i(t) \in \Omega.$

Therefore $A(\Omega) \subset \Omega$.

Now, let $v_i^0 = v_i, \ w_i^0 = w_i, \ v_i^n = Av_i^{n-1}, w_i^n = Aw_i^{n-1}, \quad n = 1,2,3,....$ From (3.3) and (3.4), we have

$$v_i^0 \le v_i^1 \le v_i^2 \le ... \le v_i^n \le ... \le w_i^n \le ... \le w_i^2 \le w_i^1 \le w_i^0,$$

$$D(t) \le \mathscr{D}^q v_i^1 \le ... \le \mathscr{D}^q v_i^n \le ... \le \mathscr{D}^q w_i^n \le ... \le \mathscr{D}^q w_i^1 \le D^*(t).$$

It is clear that the upper sequence $\{w_i^n(t)\}$ is monotone nondecreasing and bounded from below and that lower sequence $\{v_i^n(t)\}$ is monotone nondecreasing and bounded above. Moreover $\mathscr{D}^q v_i^n(t), \mathscr{D}^q w_i^n(t) \in [D(t), D^*(t)]$.

Let $B_i = \left\{ v_i^n : n = 1,2,3,... \right\}$. Now we show that the set B is relatively compact in $C_{1-q}^q([0,T])$. For any $\eta_i(t) \in \Omega$, by definition of lower and upper solutions and Lipschitz condition, we have

$$\begin{aligned}(\mathscr{D}^{2q}v_i)(t) + N_i\mathscr{D}^q v_i(t) + M_i v_i(t) &\le f_i(t, v_1, v_2, \mathscr{D}^q v_1, \mathscr{D}^q v_2) + \\ &\qquad N_i\mathscr{D}^q v_i(t) + M_i v_i(t) \\ &\le f_i(t, \eta_1, \eta_2, \mathscr{D}^q \eta_1, \mathscr{D}^q \eta_2) + N_i\mathscr{D}^q \eta_i(t) + M_i \eta_i(t) \\ &\le f_i(t, w_1, w_2, \mathscr{D}^q w_1, \mathscr{D}^q w_2) + N_i\mathscr{D}^q w_i(t) + M_i w_i(t)\end{aligned}$$

Since $B, \Omega \subset C_{1-q}^q([0,T])$ are bounded sets, therefore

$$\sigma\eta_i(t) = f_i(t, \eta_1, \eta_2, \mathscr{D}^q \eta_1, \mathscr{D}^q \eta_2) + N_i\mathscr{D}^q \eta_i(t) + M_i \eta_i(t) | \eta_i \in \Omega$$

is bounded also. Hence there exists a constant $L > 0$ such that

$$|\sigma(v_i^n)\| = max|t^{1-q}\sigma(v_i^n(t)| \le L, \quad \forall, \quad n = 1,2,...$$

This implies $\quad |\sigma(v_i^n(t))| \le Lt^{1-q}, \forall \quad t \in (0,T]$.

On the other hand, $\{v_i^n(t) | n \in N\}$ satisfies

$$\begin{aligned}v_i^n(t) = \Gamma(q)u_i^0 e_q(\lambda_2^i, t) + \Gamma(q)(u_i^1 - \lambda_2^i u_i^1)\Big[e_q(\lambda_2^i, x) * e_q(\lambda_1^i, x)\Big](t) + \\ \Big[e_q(\lambda_2^i, x) * e_q(\lambda_1^i, x) * (\sigma(v_i^{n-1})(x)\Big](t)\end{aligned} \tag{3.5}$$

$$(\mathscr{D}^q v_i^n)(t) = \Gamma(q)u_i^0\lambda_2 e_q(\lambda_2^i, t)+ \Gamma(q)(u_i^1 - \lambda_2^i u_i^0)\frac{1}{\lambda_1^i - \lambda_2^i}\Big[\lambda_1^i e_q(\lambda_1^i, x) - \lambda_2^i e_q(\lambda_2^i, x)\Big](t)+ \frac{1}{\lambda_1^i - \lambda_2^i}\Big[\lambda_1^i e_q(\lambda_1^i, x) * \sigma(v_i^{n-1})(x) - \lambda_2^i e_q(\lambda_2^i, x) * \sigma(v_i^{n-1})(x)\Big](t) \tag{3.6}$$

Let $G(\lambda_j^i, x) = x^{1-q}\Big[e_q(\lambda_j^i, x) * \sigma(v_i^{n-1})(x)\Big], \quad x \in [0, T], \quad j = 1, 2.$

Without loss of generality, assume that $0 \le x_1 < x_2 \le T$, from $\lambda_2^i < 0 \le \lambda_1^i$, we have

$$\Big|G(\lambda_2^i, x_1) - G(\lambda_2^i, x_2)\Big| \le \frac{L_i\Gamma(q)}{|\lambda_1^i|}\Big|E_{q,q}(\lambda_2^i, x_1^q) - E_{q,q}(\lambda_2^i, x_2^q)\Big|+ \frac{2L_i\Gamma(q)}{\Gamma(2q)}(x_2 - x_1)^q$$

and
$$\Big|G(\lambda_1^i, x_1) - G(\lambda_1^i, x_2)\Big| \le \left(\frac{L_i\Gamma(q)}{|\lambda_1^i|} + \frac{L_iT^q}{q}\right)\Big|E_{q,q}(\lambda_1^i, x_1^q) - E_{q,q}(\lambda_1^i, x_2^q)\Big|+ \frac{2L_i\Gamma(q)}{\Gamma(2q)}E_{q,q}(\lambda_1^i, T^q(x_2 - x_1)^q) \tag{3.7}$$

From $E_{q,q}(x) \in C([0, T]), \quad \forall \quad \epsilon > 0$, there exists $\delta = \delta(\epsilon)$, when $|x_1 - x_2| < \delta$ (without loss of generality $0 \le x_1 < x_2 \le T$), we have

$$\Big|E_{q,q}(\lambda_1^i, x_1^q) - E_{q,q}(\lambda_1^i, x_2^q)\Big| < \frac{\epsilon}{8L_i^1} \tag{3.8}$$

$$\Big|E_{q,q}(\lambda_2^i, x_1^q) - E_{q,q}(\lambda_2^i, x_2^q)\Big| < \frac{\epsilon}{8L_i^2} \tag{3.9}$$

$$(x_2 - x_1)^q < \frac{\epsilon}{8L_i^3} \tag{3.10}$$

where

$$L_i^1 = max\left\{\frac{|\Gamma(q)(u_i^1 - \lambda_2^i u_i^0)\lambda_1^i|}{|\lambda_1^i - \lambda_2^i|}, \frac{1}{|\lambda_1^i - \lambda_2^i|}\left(\Gamma(q) + \frac{|\lambda_1^i|T^q}{q}\right)\right\}$$

$$L_i^2 = max\left|\Gamma(q)(u_0^i\lambda_2^i), \frac{|\Gamma(q)(u_i^1 - \lambda_2^i u_i^0)\lambda_1^i|}{|\lambda_1 - \lambda_2|}, \frac{L_i\Gamma(q)}{|\lambda_1 - \lambda_2|}\right|$$

$$L_i^3 = \frac{2L_i\Gamma(q)}{\Gamma(2q)|\lambda_1^i - \lambda_2^i|}\Big(|\lambda_2^i| + |\lambda_1^i|\Gamma(q)E_{q,q}(|\lambda_1^i|T^q)\Big)$$

Using (3.7) to (3.8) in (3.6), we obtain

$$\Big|x_1^{1-q}(\mathscr{D}^q v_i^n)(x_1) - x_2^{1-q}(\mathscr{D}^q v_i^n)(x_2)\Big| \le \Big|\Gamma(q)u_i^0\lambda_2^i\Big|.\Big|E_{q,q}(\lambda_2^i x_1^q) - E_{q,q}(\lambda_2^i x_2^q)\Big|+$$

$$\frac{|\Gamma(q)(u_i^1-\lambda_2^i u_i^0)}{|\lambda_1^i-\lambda_2^i|}\Big\{|\lambda_1^i||E_{q,q}(\lambda_1^i x_1^q)-E_{q,q}(\lambda_1^i x_2^q)|+$$

$$,\quad |\lambda_2^i||E_{q,q}(\lambda_2^i x_1^q)-E_{q,q}(\lambda_2^i x_2^q)|\Big\}+$$

$$\frac{L_i}{|\lambda_1^i-\lambda_2^i|}\Bigg((\Gamma(q)+\frac{|\lambda|T^q}{q})|E_{q,q}(\lambda_1^i x_1^q)-E_{q,q}(\lambda_1^i x_2^q)|+$$

$$\Gamma(q)|E_{q,q}(\lambda_2^i t_1^q)-E_{q,q}(\lambda_2^i t_2^q)|\Bigg)+$$

$$\frac{2L_i\Gamma(q)}{\Gamma(2q)|\lambda_1^i-\lambda_2^i|}\Bigg(|\lambda_2^i|+|\lambda_1^i|\Gamma(q)E_{q,q}(|\lambda_1^i|T^q)\Bigg)(x_2-x_1)^q$$

$$\leq \epsilon.$$

Thus B is equicontinuous in $C_{1-q}^q([0,T])$, by Ascoli-Arzela theorem, we have that B is relatively compact set of $C_{1-q}^q([0,T])$,. Similarly we can prove that $\{w_i^n(t)\}$ is relatively compact set of $C_{1-q}^q([0,T])$. Hence, the sequences $\{v_i^n(t)\},\{w_i^n(t)\}$ converges uniformly to $v_i(t), w_i(t)$ respectively on $[0,T]$

$$\lim_{n\to\infty} v_i^n(t)=v_i(t),\qquad \lim_{n\to\infty} w_i^n(t)=w_i(t),\quad t\in[0,T]$$
$$\lim_{n\to\infty} \mathscr{D}^{2q}v_i^n(t)=\mathscr{D}^{2q}v_i(t),\quad \lim_{n\to\infty} \mathscr{D}^{2q}w_i^n(t)=\mathscr{D}^{2q}w_i(t),\quad t\in[0,T].$$

Thus by relations ($v_i^0\leq v_i^1\leq v_i^2\leq ...$), it follows that $v_i(t)$ and $w_i(t)$ satisfy

$$v_i^0\leq v_i^1\leq v_i^2\leq ...\leq v_i^n\leq v_i\leq w_i\leq w_i^n\leq ...\leq w_i^2\leq w_i^1\leq w_i^0$$
$$D(t)\leq \mathscr{D}^q v_i^1\leq \mathscr{D}^q v_i^2\leq ...\leq \mathscr{D}^q v_i^n\leq \mathscr{D}^q v_i\leq \mathscr{D}^q w_i\leq \mathscr{D}^q w_i^n\leq$$
$$...\leq \mathscr{D}^q w_i^2\leq \mathscr{D}^q w_i^1\leq D^*(t)$$

Lastly, we prove that $v_i(t), w_i(t)$ are extremal solutions of initial value problem (1.1). Since $f_i,(i=1,2)$ is continuous, the function $\sigma(\eta_i)(t)$ is continuous and is monotone nondecreasing in $v_i(t)$, the sequence $\{v_i^n(t)\}$ converges to $v_i(t)$ implies that $\{\sigma(v_i^n)(t)\}$ converges to $\sigma(v_i)(t), t\in(0,T]$. Taking limit as $n\to\infty$ of $\{v_i^n(t)\}$ and using dominated convergence theorem, $v_i(t)$ satisfies the integral equation

$$v_i(t)=(Av_i)(t)=\Gamma(q)u_i^0 e_q(\lambda_2^i,t)+\Gamma(q)(u_i^1-\lambda_2^i u_i^0)\Big[e_q(\lambda_2^i,x)*e_q(\lambda_1^i,x)\Big](t)+$$
$$\Big[e_q(\lambda_2^i,x)*e_q(\lambda_1^i,x)*(\sigma v_i)(x)\Big](t).$$

Thus $v_i(t)$ is an integral representation of the solution of initial value problem. Since $f_i, (i = 1, 2)$ is continuous and by Lemma 2.2, it follows that $v_i(t)$ is a classical solution of initial value problem (1.1). This proves that the lower sequence $\{v_i^n(t)\}$ converges to a solution $v_i(t)$ of initial value problem (1.1). Similarly we can show that the upper sequence $\{w_i^n(t)\}$ converges to a solution $w_i(t)$ of initial value problem (1.1) and satisfies

$$v_i(t) \leq w_i(t), \mathscr{D}^q v_i(t) \leq \mathscr{D}^q w_i(t), \;\; t \in (0, T].$$

Thus by standard arguments it follows that
$v_i^0 \leq v_i^1 \leq v_i^2 \leq ... \leq v_i^n \leq v_i \leq w_i \leq w_i^n \leq ... \leq w_i^2 \leq w_i^1 \leq w_i^0$
and hence $v_i(t)$ and $w_i(t)$ are minimal and maximal solutions of initial value problem (1.1) on the sector $[v^0, w^0]$ respectively.
Finally, if there exist N_i, M_i such that (3.1) holds and for $t \in (0, T]$,
$v_i \leq y_i \leq y_i^* \leq w_i, D(t) \leq z_i \leq D^*(t), D(t) \leq z_i^* \leq D^*(t)$ such that

$$f_i(t, y_1, y_2, z_1, z_2) - f_i(t, y_1^*, y_2^*, z_1^*, z_2^*) \leq N_i(z_i - z_i^*) + M_i(y_i - y_i^*)$$

then $v_i(t) = w_i(t)$ is a unique solution of initial value problem (1.1). It is sufficient to prove $v_i(t) \geq w_i(t), \mathscr{D}^q v_i(t) \geq \mathscr{D}^q w_i(t), t \in (0, T]$. For this, consider $u_i(t) = v_i(t) - w_i(t)$. Then from IVP (1.1) and above hypothesis, we have

$$\begin{aligned}
(\mathscr{D}^{2q} u_i)(t) + N_i \mathscr{D}^q u_i(t) + M_i u_i(t) &= \mathscr{D}^{2q}(v_i - w_i)(t) + N_i \mathscr{D}^q (v_i - w_i)(t) \\
&\quad + M_i (v_i - w_i)(t) \\
&= \mathscr{D}^{2q} v_i(t) - \mathscr{D}^{2q} w_i(t) + N_i \mathscr{D}^q v_i - N_i \mathscr{D}^q w_i \\
&\quad + M_i v_i(t) - M_i w_i(t) \\
&= (\phi u_i)(t) \geq 0, \quad t \in (0, T];
\end{aligned}$$

$$t^{1-q} u_i(t)|_{t=0} = 0, \quad t^{1-q} (\mathscr{D}^q u_i)(t)|_{t=0} = 0.$$

$$(\mathscr{D}^q u_i)(t) = \frac{1}{\lambda_1^i - \lambda_2^i} \left\{ \left(\lambda_1^i e_q(\lambda_1^i, x) - \lambda_2^i e_q(\lambda_2^i, x) \right) * (\phi u_i)(x) \right\}(t)$$

By Lemma 2.4, it follows that $u_i(t) \geq 0, \quad t \in (0, T]$. Thus $v_i(t) \geq w_i(t)$, $\mathscr{D}^q v_i(t) \geq \mathscr{D}^q w_i(t), \quad t \in (0, T]$.

Hence $v_i(t) = u_i(t) = w_i(t)$ is a solution of the IVP (1.1).

Next we prove the uniqueness of solution of the IVP (1.1).

Theorem 3.2. *Assume that*

(i) $v_i^0, w_i^0 \in C_{1-q}^q([0,T])$ *are ordered lower and upper solutions of IVP* (1.1), $f = (f_1, f_2) \in C([0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ *is quasimonotone nondecreasing*

(ii) f_i *satisfies Lipschitz condition*

$$\Delta \leq M_i(w_i - v_i) + N_i(\mathscr{D}^q w_i - \mathscr{D}^q v_i), \tag{3.11}$$

where

$$\Delta = f_i(t, v_1, v_2, \mathscr{D}^q v_1, \mathscr{D}^q v_2) - f_i(t, w_1, w_2, \mathscr{D}^q w_1, \mathscr{D}^q w_2);$$

$$v_i, w_i \in [v^0, w^0], \mathscr{D}^q v_i, \mathscr{D}^q w_i \in [D(t), D^*(t)],$$

and $M_i > 0$, $N_i > 0$, $N_i^2 > 4M_i$ *are Lipschitz constants such that*

$$D = \mathscr{D}^q v_i(t) + \lambda_2^i(w_i(t) - v_i(t)), \quad D^* = \mathscr{D}^q w_i(t) - \lambda_2^i(w_i(t) - v_i(t)),$$
$$\lambda_1^i = \frac{-N_i + \sqrt{N_i^2 - 4M_i}}{2} \geq 0 > \lambda_2^= \frac{-N_i - \sqrt{N_i^2 - 4M_i}}{2}$$

Then the IVP (1.1) *has unique solution in the sector* $[v^0, w^0]$.

Proof. We know $v_i^0 \leq w_i^0$. It is enough to show that $v_i^0 \geq w_i^0$. From (3.11), we have

$$-M_i(w_i - v_i) - N_i(\mathscr{D}^q w_i - \mathscr{D}^q v_i) \leq \Delta \leq M_i(w_i - v_i) + N_i(\mathscr{D}^q w_i - \mathscr{D}^q v_i)$$

where

$$\Delta = f_i(t, w_1, w_2, \mathscr{D}^q w_1, \mathscr{D}^q w_2) - f_i(t, v_1, v_2, \mathscr{D}^q v_1, \mathscr{D}^q v_2).$$

This implies $v_i^0 \geq w_i^0$. Thus the IVP (1.1) has unique solution in the sector $[v^0, w^0]$.

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