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POSITIVE SOLUTIONS FOR SYSTEMS OF NONLINEAR FOURTH-ORDER DIFFERENTIAL EQUATIONS WITH P-LAPLACIAN

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ABSTRACT: In this paper, by constructing a special set and utilizing fixed point theory, we study the existence of positive solutions for systems of nonlinear fourth-order differential equations with p-laplacian, which improved and generalized the result of related paper.

Key Words: Laplacian, cone, fixed point theorem, positive solution

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1. INTRODUCTION

The boundary value problems of differential equation with p-Laplacian arises in a variety of applied mathematics, physics and engineering, however there is still a little research about it. Recently, some results concerning the boundary value problems of nonlinear differential equations with p-Laplacian have been obtained by a variety of methods, we refer the readers to [1-4] and the references cited therein. In the thesis [5], the author investigated the following problem:

$$\begin{cases} (\phi_p(u''(t)))'' = f(t, u(t)), & t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

In thesis [6] , the author investigated the following coupled singular boundary value problems:

$$\begin{cases} (\phi_p(u''(t)))' + \omega_1(t) f_1(t, v(t)) = 0, & t \in (0,1), \\ (\phi_p(v''(t)))' + \omega_2(t) f_2(t, u(t)) = 0, & t \in (0,1), \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \gamma_1 u(1) + \delta_1 u'(1) = 0, u''(0) = 0, \\ \alpha_2 v(0) - \beta_2 v'(0) = 0, \gamma_2 v(1) + \delta_2 v'(1) = 0, v''(0) = 0. \end{cases}$$

Motivated by the thesis [5,6], in this paper, we consider the existence and multiplicity of positive solutions for the following systems of fourth-order boundary value problems:

$$\begin{cases} (\phi_p(u''(t)))'' = \omega_1(t) f_1(t, u(t), v(t)), & t \in (0,1), \\ (\phi_p(v''(t)))'' = \omega_2(t) f_2(t, u(t), v(t)), & t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0, . \end{cases} \tag{1}$$

where $\phi_p(s) = \mid s \mid^{p-2} s, p \geq 2$, $f_i \in C((0,1) \times [0,+\infty) \times [0,+\infty), [0,+\infty))$, $\omega_i(t) \in C((0,1), [0,+\infty))$ and $f_i, \omega_i(t)$ may be singular at $t = 0, 1$. In thesis [6], f_1 and f_2 are functions of one variable. In thesis [5], the control function is quasi-homogeneous. Our purpose here is to deal with more general functions than thesis [6], and the conditions are weaker than thesis [5]. The organization of this paper is as follows, we shall introduce some definitions and lemmas in the rest of this section. The main results will be stated and proved in Section 2. For the sake of convenience, we first give some conditions.

$(\mathbf{H_1}) f_i(t, u(t), v(t)) \leq g_i(t) h_i(u(t), v(t)), g_i(t) : (0,1) \to [0,+\infty)$ may be singular at $t = 0, 1$, $h_i(u, v) : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous.

$(\mathbf{H_2})$ $\omega_i \in C((0,1), [0,+\infty)), \omega_i$ may be singular at $t = 0, 1$, such that

$$0 \leq \int_0^1 G(s,s) \omega_i(s) ds < +\infty$$

and

$$0 \leq \int_0^1 G(s,s) \omega_i(s) g_i(s) ds < +\infty,$$

where Green function

$$G(t,s)=\begin{cases} s(1-t), 0\leq s\leq t\leq 1,\\ t(1-s), 0\leq t\leq s\leq 1,\end{cases}$$

($\mathbf{H_3}$)

$$0\leq \limsup_{(u,v)\to 0}\frac{h_1(u,v)}{(u+v)^{p-1}}<\eta_1^{p-1}$$

and

$$0\leq \limsup_{(u,v)\to 0}\frac{h_2(u,v)}{(u+v)^{p-1}}<\eta_2^{p-1}.$$

($\mathbf{H_4}$)

$$(4\xi_1)^{p-1}<\liminf_{(u,v)\to\infty}\frac{f_1(s,u,v)}{(u+v)^{p-1}}\leq\infty,$$

or

$$(4\xi_2)^{p-1}<\liminf_{(u,v)\to\infty}\frac{f_2(s,u,v)}{(u+v)^{p-1}}\leq\infty.$$

($\mathbf{H_5}$)

$$0\leq \limsup_{(u,v)\to \infty}\frac{h_1(u,v)}{(u+v)^{p-1}}<\eta_1^{p-1}$$

and

$$0\leq \limsup_{(u,v)\to \infty}\frac{h_2(u,v)}{(u+v)^{p-1}}<\eta_2^{p-1}.$$

($\mathbf{H_6}$

$$(4\xi_1)^{p-1}<\liminf_{(u,v)\to 0}\frac{f_1(s,u,v)}{(u+v)^{p-1}}\leq\infty$$

or

$$(4\xi_2)^{p-1}<\liminf_{(u,v)\to 0}\frac{f_2(s,u,v)}{(u+v)^{p-1}}\leq\infty,$$

where η_i and ξ_i $(i=1,2)$ in $(H_3)-(H_6)$ are constants such that

$$0<\eta_i(\max_{0\leq t\leq 1}\int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_i(s)g_i(s)ds)d\tau)\leq\frac{1}{2}$$

and

$$\xi_i\int_0^1 G(\frac{1}{2},\tau)\phi_q(\int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,s)\omega_i(s)ds)d\tau\geq 1.$$

It is easy to show the systems (1) are equivalent to the following integral equations

$$\begin{cases} u(t) = \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau, \\ v(t) = \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_2(s)f_2(s,u(s),v(s))ds)d\tau, \end{cases} \tag{2}$$

where q is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$,

Let

$$A(u,v)(t) = \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau,$$

$$B(u,v)(t) = \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_2(s)f_2(s,u(s),v(s))ds)d\tau,$$

and

$$F(u,v)(t) = (A(u,v)(t), B(u,v)(t)).$$

Then systems (2) are equivalent to the fixed point equation

$$F(u,v) = (u,v)$$

in the Banach space $E = X \times X$, where $X = \{u : u, \phi_p(u'') \in C^2[0,1]\}$. The following Fixed -Point Theorem of Cone Expansion and Compression Type will be crucial in the argument that follow.

Lemma 1.1. (see [7]) *Let K be a cone in Banach space E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $F : K \bigcap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

$$\|Fu\| \le \|u\|, u \in K \bigcap \partial\Omega_1, \qquad \|Fu\| \ge \|u\|, u \in K \bigcap \partial\Omega_2;$$

or

$$\|Fu\| \ \ge \|u\|, u \in K \bigcap \partial\Omega_1, \qquad \|Fu\| \le \|u\|, u \in K \bigcap \partial\Omega_2,$$

then F has a fixed point in $K \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$.

In what follows we set $\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \max_{t\in[0,1]} |u(t)|$. In order to apply Lemma 1.1, we will let K be the cone defined by

$$K = \{(u,v) : (u,v) \in E : u, v \geq 0, \min_{t\in[\frac{1}{4},\frac{3}{4}]}(u(t)+v(t)) \geq \frac{1}{4}(\| u \| + \| v \|),$$
$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} -(u^{''}(t)+v^{''}(t)) \geq (\frac{1}{4})^q(\| u^{''} \| + \| v^{''} \|)\}.$$

Lemma 1.2. (see [5]) *If* $p \geq 2, \frac{1}{p}+\frac{1}{q}=1,$ *then* $| \phi_q(x) - \phi_q(y) | \leq \phi_q(x-y)$.

Lemma 1.3. *Suppose that conditions* $(H_1)-(H_2)$ *hold, then* $F : K \to K$ *is completely continuous.*

Proof. First we show that $F(K) \subset K$. $\forall (u,v) \in K, t \in [0,1]$,

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau \\ &\leq \int_0^1 G(\tau,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau. \end{aligned}$$

Hence

$$\|A(u,v)\| \leq \int_0^1 G(\tau,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau.$$

For $t \in [\frac{1}{4},\frac{3}{4}], G(t,\tau) \geq \frac{1}{4}G(\tau,\tau), \tau \in [0,1]$, we have

$$\begin{aligned} \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} A(u,v)(t) &= \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau \\ &\geq \frac{1}{4}\int_0^1 G(\tau,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau \\ &\geq \frac{1}{4} \| A(u,v) \| . \end{aligned}$$

Similarly

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} B(u,v)(t) \geq \frac{1}{4} \| B(u,v) \|$$

Thus

$$\begin{aligned} \min_{\frac{1}{4}\leq t\leq\frac{3}{4}}(A(u,v)(t)+B(u,v)(t)) &\geq \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} A(u,v)(t) + \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} B(u,v)(t) \\ &\geq \frac{1}{4} \| A(u,v) \| + \frac{1}{4} \| B(u,v) \| \\ &\geq \frac{1}{4}(\| A(u,v) \| + \| B(u,v) \|). \end{aligned}$$

At the same time

$$\begin{aligned}-A''(u,v)(t) &= \phi_q(\int_0^1 G(t,s)\omega_1(s)f_1(s,u(s),v(s))ds)\\ &\le \phi_q(\int_0^1 G(s,s)\omega_1(s)f_1(s,u(s),v(s))ds).\end{aligned}$$

Hence

$$\|A''(u,v)\| \le \phi_q(\int_0^1 G(s,s)\omega_1(s)f_1(s,u(s),v(s))ds).$$

$$\begin{aligned}\min_{\frac{1}{4}\le t\le\frac{3}{4}} -A''(u,v)(t) &= \min_{\frac{1}{4}\le t\le\frac{3}{4}} \phi_q(\int_0^1 G(t,s)\omega_1(s)f_1(s,u(s),v(s))ds)\\ &\ge (\frac{1}{4})^q\phi_q(\int_0^1 G(s,s)\omega_1(s)f_1(s,u(s),v(s))ds)\\ &\ge (\frac{1}{4})^q \parallel A''(u,v) \parallel .\end{aligned}$$

Similarly

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} -B''(u,v)(t) \ge (\frac{1}{4})^q \parallel B''(u,v) \parallel$$

Thus

$$\begin{aligned}\min_{\frac{1}{4}\le t\le\frac{3}{4}} -(A''(u,v)(t)+B''(u,v)(t)) &\ge \min_{\frac{1}{4}\le t\le\frac{3}{4}} -A''(u,v)(t) + \min_{\frac{1}{4}\le t\le\frac{3}{4}} -B''(u,v)(t)\\ &\ge (\frac{1}{4})^q \parallel A''(u,v) \parallel + (\frac{1}{4})^q \parallel B''(u,v) \parallel\\ &= (\frac{1}{4})^q(\parallel A''(u,v)\| + \|B''(u,v) \|).\end{aligned}$$

We conclude that $F(K) \subset K$. Next we show that $F : K \to K$ is completely continuous.

Let $\forall D \in K$ be a bounded set, i.e. $\exists M > 0$ such that $\forall (u,v) \in K, \|(u,v)\| \le M$, then

$$\begin{aligned}A(u,v)(t) &= \int_0^1 G_1(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau\\ &\le \int_0^1 G(\tau,\tau)\phi_q(\int_0^1 G(s,s)\omega_1(s)g_1(s)h_1(u(s),v(s))ds)d\tau\\ &\le \{\max \phi_q(h_1(u(s),v(s)) : 0 \le s \le 1\}\\ &\int_0^1 G(\tau,\tau)d\tau\phi_q(\int_0^1 G(s,s)\omega_1(s)g_1(s)ds)\end{aligned}$$

$$\le \{\max \phi_q(h_1(u(s), v(s)) : 0 \le s \le 1\}$$
$$\int_0^1 G(\tau,\tau)d\tau \phi_q(\int_0^1 G(s,s)\omega_1(s)g_1(s)ds)$$
$$= N_1 < \infty.$$

So $\| A(u,v) \| \le N_1$, similarly $\| B(u,v) \| \le N_2 < \infty$. Hence

$$\| F(u,v) \| = \| A(u,v) \| + \| B(u,v) \| \le N_1 + N_2.$$

Correspondingly, $F : K \to K$ is bounded uniformly.

Now, we show that F is equicontinuous. The continuity of $G(t,s)$ on $[0,1] \times [0,1]$ implies that $G(t,s)$ is continuous uniformly. i.e. $\forall \varepsilon > 0, \exists \delta > 0$, such that $\forall \xi \in [0,1]$, $|t_1 - t_2| < \delta$, then

$$|G(t_1,\xi) - G(t_2,\xi)| < \frac{\varepsilon}{2}(\{\max \phi_q(h_i(u(s), v(s)) : 0 \le s \le 1\}$$
$$\phi_q(\int_0^1 G(s,s)\omega_i(s)g_i(s)ds))^{-1}.$$

Hence $\forall D \in K$, we have that

$$\begin{aligned}
&|F(u,v)(t_1) - F(u,v)(t_2)| \\
&= |(A(u,v)(t_1), B(u,v)(t_1)) - (A(u,v)(t_2), B(u,v)(t_2))| \\
&= |(A(u,v)(t_1) - A(u,v)(t_2)), (B(u,v)(t_1) - B(u,v)(t_2))| \\
&\le |A(u,v)(t_1) - A(u,v)(t_2)| + |B(u,v)(t_1) - B(u,v)(t_2)| \\
&= |\int_0^1 (G(t_1,\tau) - G(t_2,\tau))\phi_q(\int_0^1 G(\tau,s)\omega_1(s)g_1(s)h_1(u(s),v(s))ds)d\tau| \\
&+ |\int_0^1 (G(t_1,\tau) - G(t_2,\tau))\phi_q(\int_0^1 G(\tau,s)\omega_2(s)g_2(s)h_2(u(s),v(s))ds)d\tau| \\
&\le |\int_0^1 (G(t_1,\tau) - G(t_2,\tau)) \max\{\phi_q(h_1(u(s), v(s)) : 0 \le s \le 1\} \\
&\phi_q(\int_0^1 G(s,s)\omega_1(s)g_1(s)ds)d\tau \;| \\
&+ |\int_0^1 (G(t_1,\tau) - G(t_2,\tau)) \max\{\phi_q(h_2(u(s), v(s)) : 0 \le s \le 1\} \\
&\phi_q(\int_0^1 G(s,s)\omega_2(s)g_2(s)ds)d\tau \;| \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This means that $F(K)$ is equicontinuous, so $F(K)$ is relatively compact in K.

Finally we show $F : K \to K$ is continuous. Let $\{(u_n, v_n)\} \subset K$ be sequence such that $(u_n, v_n) \to (u_0, v_0), n \to \infty$, by Lemma 1.2, we have

$$\begin{aligned}
&|F(u_n,v_n)(t) - F(u_0,v_0)(t)| \\
&\le \int_0^1 G(\tau,\tau)|\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u_n(s),v_n(s))ds) \\
&-\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u_0(s),v_0(s))ds)|d\tau \\
&+\int_0^1 G(\tau,\tau)|\phi_q(\int_0^1 G(\tau,s)\omega_2(s)f_2(s,u_n(s),v_n(s))ds) \\
&-\phi_q(\int_0^1 G(\tau,s)\omega_2(s)f_2(s,u_0(s),v_0(s))ds)|d\tau \\
&\le \int_0^1 G(\tau,\tau)d\tau|\phi_q(\int_0^1 G(s,s)\omega_1(s)(f_1(s,u_n(s),v_n(s)) \\
&-f_1(s,u_0(s),v_0(s)))ds)| \\
&+\int_0^1 G(\tau,\tau)d\tau|\phi_q(\int_0^1 G(s,s)\omega_2(s)(f_2(s,u_n(s),v_n(s)) \\
&-f_2(s,u_0(s),v_0(s)))ds)|.
\end{aligned}$$

By (H_2) ,Lebesgue dominated convergence theorem and the continuity of f_1, f_2, we have

$$\|F(u_n,v_n) - F(u_0,v_0)\| \to 0, n \to \infty.$$

Thus F is continuous.
Sum it up, we claim F is completely continuous.

2. MAIN RESULTS

Theorem 2.1. *Suppose that conditions $(H_1) - (H_4)$ hold, then systems (1) has at least one positive solution.*

Proof. By (H_3), we may choose $r > 0$, for any $\|u\| + \|u\| \le r$ we have

$$h_1(u,v) \le \eta_1^{p-1}(u+v)^{p-1}, \qquad h_2(u,v) \le \eta_2^{p-1}(u+v)^{p-1}.$$

Set

$$\Omega_1 = \{(u,v) : (u,v) \in K; \|(u,v)\| < r\}$$

If $(u,v)\in K\bigcap\partial\Omega_1$ then $\|u\|+\|v\|\le r$, we have

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau, \\ &\le \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)g_1(s)h_1(u(s),v(s))ds)d\tau \\ &\le \eta_1\|(u,v)\| \max_{0\le t\le 1}\int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)g_1(s)ds)d\tau \\ &\le \frac{1}{2}\|(u,v)\|. \end{aligned}$$

Which implies

$$\|A(u,v)\|\le\frac{1}{2}\|(u,v)\|.$$

Similarly

$$\|B(u,v)\|\le\frac{1}{2}\|(u,v)\|.$$

Hence, for $(u,v)\in K\bigcap\partial\Omega_1$,we have

$$\|F(u,v)\|=\|A(u,v)\|+\|B(u,v)\|\le\|\ (u,v)\ \|\ .$$

By (H_4), if we further assume

$$(4\xi_1)^{p-1}<\liminf_{(u,v)\to\infty}\frac{f_1(s,u,v)}{(u+v)^{p-1}}\le\infty,$$

then there is an $\overline{R}>r$, for any $\|u\|+\|v\|>\overline{R}$, we have

$$f_1(s,u,v)\ge(4\xi_1(u+v))^{p-1}.$$

Let $R>\overline{R}$, we set

$$\Omega_2=\{(u,v):(u,v)\in K;\|(u,v)\|<R\}.$$

If $(u,v)\in K\bigcap\partial\Omega_2$, then

$$\begin{aligned} A(u,v)(\tfrac{1}{2}) &= \int_0^1 G(\frac{1}{2},\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau \\ &\ge 4\xi_1\int_0^1 G(\frac{1}{2},\tau)\phi_q(\int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,s)\omega_1(s)(u(s)+v(s))^{p-1}ds)d\tau \\ &\ge \xi_1(\|u\|+\|v)\|)\int_0^1 G(\frac{1}{2},\tau)\phi_q(\int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,s)\omega_1(s)ds)d\tau \\ &\ge \|(u,v)\|. \end{aligned}$$

Which implies

$$\|A(u,v)\| \geq \|(u,v)\|.$$

Similarly

$$\|B(u,v)\| \geq \|(u,v)\|.$$

Hence, for $(u,v) \in K \bigcap \partial\Omega_2$ we have

$$\|F(u,v)\| = \|A(u,v)\| + \|B(u,v)\| \geq \| (u,v) \| .$$

An analogous estimate holds for f_2 in condition (H_4).

By Lemma 1.1, F has a fixed point $(u,v) \in K \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$, systems (1) has at least one positive solution.

Theorem 2.2. *Suppose that conditions* $(H_1), (H_2), (H_5), (H_6)$ *hold, then systems (1) has at least one positive solution.*

Proof. By(H_5), there exist $R_0, \varepsilon_i > 0$, for $\|u\| + \|v\| \geq R_0$, we have

$$h_i(u,v) \leq (\eta_i - \varepsilon_i)^{p-1}(u+v)^{p-1}.$$

Let

$$a = \max_{i=1,2} \max\{\phi_q(h_i(u,v)) : u(t) \leq R_0, v(t) \leq R_0\},$$

$$R > \max\{\frac{a}{\varepsilon_1}, \frac{a}{\varepsilon_2}\}.$$

We set

$$\Omega_1 = \{(u,v) : (u,v) \in K; \|(u,v)\| < R\}.$$

If $(u,v) \in K \bigcap \partial\Omega_1$, then

$$\begin{aligned}
A(u,v)(t) &= \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)f_1(s,u(s),v(s))ds)d\tau \\
&\leq \int_0^1 G(t,\tau)\phi_q(\int_0^1 G(\tau,s)\omega_1(s)g_1(s)h_1(u(s),v(s))ds)d\tau \\
&= \int_0^1 G(t,\tau)\phi_q(\int_{\|u\|+\|v\|\geq R_0} G(\tau,s)\omega_1(s)g_1(s)h_1(u(s),v(s))ds)d\tau \\
&\quad + \int_0^1 G(t,\tau)\phi_q(\int_{\|u\|+\|v\|\leq R_0} G(\tau,s)\omega_1(s)g_1(s)h_1(u(s),v(s))ds)d\tau \\
&\leq (\eta_1 - \varepsilon_1)(\|u(t)\| + \|v(t)\|) \int_0^1 G(t,\tau)
\end{aligned}$$

$$\phi_q(\int_{\|u\|+\|v\|\geq R_0} G(\tau,s)\omega_1(s)g_1(s)ds)d\tau$$
$$+ a\int_0^1 G(t,\tau)\phi_q(\int_{\|u\|+\|v\|\leq R_0} G(\tau,s)\omega_1(s)g_1(s)ds)d\tau$$
$$\leq [(\eta_1-\varepsilon_1)(\|u\|+\|v\|)+a]\max_{0\leq t\leq 1}\int_0^1 G(t,\tau)$$
$$\phi_q(\int_0^1 G(\tau,s)\omega_1(s)g_1(s)ds)d\tau$$
$$\leq \frac{1}{2}\|(u,v)\|.$$

Which implies

$$\|A(u,v)\| \leq \frac{1}{2}\|(u,v)\|.$$

Similarly

$$\|B(u,v)\| \leq \frac{1}{2}\|(u,v)\|.$$

Hence, for any $(u,v)\in K\bigcap\partial\Omega_1$, we have

$$\|F(u,v)\| = \|A(u,v)\| + \|B(u,v)\| \leq \| (u,v) \| .$$

On the other hand, by (H_6), if we further assume

$$(4\xi_1)^{p-1} < \liminf_{(u,v)\to 0}\frac{f_1(s,u,v)}{(u+v)^{p-1}} \leq \infty,$$

then there is an $0<r<R$, for any $\|u\|+\|v\|\leq r$, we have

$$f_1(s,u,v)\geq (4\xi_1(u+v))^{p-1}.$$

Set

$$\Omega_2 = \{(u,v) : (u,v)\in K; \|(u,v)\| < r\},$$

similar to the proof of Theorem 2.1, for $(u,v)\in K\bigcap\partial\Omega_2$, we have

$$\|F(u,v)\| = \|A(u,v)\| + \|B(u,v)\| \geq \| (u,v) \| .$$

An analogous estimate holds for f_2 in condition (H_6).

By Lemma 1.1, F has a fixed point $(u,v)\in K\bigcap(\overline{\Omega}_1\setminus\Omega_2)$, so system (1) has at least one positive solution.

Theorem 2.3. *Suppose that conditions $(H_1)-(H_4)$ and (H_6) hold, then systems (1) has at least two positive solutions (u_1,v_1) and (u_2,v_2) with $\|(u_1,v_1)\| < r < \|(u_2,v_2)\|$.*

Proof. By (H_4) if we further assume that

$$(4\xi_1)^{p-1} < \liminf_{(u,v)\to\infty} \frac{f_1(s,u,v)}{(u+v)^{p-1}} \leq \infty,$$

then we choose a $R_1 > r$ large sufficiently, for any $\|u\|+\|v\| > R_1$, we have

$$f_1(s,u,v) \geq (4\xi_1(u+v))^{p-1}.$$

Set

$$\Omega_{R_1} = \{(u,v) : (u,v) \in K; \|(u,v)\| < R_1\},$$

similarly, for any $(u,v) \in K \bigcap \partial\Omega_{R_1}$, we have

$$\|F(u,v)\| = \|A(u,v)\| + \|B(u,v)\| \geq \| (u,v) \| .$$

An analogous estimate holds for f_2 in condition (H_4).

On the other hand, by (H_6) we assume that

$$(4\xi_1)^{p-1} < \liminf_{(u,v)\to 0} \frac{f_1(s,u,v)}{(u+v)^{p-1}} \leq \infty,$$

we choose $R_2 < r$ small sufficiently, for any $\|u(t)\|+\|v(t)\| < R_2$, we have

$$f_1(s,u,v) \geq (4\xi_1(u+v))^{p-1}.$$

Set

$$\Omega_{R_2} = \{(u,v) : (u,v) \in K; \|(u,v)\| < R_2\},$$

similarly, for any $(u,v) \in K \bigcap \partial\Omega_{R_2}$, we have

$$\|F(u,v)\| = \|A(u,v)\| + \|B(u,v)\| \geq \| (u,v) \| .$$

An analogous estimate holds for f_2 in condition (H_6).

By Lemma 1.1, F has at least twice fixed points $(u_1,v_1) \in \overline{\Omega}_{R_1} \setminus \Omega_1$ and $(u_2,v_2) \in \overline{\Omega}_1 \setminus \Omega_{R_2}$, so systems (1) has at least twice positive solutions with $\|(u_1,v_1)\| < r < \|(u_2,v_2)\|$.

Theorem 2.4. *Suppose that conditions $(H_1)-(H_5)$ hold, then problem (1) has at least two positive solutions (u_1,v_1) and (u_2,v_2) with $\|(u_1,v_1)\| < R < \|(u_2,v_2)\|$.*

The proof is similar to the Theorem 2.3, we omit it.

3. EXAMPLE

$$\begin{cases} (\phi_p(u''(t)))'' = \dfrac{(u(t)+v(t))^{\alpha_1}}{6\sqrt{(t(1-t))^3}}, \\ (\phi_p(v''(t)))'' = \dfrac{(u(t)+v(t))^{\alpha_2}}{6\sqrt{(t(1-t))^3}}, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0. \end{cases}$$

where $p \geq 2$, if $\alpha_1 \geq \alpha_2 > p-1$. $\omega_i(t) = \dfrac{1}{\sqrt[4]{(t(1-t))^3}}$, $f_i(t,u,v) = \frac{(u+v)^{\alpha_i}}{6\sqrt[4]{(t(1-t))^3}}$.

Let

$$g_i(t) = \frac{1}{3\sqrt[4]{t(1-t)^3}}, h_i(u,v) = (u+v)^{\alpha_i},$$

since

$$\int_0^1 G(s,s)\omega_i(s)g_i(s)ds = \int_0^1 \frac{1}{3\sqrt{t(1-t)}} = \frac{\pi}{3},$$

$$\int_0^1 G(s,s)\omega_i(s)ds = \int_0^1 \sqrt[4]{t(1-t)}ds < +\infty,$$

the theorem 2.1 holds. If $\alpha_1 \leq \alpha_2 < p-1$, then theorem 2.2 holds.

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REFERENCES

[1] Z.Z. Li, W.G. Ge, Positive solution for p-laplacian singular Sturm-Liouville boundary value problem, *Math. Appl.*, **15**, No. 3 (2002), 13-17.

[2] F.H. Wong, Existence of positive solutions for m-Laplacian boundary value problem, *Appl. Math. Lett.*, **12** (1999), 11-17.

[3] P.R. Agarwal, D. O'Regan, P.J. Wong, F.H. Wong, *Positive Solutions of Differential and Integral Equations*, Singapore, Springer-Verlag, 2000.

[4] X.H. Ni, W.G. Ge, Existence of positive solutions for one-dimentional p-Laplacian coupled boundary value problem, *J. Math. Research and Exposition*, **25**, No. 3 (2005), 489-494.

[5] Y.L. Wang, G.W. Shi, Positive solutions of fourth-order singular superlinear p-Laplacian BVP, *ACTA Mathematicae Scientia*, **29A**, No. 2 (2009), 344-352.

[6] Z.X. Cai, X.Z. Zhang, Positive solutions for third-order p-Laplacian coupled singular for boundary value problems, *ACTA Mathematicae Applicatae Sinica*, **35**, No. 3 (2012), 421-429.

[7] D.J. Guo, *Nonlinear Functional Analysis*, Jinan Shandong Science Technical Publishers, 2000.