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DISTRIBUTION FREE CONTINUOUS-REVIEW INVENTORY MODEL WITH A SERVICE LEVEL CONSTRAINT USING PIECEWISE LINEAR LEAD TIME CRASHING COST

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ABSTRACT: In this paper, we have framed a continuous-review model (Q, R, L) inventory model with service level constraint. Lead time is becoming more and more vital for industry, in particular for style supply chains, which has appealed more and more researchers and businessmen's notice. The length of lead time represents speed and service level.

In this paper, lead time is controllable and the compressing cost of lead time follows piecewise linear function. The main contribution of this proposed model is to find minimizing the total cost. A mathematical model is developed to obtain an improved result. An efficient iterative algorithm is designed to obtain the optimal solution of the order quantity, safety stock and lead time.

Furthermore, numerical examples are used to demonstrate the benefits of the model by using Matlab 2008 sofware. Sensitivity analysis of the optimal solution with respect to major parameters is carried out and managerial implications are also incorporated. Graphical representation is presented to show the convexity of the total cost.

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1. INTRODUCTION

Inventory is an asset that is owned by a business that has the express purpose of being sold to a customer. This includes items sold to end customers or distributors. It includes raw materials, work in process, and finished goods. The management of inventory is a key concern of all businesses. If a company's inventory level is too low, it risks delays in fulfilling its customer's orders. If the inventory is too high, it is tying up dollars that can be better used in other areas. It also risks obsolescence and spoilage. Successful businesses keep their inventory turns high, but also keep their service level at or above the industry standard.

A more conventional definition of lead time in the supply chain management realm is the time from the moment the customer places an order (the moment you learn of the requirement) to the moment it is ready for delivery. In the absence of finished goods or intermediate (work in progress) inventory, it is the time it takes to actually manufacture the order without any inventory other than raw materials.

In the continuous review process, the inventory levels are continuously reviewed, and as soon as the stocks fall below a pre-determined level (usually called, reorder point, or reorder level), replenishment order is placed. As more and more companies start using sophisticated IT systems to track their inventories in real-time, the continuous review method becomes a viable and optimal way to plan for replenishment.

In inventory management, the length of lead time has direct influence on customer service level and total inventory cost. With the increasing competition in today's business environment, plenty of enterprises have devoted their efforts to pursuing a short lead time to enhance market competition ability. It is no doubt that the achievement of a shortened lead time requires a number of capital investments. Thus, some researchers have paid their attentions to balancing benefits and costs resulting from the reduction of lead time, and developed some theoretical models for possible decision aid.

Service level is used in supply chain management and in inventory management to measure the performance of inventory replenishment policies. Under consideration, from the optimal solution of such a model also the optimal size of backorders can be derived. Unfortunately, this optimization approach requires that the planner know the optimal value of the backorder costs. As these costs are difficult to quantify in practice, the logistical performance of an inventory node in a supply network is measured with the help of technical performance measures. The target values of these measures are set by the decision maker.

A lead time is the latency between the initiation and execution of a process. For example, the lead time between the placement of an order and delivery of a new car from a manufacturer may be anywhere from 2 weeks to 6 months. In industry, lead time reduction is an important part of lean manufacturing. Inventory control is one of the main issues in logistic and supply chain management in which various types of uncertain and imprecise parameters exist. The(Q, R)model is widely used to explore the continuous review inventory control problem.

The continuous-review (Q, R)inventory model has been extensively studied in the literature and is commonly used in practice (Hadley et al. [5]; Nahmias [16]; Silver et al. [24]). To obtain the optimal parameters, however, one requires the distributional form of demand during the lead time, which is very often not available in practice. The mean and standard deviation of lead time demand are usually the only information accessible to practitioners.

We presented the distribution free continuous-review (Q, r, L) inventory system with a service level. The purpose of this study is to obtain the optimal cost with respect to optimal order quantity, reorder point and lead time under the effect of a service level constraint by applying the distribution free approach. Our main goal in developing this model is to reduce total cost by using lead time can be decomposed into n mutually independent components each having a different crashing cost for reducing lead time. Graphical representation is also presented to illustrate the proposed model. Besides, an efficient algorithm is developed to determine the optimal solution, and our approach is illustrated through a numerical example.

2. LITERATURE REVIEW

Among the modern production management, the Japanese successful experiences of using Just-In-Time (JIT) production show that the advantages and benefits associated with efforts to control the lead time can be clearly perceived. The goal of JIT inventory management philosophies is the focus which emphasizes high quality, keeps low inventory level and lead time to a practical minimum. In 1983, Monden [11] studied Toyota production system, and clearly declared that shortening lead time is a crux of elevating productivity.

In most of the early literature dealing with inventory problems, either in deterministic or probabilistic model, lead time is viewed as a prescribed constant or a stochastic variable, which therefore, is not subject to control (see, e.g., Naddor [15] and Silver et al. [23]). Recently, there have been some inventory literatures which consider lead time as a decision variable. Liao et al. [9] first presented a continuous review inventory model in which the order quantity was predetermined and lead time was a decision unique variable. Ben-Daya et al. [1] extended Liao et al. [9] model by considering both the lead time and order quantity to be decision variables where shortages were neglected. Ouyang et al. [18] generalized Ben-Daya et al. [1] model by allowing shortages and the total amount of stock out is considered as a mixture of backorders and lost sales.

In a recent research article, Ouyang et al. [19] considered an inventory model with a mixture of backorders and lost sales in which a service level constraint was used instead of shortage cost in the objective function. However, in those models previously mentioned [1,18,19] reorder point had not been taken into account, and merely focused on the relationship between lead time and order quantity; that is, they neglected the possible impact of reorder point on the economic ordering strategy.

Lead time can be reduced by an additional crashing cost; In other words, lead time is controllable. According to Hsu et al. [8] crashing cost could be expenditures on equipment improvement, information technology, order expedite, or special shipping and handling. By shortening lead time, buyers can lower the safety stock, reduce the out-of-stock loss, improve the customer service level and increase the competitive advantage of business. Therefore, lead time reduction has been one of the most offered themes for both researchers and practitioners. Liao et al. [9] first devised a probabilistic inventory model in which lead time was considered as a decision variable. In a recent study, Pan et al. [21] considered a continuous review inventory system in which the shortage is allowed and the total quantity of stock-out is a mixture of backorder and lost sale.

Later, many researchers (see [4], [7], [17] and [20]) investigated various integrated production- distribution inventory models for lead time reduction in single-vendor single-buyer supply chain. Controlling the lead time plays an important role for any inventory model. Tersine [26] introduced the lead time as a partition of five components as the supplier's lead time, order transit, delivery time and setup time. Liao et al. [9] considered the lead time as a unique decision variable in their inventory model. Ben-Daya et al. [1] explained both the ordering quantity and the lead time as a decision variable without shortages. Ouyang et al. [18] extended Ben-Daya et al. [1] model in view of shortages but they made a mistake which is corrected by Moon et al. [14]. Hariga et al. [6] developed some stochastic inventory models with variable lead time.

Pan et al. [22] considered the lead time as a controllable factor to obtain the joint total expected cost. To apply probability inventory models in practice, inventory managers have to know the distribution function of the lead time demand. However, in reality, it is quite difficult and time- consuming to know exactly what the distribution is. On the topic, Scarf [22] developed a min-max solution to the newsvendor problem in which only the first two moments of the lead time demand distribution are assumed to be known. It was beautifully expressed but lengthy and quite difficult to understand. Gallego et al. [3], on other hand, made Scraf'[22] ordering rule very easy to understand. Moon et al. [12] found out some valuable applications of the distribution free approach for different types of inventory models.

Moon et al. [13] developed a model on the distribution free continuous review inventory system with a service level constraint. Many researchers have tried to extend this model with different approach. Ouyang et al. [19] discussed a mixture inventory model involving variable lead time with a service level constraint. Chu et al. [2] extended the distribution free model to an improved inventory model with a service level constraint and variable lead time. Lee et al. [10] found out a computational algorithm for an inventory model with a service level constraint, and a lead time demand with a mixture of distributions. Tajbakhsh [25] studied the distribution free continuous-review inventory model with a service level constraint in which Moon et al. [13] derived an iterative procedure to find an optimal solution. He derived some closed form expressions for the model.

The purpose of this study is to obtain the optimal cost with respect to optimal order quantity and lead time under the effect of a service level constraint by applying the distribution free approach. To the best of our knowledge, the author has developed a distribution free continuous-review model with a service level constraint using piecewise linear lead time crashing cost.

A solution procedure is developed to find the optimal solution and numerical solution is presented to illustrate the proposed model. The solution process is furnished to establish the optimal solution and the sensitivity analysis has been carried out to illustrate the behaviours of the planned model.

This paper is organized as follows: The primary assumptions and notations are provided in Section 3. In Section 4, mathematical model is developed. An efficient algorithm is developed to obtain the optimal solution in Section 5. A numerical example is provided in Section 6, to illustrate the results. In Section 7 sensitivity analysis of the parameters is provided has been performed. In Section 8, managerial implications are also included. Finally, conclusion and suggestion is given in Section 9.

3. NOTATIONS AND ASSUMPTIONS

To establish the mathematical model, the following notations and assumptions are used as follows

3.1. NOTATIONS

To develop the proposed model, we adopt the following notations

Q Order quantity

R Reorder point

L Lead time in weeks

h Holding cost per unit time

A Fixed ordering cost per order

D Average demand per year

X Lead time demand which has a probability distribution function F

μ Mean of the lead time demand

σ Standard deviation of the lead time demand

$E(x)$ Expected value of x

x^+ Maximum value of x and 0, i.e. $x^+ = \max\{x, 0\}$

$E(X-r)^+$ Expected shortage per replenishment cycle

$f(x)$ Density of demand during the lead-time

$F(x)$ Cumulative distribution of lead time demand

β Expected demand satisfied per replenishment cycle/expected demand per replenishment cycle.

3.2. ASSUMPTIONS

To develop the proposed model, we adopt the following assumptions

1. The inventory is continuously reviewed. The buyers place an order when the on hand inventory reaches the reorder point$R$.

2. The reorder point (ROP) equals the sum of the expected demand during lead time and the safety stock. The reorder point R= expected demand during lead time +safety stock(SS), and SS=k× (standard deviation of lead time), that is, $R = DL + k\sigma\sqrt{L}$ where k is safety factor.

3. The lead time$L$ consists of n mutually independent components. The ith component has a normal duration b_i, minimum duration a_i, and crashing cost per unit time c_i. For convenience, we rearrange $c_i$such that$c_1 < c_2 < c_3 < .... < c_n$.

4. The components of lead time are crashed one at a time starting with the component of least c_i and so on. Inter-dependence among the components of lead time may require crashing of more than one component at a time. Because, in many realistic cases, the lead time components are independent, and for the sake of simplicity otherwise, we assume that there is no inter-dependence among the components of lead time.

5. Let $L_0 = \sum_{i=1}^{n} b_i$and L_i be the length of lead time with components 1, 2, 3, ..., i crashed to their minimum duration, then L_i can be expressed as $L_i = L_0 - \sum_{j=1}^{n}(b_j - a_j)$, $i = 1, 2, ..., n$; and the lead time crashing cost per cycle $C(L)$ is given by $C(L) = c_i = (L_{i-1} - L) +$

$\sum_{j=1}^{i-1} c_j (b_j - a_j)$, $L \in (L_i, L_{i-1})$. In addition, the length of lead time is equal for all shipping cycles, and the lead time crashing costs occur in each shipping cycle. (See Liao et al. [9]).

6. The extra costs incurred by the vendor will be fully transferred to the buyer if shortened lead time is required.

4. MODEL FORMULATION

A probabilistic inventory model in which the lead time is a decision variable is presented. It is assumed that the demand follows distribution free and the lead time consists of n components each having a different cost for reduced lead time. The total crashing cost is represented

$$C(L) = c_i (L_{i-1} - L) + \sum_{j=1}^{i-1} c_j (b_j - a_j), \quad L \in (L_i, L_{i-1}).$$

The values of lead time crashing cost for the values of L consist of n mutually independent components. The service level is more than 50% and this constraint is used to reduce the cost on the distribution free continuous review inventory model.

We consider the cost function of Moon et al. [12] as

$$TC(Q, R) = \frac{AD}{Q} + h\left(\frac{Q}{2} + R - \mu\right). \tag{1}$$

Subject to $n(r) \leq \beta Q$. Here $n(r) = \int_r^\infty (X - r) f(x)\, dx$is the expected number of stock outs per cycle.

Then, we introduce the total cost function, which is done through the lead time$L$consists of n mutually independent components as described in assumption [3]. The total cost expression is as follows

$$TC(Q, R, L) = \frac{AD}{Q} + h\left(\frac{Q}{2} + R - \mu\right) + \frac{D}{Q} C(L). \tag{2}$$

Because the lead time crashing cost follows a lead time Lconsists of n mutually independent components, the cost function can be rewritten as

$$TC\left(Q,R,L\right)=\frac{AD}{Q}+h\left(\frac{Q}{2}+R-\mu\right)+\frac{D}{Q}\left[c_i\left(L_{i-1}-L\right)+\sum_{j=1}^{i-1}c_j\left(b_j-a_j\right)\right]. \quad (3)$$

The first term in (3) is the ordering cost, the second term represents the holding cost and the third term represents the lead time crashing cost.

Note that here $R-\mu$ is the safety stock. Our objective is to minimize the total cost $TC\left(Q,R,L\right)$subject to a specified fill rate. The fill rate is defined as the partial demand satisfied directly from inventory. This service measure is denoted by β and is obtained by

$$\beta=\frac{\text{expected demand satisfied per replenshment cycle}}{\text{expected demand per replenishment cycle}}. \quad (4)$$

Not only does β consider the probability of stock outs but also it takes the size of shortage into account. That is why the fill rate is considerably appealing to practitioners. Using (4), we now write

$$\beta=1-\frac{E\left(X-r\right)^{+}}{Q}.$$

This reduces to

$$\begin{aligned}\beta&=\frac{Q-E\left(X-r\right)^{+}}{Q},\\ \beta Q&=Q-E\left(X-r\right)^{+},\\ E\left(X-r\right)^{+}&=Q-\beta Q,\\ E\left(X-r\right)^{+}&=Q\left(1-\beta\right).\end{aligned} \quad (5)$$

Let $\Delta=r-\mu$, where Δ is the safety stock. We can rewrite (3) as

$$TC\left(Q,\Delta,L\right)=\frac{AD}{Q}+h\left(\frac{Q}{2}+\Delta\right)+\frac{D}{Q}C\left(L\right).$$

The objective is to minimize $TC\left(Q,R,L\right)$ subject to a specified service level constraint the least favourable distribution. This model is solved by the min-max distribution free approach, which was suggested by Scarf [21] and further explained easily by Gallego et al. [3]. Using this concept, we obtain the minimum cost at the optimal $TC\left(Q,\Delta,L\right)$for the least favourable distribution function in F (see Gallego et al. [3]).

Lemma 1. *Now we consider the distribution free approach. We make no assumption on the distribution F of X other than saying that it belongs to the class $\Im$ of cumulative distribution functions with mean μ and variance σ^2. If we replace $n(r)$ in equation (1) by its worst case upper bound, we can keep the service level against the worst possible distribution in $\Im$. To this end, we need the following proposition.*

Proposition 1.

$$E(X-r)^{+} \leq \frac{\sqrt{\sigma^2 L + (r-\mu)^2} - (r-\mu)}{2}. \tag{6}$$

Moreover, the upper bound (6) is tight. That is, for every r, there exists a distribution $F^* \in \Im$ where the bound (6) is tight.

Proof. Notice that $E(X-r)^{+} = \frac{|X-r|+(X-r)}{2}$. If we take expectations on the above and use the following Cauchy-Schwarz inequality, we can obtain (6).

$$E|X-r| \leq \sqrt{E(X-r)^2} = \sqrt{\sigma^2 + (r-\mu)^2}$$

Now we prove the tightness of the upper bound. For every r, consider the two point cumulative distribution F^* assigning weight

$$\alpha = \frac{\sqrt{\sigma^2 + (r-\mu)^2} + (r-\mu)}{2\sqrt{\sigma^2 + (r-\mu)^2}}$$

To

$$\mu - \sigma\sqrt{\frac{1-\alpha}{\alpha}} = r - \sqrt{\sigma^2 + (r-\mu)^2}$$

And weight

$$1-\alpha = \frac{\sqrt{\sigma^2 + (r-\mu)^2} - (r-\mu)}{2\sqrt{\sigma^2 + (r-\mu)^2}}$$

To

$$\mu + \sigma\sqrt{\frac{1-\alpha}{\alpha}} = r + \sqrt{\sigma^2 + (r-\mu)^2}$$

Clearly (6) holds with equality and it is easy to verify that $F^* \in \Im$.

Moreover, this upper bound is tight. By considering Δ_β as the safety stock with respect to β, we can obtain Δ_β as follows

$$\frac{\sqrt{\sigma^2 L + \Delta_\beta^2} - \Delta_\beta}{2} = (1-\beta)\,Q$$

$$\sqrt{\sigma^2 L+\Delta_\beta^2}=2\left(1-\beta\right)Q+\Delta_\beta$$

Taking square root on both sides

$$\begin{aligned}
\Delta_\beta^2 &= \left(2\left(1-\beta\right)Q+\Delta_\beta\right)^2,\\
\Delta_\beta^2 &= \left(2\left(1-\beta\right)Q+\Delta_\beta\right)^2-\sigma^2 L,\\
\Delta_\beta^2 &= 4\left(1-\beta\right)^2 Q^2+\Delta_\beta^2+4Q\Delta_\beta\left(1-\beta\right)-\sigma^2 L,\\
\Delta_\beta^2-\Delta_\beta^2 &= 4\left(1-\beta\right)^2 Q^2+4Q\Delta_\beta\left(1-\beta\right)-\sigma^2 L,\\
-4Q\Delta_\beta\left(1-\beta\right) &= 4\left(1-\beta\right)^2 Q^2-\sigma^2 L,\\
\Delta_\beta &= -\left[\frac{4\left(1-\beta\right)^2 Q^2-\sigma^2 L}{4Q\left(1-\beta\right)}\right]\\
&\quad -\left[\frac{4\left(1-\beta\right)^2 Q^2}{4Q\left(1-\beta\right)}\right]+\frac{\sigma^2 L}{4Q\left(1-\beta\right)},\\
\Delta_\beta &= \frac{\sigma^2 L}{4Q\left(1-\beta\right)}-Q\left(1-\beta\right).
\end{aligned}\tag{7}$$

Using Δ_β, we obtain

$$\begin{aligned}
TC\left(Q,\Delta_\beta,L\right) &= \frac{AD}{Q}+h\left(\frac{Q}{2}+\Delta_\beta\right)+\frac{D}{Q}C\left(L\right)\\
&= \frac{AD}{Q}+h\left(\frac{Q}{2}+\frac{\sigma^2 L}{4Q\left(1-\beta\right)}-Q\left(1-\beta\right)\right)+\frac{D}{Q}C\left(L\right)\\
&= \frac{AD}{Q}+\frac{hQ}{2}+\frac{h\sigma^2 L}{4Q\left(1-\beta\right)}-hQ\left(1-\beta\right)+\frac{D}{Q}C\left(L\right)\\
&= \frac{AD}{Q}+hQ\left(\frac{1}{2}-\left(1-\beta\right)\right)+\frac{h\sigma^2 L}{4Q\left(1-\beta\right)}+\frac{D}{Q}C\left(L\right)\\
&= \frac{AD}{Q}+hQ\left(\frac{1}{2}-1+\beta\right)+\frac{h\sigma^2 L}{4Q\left(1-\beta\right)}+\frac{D}{Q}C\left(L\right)\\
&= \frac{AD}{Q}+hQ\left(\beta-\frac{1}{2}\right)+\frac{h\sigma^2 L}{4Q\left(1-\beta\right)}+\frac{D}{Q}C\left(L\right).
\end{aligned}\tag{8}$$

Using this and taking Qas Q_β

We want to minimize $TC\left(Q_\beta,\Delta_\beta,L\right)$over $Q_\beta>0$ and $L\in\left(L_i,\,L_{i-1}\right)$

$$\frac{\partial TC\left(Q_\beta,\Delta_\beta,L\right)}{\partial Q_\beta^2}=\frac{2AD}{Q_\beta^3}+\frac{2h\sigma^2 L}{4Q_\beta^3\left(1-\beta\right)}+\frac{2D}{Q_\beta^3}C\left(L\right)>0.$$

Because it can be verified that $\frac{\partial TC\left(Q_\beta,\Delta_\beta,L\right)}{\partial Q_\beta^2}>0$, for $Q_\beta>0$, $L\in\left(L_i,\,L_{i-1}\right)$, $TC\left(Q_\beta,\Delta_\beta,L\right)$ is convex function inQ_β.

To obtain a unique global minimum of$TC\left(Q_\beta,\Delta_\beta,L\right)$ and $L\in\left(L_i,\,L_{i-1}\right)$ we solve

$$\frac{\partial TC\left(Q_\beta,\Delta_\beta,L\right)}{\partial Q_\beta}=-\frac{AD}{Q_\beta^2}+h\left(\beta-\frac{1}{2}\right)-\frac{h\sigma^2L}{4Q_\beta^2\left(1-\beta\right)}-\frac{D}{Q_\beta^2}C\left(L\right)$$
$$\frac{\partial TC\left(Q_\beta,\Delta_\beta,L\right)}{\partial Q_\beta}=0.$$

By assumption, the service level β is $0.5<\beta<1$. Hence, we obtain a unique global minimum as follows

$$\frac{\partial TC\left(Q_\beta,\Delta_\beta,L\right)}{\partial Q_\beta}=0$$

$$-\frac{AD}{Q_\beta^2}+h\left(\beta-\frac{1}{2}\right)-\frac{h\sigma^2L}{4Q_\beta^2\left(1-\beta\right)}-\frac{D}{Q_\beta^2}C\left(L\right)=0$$

$$-\frac{AD}{Q_\beta^2}-\frac{h\sigma^2L}{4Q_\beta^2\left(1-\beta\right)}-\frac{D}{Q_\beta^2}C\left(L\right)=-h\left(\beta-\frac{1}{2}\right)$$

$$\frac{-4AD\left(1-\beta\right)-h\sigma^2L-4\left(1-\beta\right)DC\left(L\right)}{4Q_\beta^2\left(1-\beta\right)}=-h\left(\beta-\frac{1}{2}\right)$$

$$\left(\frac{4AD\left(1-\beta\right)+h\sigma^2L+4\left(1-\beta\right)DC\left(L\right)}{4Q_\beta^2\left(1-\beta\right)}\right)=h\left(\beta-\frac{1}{2}\right)$$

$$\left(\frac{4D\left(1-\beta\right)\left(A+C\left(L\right)\right)+h\sigma^2L}{4Q_\beta^2\left(1-\beta\right)}\right)=h\left(\beta-\frac{1}{2}\right)$$

$$\left(\frac{4D\left(1-\beta\right)\left(A+C\left(L\right)\right)+h\sigma^2L}{4Q_\beta^2\left(1-\beta\right)}\right)=h\left(\frac{2\beta-1}{2}\right)$$

$$4D\left(1-\beta\right)\left(A+C\left(L\right)\right)+h\sigma^2L=\left(2\beta-1\right)h2Q_\beta^2\left(1-\beta\right)$$

$$\frac{4D\left(1-\beta\right)\left(A+C\left(L\right)\right)+h\sigma^2L}{2h\left(2\beta-1\right)\left(1-\beta\right)}=Q_\beta^2$$

$$Q_\beta^2=\frac{4D\left(1-\beta\right)\left(A+C\left(L\right)\right)+h\sigma^2L}{2h\left(2\beta-1\right)\left(1-\beta\right)}$$

$$Q_\beta=\sqrt{\frac{4D\left(1-\beta\right)\left(A+C\left(L\right)\right)+h\sigma^2L}{2h\left(2\beta-1\right)\left(1-\beta\right)}}$$

Again, by using the condition$0 < 2\beta - 1 < 1$, we find

$$Q_\beta > \sqrt{\frac{2AD}{h\,(2\beta - 1)}} > \sqrt{\frac{2AD}{h}} = \text{EOQ}.$$

To find the necessary conditions for the minimum total cost$TC\,(Q, \Delta_\beta, L)$ and $L \in (L_i,\, L_{i-1})$we have

$$\frac{\partial TC\,(Q_\beta, \Delta_\beta, L)}{\partial Q_\beta} = -\frac{AD}{Q_\beta^2} + h\left(\beta - \frac{1}{2}\right) - \frac{h\sigma^2 L}{4Q_\beta^2\,(1-\beta)} - \frac{D}{Q_\beta^2} C\,(L)\,, \quad (9)$$

$$\frac{\partial TC\,(Q_\beta, \Delta_\beta, L)}{\partial L} = \frac{h\sigma^2 L}{4Q_\beta\,(1-\beta)} - \frac{DC_i}{Q_\beta^2}. \quad (10)$$

Therefore, we obtain the optimal values of our decision variables as

$$Q_{i\beta} = \sqrt{\frac{4D\,(1-\beta)\,(A + C\,(L)) + h\sigma^2 L}{2h\,(2\beta - 1)\,(1-\beta)}}, \quad (11)$$

$$\Delta_{i\beta} = \frac{\sigma^2 L}{4Q\,(1-\beta)} - Q\,(1-\beta)\,. \quad (12)$$

Further, based on the convexity and concavity behaviour of the objective function with respect to the decision variable, the following algorithm is designed to find the optimal values of order quantity Q, lead time L and safety stock Δ_β which minimizes the total cost $TC\,(Q^*,\, \Delta^*,\, L^*)$.Therefore we establish the following iterative algorithm to obtain the optimal solution.

5. ALGORITHM

Step 1. For each $L_{i,}\, i = 1,\, 2,\, 3......, n$compute $Q_{i\beta}$and $\Delta_{i\beta}$using the equations (11) and (12).

Step 2. For each$(Q_{i\beta},\, \Delta_{i\beta})$,compute$TC\,(Q_{i\beta}, \Delta_{i\beta}, L_i)$, for $i = 1,\, 2,\, 3......, n$ by using the equation (8).

Step 3. Set$TC\,(Q_i^*,\, \Delta^*,\, L_i^*) = \min_{\,i=0,\,1,......n}\; TC\,(Q_{i\beta}, \Delta_{i\beta}, L_i)$. Then $TC\,(Q^*,\, \Delta^*,\, L^*)$is the optimal solution.

6. NUMERICAL EXAMPLE

In this section, a numerical example is given to illustrate the above solution procedure. Let us consider the inventory system with the following data

D= 600 units per year, A=\$200 per order, h=\$20 per unit per year, $\sigma = 6$ units per week,$\beta = 0.95$

The lead time has three components with data shown in Table 1 and summarized in the lead time data in Table 2. Applying the solution procedure of the above algorithm, the computational results are presented in Table 3. The optimal solutions from Table 3 can be read off as lead time $L^* = 6$ weeks, order quantity $Q^* = 127$ units, safety stock$\Delta_\beta^* = 336$and the corresponding total cost $TC\left(Q_\beta, \Delta_\beta, L\right) = \$2284$. A graphical representation is presented to show the convexity of $TC\left(Q_\beta^*, \Delta_\beta^*, L^*\right)$ in Figure 2 and the graphical representation of the total cost is given in the Figure 3.

Table 1. Lead time components with data

Lead time component i	Normal duration b_i (days)	Minimum duration a_i (days)	Unit crashing cost c_i (days)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Table 2. Summarized lead time data

Lead time in (week)	**R(L)**
8	0
6	5.6
4	22.4
3	57.4

Table 3. Illustration of the solution procedure for the numerical example

L	Q_β	Δ_β	$TC\left(Q_\beta, \Delta_\beta, L\right)$
8	129	456	2315
6	**127**	**336**	**2284**
4	128	224	2307
3	135	176	2439

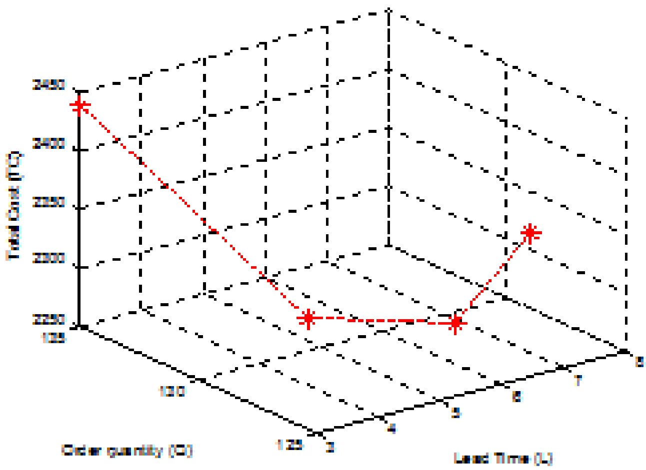

Figure 1. Graph representing the convexity of $TC\,(Q_\beta, \Delta_\beta, L)$

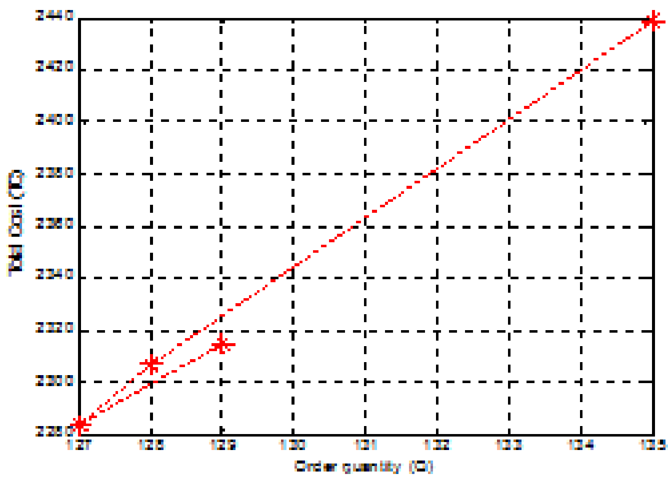

Figure 2. Graphical representation of the optimal solution in $TC\,(Q_\beta, \Delta_\beta, L)$.

7. SENSITIVITY ANALYSIS

We now study the effects of changes in the system parameters Demand, Ordering cost and Holding cost on the optimal order quantity Q, lead time L, and safety stock Δ_β in order to minimize the cost $TC(Q_\beta, \Delta_\beta, L)$ of the given example.

7.1. EFFECTS OF DEMAND ON THE OPTIMAL SOLUTION

In order to study how various demand$D$ affect the optimal solution of the proposed model, the demand sensitivity analysis is performed by changing the parameter of D by 100, 200, 300, 400, 500 and keeping remaining parameter unchanged.

The results of the demand analysis are shown in Table 4 and the corresponding curves of the minimum total cost are plotted in Figure 3.

Table 4. Effects of demand on optimal solution of the given example

Demand (D)	L weeks	R(L)	Q	Δ_β	$TC(Q_\beta, \Delta_\beta, L)$
500	8	0	120	425	2153
	6	**5.6**	**118**	**312**	**2116**
	4	22.4	118	207	2126
	3	57.4	125	162	2241
400	8	0	110	390	1979
	6	5.6	107	285	1934
	4	**22.4**	**107**	**188**	**1929**
	3	57.4	112	146	2024
300	8	0	99	353	1788
	6	5.6	96	255	1732
	4	**22.4**	**95**	**166**	**1709**
	3	57.4	99	129	1780
200	8	0	87	310	1574
	6	5.6	83	221	1503
	4	**22.4**	**81**	**142**	**1456**
	3	57.4	83	108	1497
100	8	0	74	261	1325
	6	5.6	68	181	1232
	4	22.4	64	112	1149
	3	**57.4**	**64**	**83**	**1147**

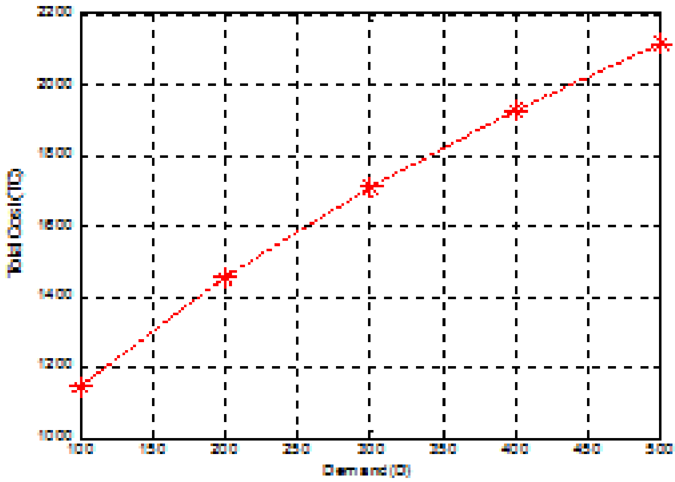

Figure 3. Curves of minimum total cost TC for various demands

7.2. EFFECTS OF ORDERING COST ON OPTIMAL SOLUTION

In order to study how various ordering cost$A$ affect the optimal solution of the proposed model, the ordering cost sensitivity analysis is performed by changing the parameter of A by 50, 100, 150 and keeping remaining parameter unchanged.

The results of the ordering cost analysis are shown in Table 5 and the corresponding curves of the minimum total cost are plotted in Figure 4.

Table 5. Effects of ordering cost on optimal solution of the given example

Ordering cost (A)	L weeks	R(L)	Q	Δ_β	$TC\,(Q_\beta, \Delta_\beta, L)$
50	8	0	81	287	1455
	6	**5.6**	**78**	**207**	**1407**
	4	22.4	80	140	1443
	3	57.4	91	119	1646
100	8	0	99	353	1788
	6	**5.6**	**97**	**257**	**1749**
	4	22.4	99	173	1778
	3	57.4	108	141	1946
150	8	0	115	408	2068
	6	**5.6**	**113**	**300**	**2034**
	4	22.4	114	200	2060
	3	57.4	123	159	2207

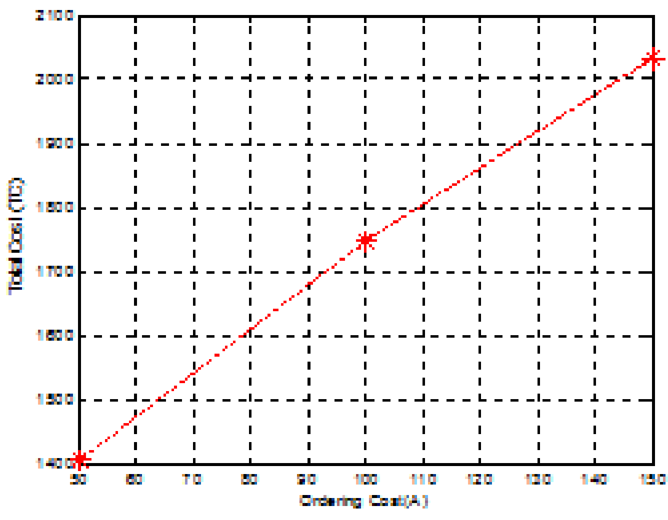

Figure 4. Curves of minimum total cost TC for various ordering cost

7.3. EFFECTS OF HOLDING COST ON OPTIMAL SOLUTION

In order to study how various holding cost h affect the optimal solution of the proposed model, the holding cost sensitivity analysis is performed by changing the parameter of h by 5, 10, 15 and keeping remaining parameter unchanged.

The results of the holding cost analysis are shown in Table 6 and the corresponding curves of the minimum total cost are plotted in Figure 5.

Table 6. Effects of holding cost on optimal solution of the given example

Holding cost (h)	L weeks	R(L)	Q	Δ_β	$TC\,(Q_\beta, \Delta_\beta, L)$
5	**8**	**0**	**238**	**844**	**1070**
	6	5.6	239	634	1077
	4	22.4	247	432	1111
	3	57.4	264	344	1189
10	8	0	173	614	1555
	6	**5.6**	**173**	**458**	**1554**
	4	22.4	177	309	1591
	3	57.4	188	245	1696
15	8	0	145	514	1955
	6	**5.6**	**144**	**381**	**1941**
	4	22.4	146	256	1973
	3	57.4	155	202	2095

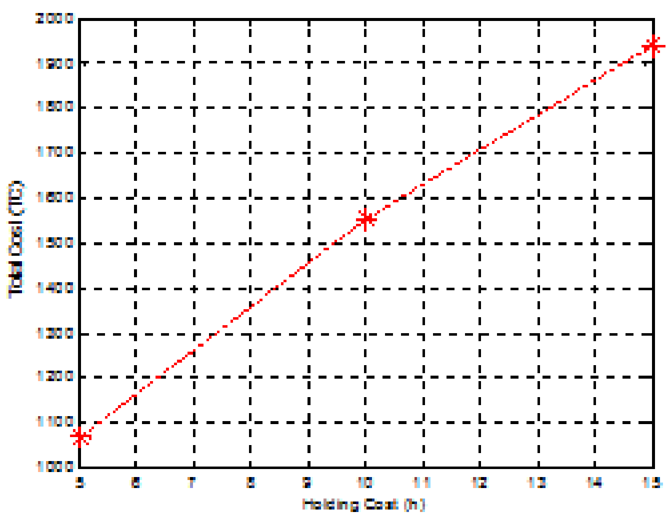

Figure 5. Curves of minimum total cost TC for various holding cost

7.4. EFFECTS OF SERVICE LEVEL CONSTRAINT ON OPTIMAL SOLUTION

In order to study how various service level constraint β affect the optimal solution of the proposed model, the service level constraint sensitivity analysis is performed by changing the parameter of β by 0.90, 0.91, 0.92, 0.93, 0.94 and keeping remaining parameter unchanged. The results of the service level constraint analysis are shown in Table 7 and the corresponding curves of the minimum total cost are plotted in Figure 6.

Table 7. Effects of service level constraint on optimal solution of the given example

Service level constraint (β)	L weeks	R(L)	Q	Δ_β	$TC\,(Q_\beta, \Delta_\beta, L)$
0.90	8	0	130	920	2074
	6	**5.6**	**129**	**686**	**2072**
	4	22.4	133	464	2121
	3	57.4	141	368	2262
0.91	8	0	129	823	2112
	6	**5.6**	**128**	**612**	**2107**
	4	22.4	131	414	2154
	3	57.4	140	327	2294
0.92	8	0	128	728	2153
	6	**5.6**	**128**	**541**	**2144**
	4	22.4	130	365	2188
	3	57.4	139	288	2327
0.93	8	0	128	635	2199
	6	**5.6**	**127**	**471**	**2185**
	4	27.2	129	317	2224
	3	55.4	137	250	2362
0.94	8	0	128	545	2251
	6	**5.6**	**127**	**403**	**2231**
	4	22.4	129	270	2263
	3	57.4	136	213	2399

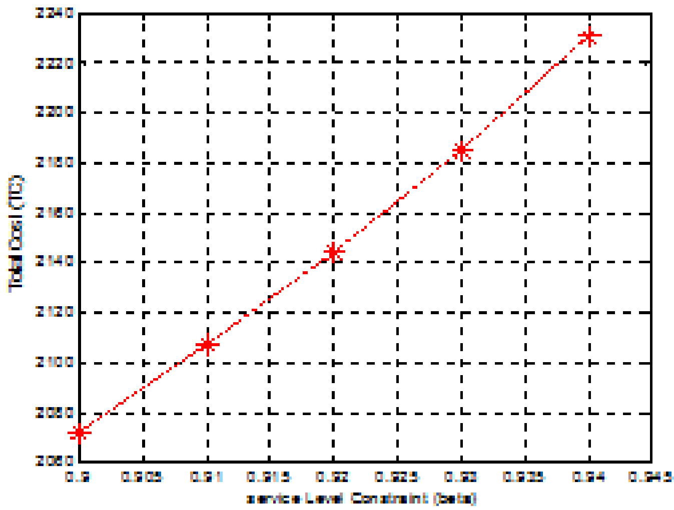

Figure 6. Curves of minimum total cost TC for various service level constraints

8. MANAGERIAL IMPLICATIONS

There are some interesting managerial implications in the above analyses. We make the following observations.

1. Tale (4) shows that when the demand D is decreasing, the total cost TC of the system is also decreased.

2. Tale (5) shows that when the ordering cost A is increasing, the total cost TC of the system is also increased.

3. Tale (6) shows that when the holding cost h is increasing, the total cost TC of the system is also increased.

4. Tale (7) shows that when the service level constraint β is increasing, the total cost TC of the system is also increased.

9. CONCLUSION

We consider the distribution free continuous-review model with a service level constraint. We developed closed form expression for the order quantity, safety stock and lead time.

Our major purpose in developing this model is to reduce total cost by using the lead time Lconsists of n mutually independent components. Consequently, the lead time crashing cost function is described using a piecewise linear function.

In certain cases, lead time can be reduced but at an added cost. In this article we discuss inventory models where lead time is one of the decision variables.A solution algorithm was developed to obtain an improved result.

A mathematical model is employed in this study for optimizing the lead time, order quantity and safety stock in order to minimize the total cost. In addition, sensitivity analysis has been carried out to demonstrate the behaviors of the proposed model and some managerial implications are also included.

The algorithm with the help of the software Matlab 2008 is furnished to determine the optimal solution. In future research work, in this model can be extended fuzzy demand and budget constraints might be considered. Moreover, this model can be extended to consider different lead time crashing cost.

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