Communications in Applied Analysis, **22**, No. 2 (2018), 233-244 ISSN: 1083-2564

SIMULTANEOUS FARTHEST POINTS AND NORM DERIVATIVES IN BANACH SPACES

M. IRANMANESH[1] AND F. SOLEIMANY[2]

[1,2]Department of Mathematical Sciences
Shahrood University of Technology
P.O. Box 3619995161-316, Shahrood, IRAN

ABSTRACT: In this paper, we present various characterizations of the simultaneous farthest point of elements by bounded sets in normed spaces. First, we express short proofs for previous results. We did this work by using norm derivatives method. Then we extend to C^*-algebras some results on the simultaneous farthest point that found in $B(H)$ the algebra of all bounded operators on a Hilbert space by applying of the concept of numerical rang and Gelfand-Naimark theorem.

AMS Subject Classification: 41A28, 46L05, 46B28

Key Words: simultaneous farthest points, Gateaux differentiable, C^*-algebras

Received: September 23, 2017; **Accepted:** January 14, 2018;
Published: March 22, 2018. **doi:** 10.12732/caa.v22i2.6
Dynamic Publishers, Inc., Acad. Publishers, Ltd. http://www.acadsol.eu/caa

1. INTRODUCTION

Let X be a Banach space and S be a closed bounded subset of X. For $x \in X$; a point $s_0 \in S$ is called the farthest point to x from S if $\|s - x\| = s(x, S)$ where $s(x, S) = \sup_{s \in S} \|x - s\|$. The mapping $F_S : X \longrightarrow 2^S$ defined by $F_S(x) = \{s \in S : \|s - x\| = s(x, S)\}$ is called the farthest point map. We call S a remotal set if for each $x \in X$; the set $F_S(x)$ is not empty. Several results related to farthest points in the context of normed linear space can be obtained in [2, 6, 7, 8, 14, 15, 16] respectively. Recently the authors interested in formulating and studying the notion of simultaneously remotal see [3, 13]. In this regard, the characterizations of simultaneous farthest point of elements by bounded sets established by Naraghirad [13]. He did this work in terms of the extremal points of the closed unit ball B_{X^*} of X^* where X^* is the dual space of X. In this paper, we did this work in terms of norm derivatives and get some results that have been inspired by [13]. The structure of this paper is as follows. In Section 2, we give some preliminary results on simultaneous remotal sets and obtain characterizations of the simultaneous farthest point of elements by bounded sets based on norm derivatives such a tool seems to be profitable, at least to formulate in a simple way concerning the previous results in normed spaces. We investigate the simultaneous farthest point for bounded sets in $C(X)$, the space of all continuous functions on X and $L_1(X,\mu)$ the space of all μ-integrable function defined on the X. We obtain results concerning simultaneous farthest point by elements of bounded sets in $B(H)$, the algebra of all bounded operators on a Hilbert space H and C^*-algebras. Our main tools are the concept of numerical rang and the Gelfand-Naimark theorem.

2. CHARACTERIZATION OF SIMULTANEOUS FARTHEST POINTS

Let X be a normed vector space, $W \subseteq X$ and $x \in X$. The number $\mathbf{d}(x,W) = \inf\limits_{w\in W} \|x-w\|$ is called the distance of x to the set W. Every $w_0 \in W$ for which $\|x - w_0\| = \mathbf{d}(x,W)$ is the best approximation from x by elements of W. The set of all best approximations to x from W is denoted by $\mathbf{P}_W(x)$. Thus

$$\mathbf{P}_W(x) := \{w \in W \mid \|x-w\| = \mathbf{d}(x,W)\}. \tag{2.1}$$

We recall that the deviation of a set $S \subset X$ from a set $W \subseteq X$ is defined by

$$\mathbf{S}(W,S) := \sup_{s\in S} \inf_{\omega\in W} \|s-w\|. \tag{2.2}$$

An element $s_0 \in S$ is called a simultaneous farthest point to W from S if $\inf_{\omega\in w} \|s_0 - w\| = \mathbf{S}(W,S)$. The set of all simultaneous farthest points to W

from S will be denoted by $\mathbf{F}_S(W)$, e, i.

$$\mathbf{F}_S(W) := \{s \in S : \inf_{\omega \in w} \|s - w\| = \mathbf{S}(W, S)\}. \tag{2.3}$$

If there exists at least one simultaneous farthest point to W from the bounded set S, then S is called a simultaneous remotal subset of X respect to W. Of course, when W is only one point the preceding concepts are just remotal set and farthest point.

Let X^* be dual space of X. For $x \in X$, we define the subdifferential of norm by

$$\delta\|x\| = \{v \in X^* : v(x) = \|x\|, \|v\| \leq 1\},$$

We say the norm of X is Gateaux differentiable at $x \neq 0$ if for all $y \in X$:

$$\tau(x, y) = \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t} \textit{ exists.} \tag{2.4}$$

It is well known that

$$\tau(x, y) = \max_{v \in \delta\|x\|} v(y).$$

We say the norm of X is Frechet differentiable at x if the convergence to the limit in (2.4) is uniform for all $y \in X$. In general the norm is not Gateaux differentiable at $x \neq 0$ in X. Nevertheless, it is known [5] that the limits always exist. Norm derivatives of the spaces: $L^p(d\mu), 1 \leq p < \infty$, and $C(X)$ space of real continuous functions on a compact Hausdorff space X has been studied extensively, see for example in [9]. In [1] given necessary and sufficient conditions for the existence of the Gateaux derivative of the norm in $B(H)$ the space of bounded operators, on a Hilbert space H, with the uniform norm. They show that the norm in $B(H)$ at the point f is Frechet differentiable if and only if f attains its norm at $\pm x$, $\|x\| = 1$ and $\sup_{\substack{x \perp y \\ |y\| = 1}} \|f(y)\| \leq \|f\|$. The function f is convex on X if for each $x, y \in X$ and for each $\lambda \in (0, 1)$

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

Lemma 1. [9] Let $f : X \longrightarrow \mathbf{R}$ be a convex function. For $x_0, x \in X$ the function

$$t \mapsto \frac{f(x_0 + tx) - f(x_0)}{t}, \tag{2.5}$$

is non-decreasing for $t > 0$.

Lemma 2. [4] Let X be a Banach space and let x and t be in X such that $\|x\| \geq \|x-t\|$, then $k\|x\| \geq \|kx-t\|$ for any $k \geq 1$.

In the following, we give a characterization of simultaneous farthest point in X.

Theorem 3. *Let S be a bounded subset of X and W be compact set such that $W \cap S = \emptyset$ and $s_0 \in S$. The following statements are true:*

i) $s_0 \in \mathbf{F}_S(W)$ if there exists $w_0 \in \mathbf{P}_W(s_0)$ such that

$$\tau(w_0 - s, s - s_0) \geq 0, \quad (s \in S). \tag{2.6}$$

ii) If $s_0 \in \mathbf{F}_S(f)$ then there exists $w_0 \in \mathbf{P}_W(s_0)$ such that

$$\tau(w_0 - s_0, s - s_0) \geq 0, \quad (s \in S). \tag{2.7}$$

Moreover if $\tau(w_0 - s_0, s - s_0) \geq \|s - s_0\|^2$ then $s_0 \in \mathbf{F}_S(W)$,

Proof. i) The inequality (2.6) implies that for $t \geq 0$,

$$\|w_0 - s + t(s - s_0)\| \geq \|w_0 - s\|.$$

Since τ is increasing function, then $\|w_0 - s_0\| \geq \|w_0 - s\|$ for $t = 1$, the meaning $\sup_{s\in S} \|s - w_0\| = \|s_0 - w_0\|$. Now,

$$\begin{aligned} \sup_{s\in S} \inf_{w\in W} \|s - w\| &\leq \inf_{w\in W} \sup_{s\in S} \|s - w_0\| \\ &\leq \sup_{s\in S} \|s - w_0\| = \|s_0 - w_0\| \\ &= \inf_{w\in W} \|s_0 - w\| \\ &\leq \sup_{s\in S} \inf_{w\in W} \|s - w\|. \end{aligned}$$

Hence $\inf_{w\in W} \|s_0 - w\| = \sup_{s\in S} \inf_{w\in W} \|s - w\|$. This implies that $s_0 \in \mathbf{F}_S(W)$.

ii) Let $s_0 \in \mathbf{F}_S(W)$ then $\inf_{w\in W} \|s_0 - w\| = \mathbf{S}(W, S)$. Since W is compact set, there exists $w_0 \in W$ such that

$$\|s_0 - w_0\| = \inf_{w\in W} \|s_0 - w\|.$$

The meaning that $w_0 \in \mathbf{P}_W(s_0)$. By Lemma 2 for every $s \in S$ and $t \geq 1$ we have

$$\|t(w_0 - s_0)\| \geq \|t(w_0 - s_0) - (s - s_0)\|,$$

In particular, for $t = 1$ implies that

$$\|w_0 - s_0 - (s - s_0)\| - \|w_0 - s_0\| \leq 0. \tag{2.8}$$

By Lemma 1 and (2.8) we have

$$\tau(w_0 - s_0, s_0 - s) \leq 0,$$

i.e;

$$\tau(w_0 - s_0, s - s_0) \geq 0, \ (\forall s \in S).$$

Which is exactly (2.7). Now if $\tau(w_0 - s_0, s - s_0) \geq \|s - s_0\|^2$, then

$$0 \leq \tau(w_0 - s_0, s - s_0) - \|s - s_0\|^2 \leq \tau(w_0 - s, s - s_0).$$

Hence by part (i), we have $s_0 \in \mathbf{F}_S(W)$.

□

Proposition 4. *Let S be a bounded subset of X, W be compact set such that $W \cap S = \emptyset$ and $s_0 \in S$. If there exists $w_0 \in \mathbf{P}_W(s_0)$ and $k_{w_0} > 0$ such that*

$$\tau(w_0 - s, s - s_0) \geq k_{w_0}\|s_0 - s\|. \ (s \in S) \tag{2.9}$$

Then $\mathbf{F}_S(W) = \{s_0\}$.

Proof. By part (i) Theorem 3 s_0 is the farthest point to S from B. Now if possible, assume s_1 be another simultaneous farthest point to W from S. Then $\inf_{w \in W} \|w - s_1\| = \inf_{w \in W} \|f - s_0\|$. From (2.9) and Lemma 1 for $t = 1$ we get

$$||s_0 - w_0|| \geq ||w_0 - s|| + K_{w_0}||s - s_0||.$$

For $s = s_1$,

$$\begin{aligned} \inf_{w \in W} ||w - s_1|| = ||s_0 - w_0|| &\geq ||w_0 - s_1|| + K_{w_0}||s_1 - s_0|| \\ &\geq \inf_{w \in W} ||w - s_1|| + K_{w_0}||s_1 - s_0|| \\ &\Rightarrow K_{w_0}||s_1 - s_0|| = 0 \\ &\Rightarrow s_1 = s_0. \end{aligned}$$

Therefore s_0 is the unique farthest point to W from S i.e $\mathbf{F}_S(W) = \{s_0\}$. □

Corollary 5. *Let $S \subseteq B_X = \{x \in X : \|x\| \leq 1\}$, and W be compact set such that $W \cap S = \emptyset$ and $s_0 \in S$. The following statements are true:*

i) $s_0 \in \mathbf{F}_S(W)$ *if there exists* $w_0 \in \mathbf{P}_W(s_0)$ *such that*

$$\tau(w_0, -s_0) \geq 1. \tag{2.10}$$

ii) If $s_0 \in \mathbf{F}_S(W)$ *and* $0 \in S$*, then there exists* $w_0 \in \mathbf{P}_W(s_0)$ *such that*

$$\tau(w_0, -s_0) \geq -1. \tag{2.11}$$

Proof. i) From (2.10) and Lemma 1 for $t = 1$ we have

$$\|w_0 - s\| \leq \|w_0\| + 1 \leq \|w_0 - s_0\|, \forall s \in S,$$

then $\|w_0 - s_0\| = \sup_{s\in S} \|w_0 - s\|$, now similar to proof of part (i)Theorem 3, can show that $s_0 \in \mathbf{F}_S(W)$.

ii) Let $s_0 \in \mathbf{F}_S(f)$ then by part (ii) Theorem 3, there exists $w_0 \in \mathbf{P}_W(s_0)$ such that

$$\tau(w_0 - s_0, s - s_0) \geq 0, \ (s \in S). \tag{2.12}$$

then for $s = 0$,

$$0 \leq \tau(w_0 - s_0, -s_0) \leq \tau(w_0, -s_0) + \tau(-s_0, -s_0).$$

Thus

$$-1 = -\tau(-s_0, -s_0) \leq \tau(w_0, -s_0).$$

$\square$

In the following, we give some results by use of above theorems.

Let X be a compact Hausdorff space. It is well know that for any $f, g \in C(X)$,

$$\tau(f, g) = \max_{x\in M_f}(sign f(x)g(x)).$$

Corollary 6. *Let S be a bounded subset of $C(X)$ and W be compact set $W \cap S = \emptyset$ and $s_0 \in S$. The following statements are true:*

i) $g_0 \in \mathbf{F}_S(W)$ *if there exists* $w_0 \in \mathbf{P}_W(x)$ *such that*

$$\max_{x \in M_{w_0 - s}} (sign(w_0 - s)(x)(s - s_0)(x)) \geq 0, \ (s \in S). \tag{2.13}$$

ii) *If* $g_0 \in \mathbf{F}_S(f)$ *then there exists* $w_0 \in \mathbf{P}_W(x)$ *such that*

$$\max_{x \in M_{w_0 - s_0}} (sign(w_0 - s_0)(x)(s - s_0)(x)) \geq 0, \ (s \in S). \tag{2.14}$$

Proof. It is a consequently of Theorem 3. □

Let (X, μ) be a positive measure space and $L_1(X, \mu)$ denote the space of all μ-integrable function defined on the X with the norm $\|.\| = \int_X |.| d\mu$. It is well know that

$$\delta\|f\| = \{v \in X^* : |v(x)| \leq 1, x \in X, v(x) = sign(f(x)), f(x) \neq 0\}.$$

Since $L_1(X, \mu)^* \equiv L^\infty(X, \mu)$ is given by the correspondence $f \to \beta$, where

$$f(g) = \int_X g(x)\beta(x)d\mu(x) \quad (g \in L_1(X, \mu)).$$

Corollary 7. *Let* S *be a bounded subset of* $L(X, \mu)$, *and* W *be compact set* $W \cap S = \emptyset$ *and* $s_0 \in S$. *The following statements are true:*

i) $g_0 \in \mathbf{F}_S(W)$ *if there exists* $w_0 \in \mathbf{P}_W(x)$ *such that*

$$\int (sign(w_0 - s)(x)(s - s_0)(x)) d\mu \geq 0, \ (s \in S). \tag{2.15}$$

ii) *If* $g_0 \in \mathbf{F}_S(f)$ *then there exists* $w_0 \in \mathbf{P}_W(x)$ *such that*

$$\int (sign(w_0 - s_0)(x)(s - s_0)(x)) d\mu \geq 0, \ (s \in S). \tag{2.16}$$

Proof. It is a consequently of Theorem 3. □

Suppose that A is a unital algebra with the unit e. If A be a Banach space with respect to a norm which satisfies the multiplicative inequality

$$\|xy\| \leq \|x\|\|y\| \ (x, y \in A), \tag{2.17}$$

then the pair $(A, \|.\|)$ is called a normed algebra. A complete unital normed algebra is called unital Banach algebra. An involution $*$ on an algebra $\mathbb{A}$ is a mapping $x \longrightarrow x^*$ from $\mathbb{A}$ onto $\mathbb{A}$ such that $(\lambda x+y)^* = \overline{\lambda}x^* + y^*, (xy)^* = y^*x^*$ and $(x^*)^* = x$, for all $x, y \in \mathbb{A}, \lambda \in \mathbb{C}$. An involutive Banach algebra is called a Banach $*$-algebra. A Banach $*$-algebra $\mathbb{A}$ is said to be a C^*-algebra if $\|xx^*\| = \|x\|^2$, for each $x \in \mathbb{A}$.
In this paper, H will denote a Hilbert space and $B(H)$ will denote the bounded linear maps on H. It is routine to show that $B(H)$ is a C^*-algebra relative to involution $T \rightarrow T^*$ defined by:

$$\langle x, Ty\rangle = \langle T^*x, y\rangle \; for \; x, y \in H.$$

Let $f, g \in B(H)$. We denote by Z_f the following set:

$$Z_f := \{\{x_n\} \in H : \|x_n\| = 1, \; \lim_{n\to\infty} \|f(x_n)\| = \|f\|\}. \tag{2.18}$$

The numerical range of g^*f relative to f which is denoted by $W(g^*f)$ is defined as follows:

$$W(g^*f) := \{\lambda \in \mathbb{C} : \lambda = \lim_{n\to\infty} \langle g^*f(x_n), x_n\rangle, \;\; \{x_n\} \in Z_f\}. \tag{2.19}$$

It is well Known that $W(g^*f)$ is a compact set subset of the complex plane [11].
Let A be a subset of $\mathbb{C}$ and $ReA := \{Re(z) : \; z \in A\}$. For $f, g \in B(H)$,we denote the directional derivatives by

$$\tau_2(f, g) := \lim_{t\to 0} \frac{\|f + tg\|^2 - \|f\|^2}{2t}.$$

and upper directional derivatives respectively, by

$$\tau_2^+(f, g) := \limsup_{t\to 0^+} \frac{\|f + tg\|^2 - \|f\|^2}{2t}.$$

Lemma 8. Let $f, g \in B(H)$. Then

$$-\tau_2^+(f, -g) \le \min ReW(g^*f) \le \max ReW(g^*f) \le \tau_2^+(f, g).$$

Proof. Suppose $\{x_n\} \in Z_f$, we get

$$\|f + tg\|^2 \;\; \ge \;\; \lim_{n\to\infty} \|f(x_n) + tg(x_n)\|^2$$

$$\begin{aligned} &= \lim_{n\to\infty} (\|f(x_n)\|^2 + t^2\|g(x_n)\|^2 + 2tRe\langle f(x_n), g(x_n)\rangle) \\ &= \|f\|^2 + t^2 \lim_{n\to\infty} \|g(x_n)\|^2 + 2t \lim_{n\to\infty} Re\langle f(x_n), g(x_n), \end{aligned}$$

hence

$$\frac{\|f+tg\|^2 - \|f\|^2}{t} \geq t \lim_{n\to\infty} \|g(x_n)\|^2 + 2 \lim_{n\to\infty} Re\langle f(x_n), g(x_n)\rangle$$

setting $t \to 0^+$ and taking $\limsup$ then

$$\tau_2^+(f,g) \geq \lim_{n\to\infty} Re\langle f(x_n), g(x_n)\rangle.$$

Thus $\tau_2^+(f,g) \geq \max ReW(g^*f)$. For the other inequality replace g by $-g$. $\square$

Lemma 9. Let $f \in B(H)$. If the norm of $B(H)$ is Gateaux differentiable at f then $\tau_2(f,g)$ exists for all $g \in X$. Moreover $\tau_2(f,g) \geq 0$ if and only if $\tau(f,g) \geq 0$

Proof. Since the function norm is continues then $\lim_{t\to 0} \frac{\|f+tg\|+\|f\|}{2} = \|f\|$. Since the norm of $B(H)$ is Gateaux differentiable at f, we can suppose that $\lim_{t\to 0} \frac{\|f+tg\|-\|f\|}{t} = \beta$. Now for $g \in X$,

$$\begin{aligned} \tau_2(f,g) = \lim_{t\to 0} \frac{\|f+tg\|^2 - \|f\|^2}{2t} &= \lim_{t\to 0} \frac{\|f+tg\| - \|f\|}{t} . \frac{\|f+tg\| + \|f\|}{2} \\ &= \|f\|\tau(f,g) = \|f\|\beta. \end{aligned}$$

Thus $\tau_2(f,g) \geq 0$ if and only if $\tau(f,g) \geq 0$. $\square$

Theorem 10. *Let B be a bounded subset of $B(H)$, W be compact set such that $W \cap B = \emptyset$ and $s_0 \in B$. Suppose that the norm of $B(H)$ is Gateaux differentiable on $W - B$. The following statements are true:*

i) $b_0 \in \mathbf{F}_B(W)$ if there exists $g_0 \in \mathbf{P}_W(b_0)$ such that

$$\max ReW(h - b_0)^*(g_0 - h)) \geq 0, \quad (h \in B). \tag{2.20}$$

ii) If $b_0 \in \mathbf{F}_B(W)$ then there exists $g_0 \in \mathbf{P}_W(b_0)$ such that

$$\max ReW((b_0 - h)^*(g_0 - b_0)) \leq 0, \quad (h \in B). \tag{2.21}$$

Proof.

i) Let $g_0 \in \mathbf{P}_W(b_0)$ exists such that the inequity (2.20) holds, then by Lemma 8 $\tau_2(g_0 - h, h - b_0) \geq 0$, therefore, $\tau(g_0 - h, h - b_0) \geq 0$ as Lemma 9. Hence $b_0 \in \mathbf{F}_B(W)$ by Theorem 3.

ii) The proof is similar part (i). □

Proposition 11. *Let S be a bounded subset of $B(H)$, and W be compact set such that $W \cap S = \varnothing$ and $s_0 \in S$. If there exists $w_0 \in \mathbf{P}_W(s_0)$ and $k_{w_0} > 0$ such that for every $s \in S$,*

$$\tau_2(w_0 - s, s - g_0) \geq K_f \|s - s_0\|^2. \tag{2.22}$$

then $\mathbf{F}_S(W) = \{s_0\}$.

Proof. Since the function ϕ by setting $\phi(f) = \frac{1}{2}\|f\|^2$ is convex function on $B(H)$, then τ_2 is increases on $(0, \infty)$. By Lemma 1 now for $\alpha = 1$ we get

$$||s_0 - w_0||^2 \geq ||w_0 - s||^2 + K_{w_0}||s - s_0||^2.$$

Now similar to proof of Proposition 4 we have $\mathbf{F}_S(W) = \{s_0\}$. □

The definition of a C^*-algebra shows its elements behave "like" operators in $B(H)$. This is expressed precisely by the Gelfand-Naimark theorem.

Theorem 12. *(Gelfand-Naimark)[12] A unital C^*-algebra $\mathbb{A}$ is isometrically $*$ isomorphic to a C^*-subalgebra of $B(H)$ for some H.*

Let $\mathbb{A}$ be a C^*-algebra, and (π, H) be a faithful representation for $\mathbb{A}$. i.e. $\mathbb{A}$ is isometrically isomorphic to a concrete C^*-algebra of operators on a Hilbert space H. Suppose that $a, b \in \mathbb{A}$, The numerical range of a^*b relative to a which denoted by $W_{\mathbb{A}}(a^*b)$ is defined by:

$$W_{\mathbb{A}}(a^*b) := \{\lambda \in \mathbb{C} : \lambda \in W(\pi(a)^*\pi(b))\}. \tag{2.23}$$

Corollary 13. *Let $\mathbb{B}$ be a bounded subset of a C^*-algebra $\mathbb{A}$, and W be compact set such that $W \cap \mathbb{B} = \emptyset$ and $s_0 \in \mathbb{B}$. Suppose that the norm of $\mathbb{A}$ is Gateaux differentiable on $W - \mathbb{B}$. The following statements are true:*

i) $b_0 \in \mathbf{F}_{\mathbb{B}}(W)$ if there exists $w_0 \in \mathbf{P}_W(b_0)$ such that for each $b \in \mathbb{B}$,

$$\max ReW_{\mathbb{A}}((b - b_0)^*(w_0 - b)) \geq 0. \tag{2.24}$$

ii) If $b_0 \in \mathbf{F}_{\mathbb{B}}(W)$ *then there exists* $w_0 \in \mathbf{P}_W(b_0)$ *such that*

$$\max ReW_{\mathbb{A}}((b_0 - b)^*(w_0 - b_0)) \leq 0, \quad (b \in \mathbb{B}). \tag{2.25}$$

Proof.

i) Since $w_0 \in \mathbf{P}_W(b_0)$ and is π isometrically isomorphism, we have

$$\pi(w_0) \in \mathbf{P}_{\pi(W)}(\pi(b_0)).$$

Since for $b \in \mathbb{B}$, $\max ReW_{\mathbb{A}}((b - b_0)^*(w_0 - b)) \geq 0$, by Definition 2.23 for $\pi(b) \in \pi(\mathbb{B})$, we have

$$\max ReW(\pi(b - b_0)^* \pi(w_0 - b)) \geq 0, \quad (\forall b \in B).$$

Hence $\pi(b_0) \in \mathbf{F}_{\pi(\mathbb{B})}(\pi(W))$ by Theorem 3, thus $b_0 \in \mathbf{F}_{\mathbb{B}}(W)$ since π is isometrically isomorphism.

ii) The proof is similar part (i). □

ACKNOWLEDGMENTS

The authors would like to acknowledge the financial support of Shahrood University of Technology for this research under project No: 23097.

REFERENCES

[1] T. J. Abatzoglu, Norm derivatives on spaces of operators, *Mathematical Annlen*, **239** (1979), 129-135.

[2] G.C. Ahuja, T.D. Narang, T. Swaran, On farthest points, *Journal of Indian Mathematical Society*, **39** (1975), 293-297.

[3] Sh. Al-Sharif and M. Rawashdeh, Simultaneously remotal sets in $L_\infty(I; X)$, *International Journal of Mathematics and Mathematical Sciences* (2011), Article ID 890598, 10 pages.

[4] M. Baronti, A note on remotal sets in Banach space, *Publications de L'institut Mathématique*, **67** (1993), 95-98.

[5] N. Dunford, J. Schwartz, *Linear Operator*, Part I, General theory Interscienece Publ, New York, 1958.

[6] M. Edelstein, Farthest points of sets in uniformly convex Banach spaces, *Israel J. Math.*, **4** (1966), 171-176.

[7] C. Franchetti, I. Singer, Deviation and farthest points in normed linear spaces, *Romanian Journal of Pure and Applied Mathematics*, **24** (1979), 373-381.

[8] S. Hejazian, A. Niknam, and S. Shadkam, Farthest Points and Subdifferential in p-Normed Spaces, *International Journal of Mathematics and Mathematical Sciences* (2008), Article ID 196326, 6 pages.

[9] R. B. Holmes, *The Geometric Functional Analysis and its Applications*, Berlin-Heidelberg-New York, Springer-Verlag, 1975.

[10] R. Khalil, Sh. Al-Sharif, Remotal sets in vector valued function spaces, *Scientiae Mathematicae Japonica*, **63**, No. 3 (2006), 433-441.

[11] B. Magajna, On the distance to the finite-dimensional subspaces in operator algebras, *J. London Math. Soc.*, **47**, No. 2 (1993), 516-532.

[12] G.I. Murphy, *The C^*-Algebra and Operator Theory*, Academic. Press, London, 1990.

[13] E. Naraghirad, Characterizations of simultaneous farthest point, *Optimization Letters*, **3** (2009), 89-100.

[14] T. D. Narang, A study of farthest point, *New Archive for Mathematics*, **25** (1977), 54-79.

[15] B.B. Panda, O.P. Kapoor, On farthest points of sets, *Journal of Mathematical Analysis and Applications*, **62** (1978), 345-353.

[16] V. Zizler, On some extremal problems in Banach spaces, *Mathematical Scand.*, **32** (1973), 214-224.