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ON THE BEHAVIOR OF SOLUTIONS OF HILFER-HADAMARD TYPE FRACTIONAL NEUTRAL PANTOGRAPH EQUATIONS WITH BOUNDARY CONDITIONS

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ABSTRACT: In this paper, we study the dynamics and stability of Hilfer-Hadamard type fractional neutral pantograph equations with boundary conditions using Schaefer's fixed point theorem. In addition, we discuss Ulam stability of the system by employing Banach contraction principle.

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1. INTRODUCTION

Recently, fractional differential equations (FDEs) have attracted great attention. It has been proved that FDEs are valuable tools in modeling of many phenomena in various fields of engineering, physics, and economics. For more recent development on this topic, one can see the monographs of Hilfer [22], Kilbas [25] and Podlubny [27]. There are some works on FDEs with Hadamard fractional derivative, even if it has been studied many years ago (see for example [2, 7, 8]). A study of Hilfer type of equation has received a significant amount of interests, we refer to [9, 10, 11, 22, 23, 24, 26].

Stability analysis is always one of the most important issues in the theory of differential equations and their applications. Recently, Ulam stability of FDEs has attracted increasing interest. The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. Thereafter, this type of stability is called the Ulam-Hyers stability [5, 12, 14, 18]. In 1978, Rassias [18] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. The stability properties of all kinds of equations have attracted the attention of many mathematicians. In particular, the Ulam-Hyers stability and Ulam-Hyers-Rassias stability have been taken up by a number of mathematicians and the study of this area has developed to be one of the central subjects in the mathematical analysis area. For more details on the Ulam- Hyers stability and Ulam-Hyers-Rassias stability of differential equations, see [12, 20, 29, 30].

Delay differential equations (DDEs) have an extensive range of applications in sciences, engineering and economics. DDE is a differential equation for an unknown function which involves derivatives of the function and in which the function, and possibly its derivative(s), occur with delay arguments. When the derivative(s) occur with the delay arguments, the equation is known as neutral delay differential equation (NDDE).

The pantograph type is one of the special types of DDEs, and growing attention is given to its analysis and numerical solution. Pantograph type always has the delay term fall after the initial value but before the desire approximation being calculated. When the delay term of pantograph type involved with the derivative(s), the equation is named as neutral delay differential equation of pantograph type.

The pantograph equations have been studied extensively (see,[6, 13, 17] and references therein) since they can be used to describe many phenomena arising in number theory, dynamical systems, probability, quantum mechanics, and electro dynamics. Recently, fractional pantograph differential equations have been studied by many researchers. One of interesting subjects in this area, is the investigation of the existence of solutions by fixed point theorems, we refer to [6]. Vivek et.al. [28] studied dynamics and Ulam stability of pantograph equations with Hilfer fractional derivative. Recently, Kassim et. al. [16] investigated well-posedness and stability for a differential equations with Hilfer-Hadamard fractional derivative. Unfortunately, existence, uniqueness and Ulam stability of boundary value problem(BVP) for fractional neutral pantograph equations with Hilfer-Hadamard derivative is still not studied. The problem of the existence of solutions for FDEs with boundary conditions has been recently treated in the literature in [2, 4, 21, 19].

Motivated by the above approach, the goal in the present paper is to study existence, uniqueness and stability analysis of Hilfer-Hadamard type fractional neutral pantograph equations with boundary conditions of the form

$$\begin{cases} HD_{1^+}^{\alpha,\beta}x(t) = f(t, x(t), x(\lambda t), {}_H D_{1^+}^{\alpha,\beta}x(\lambda t)), & t \in J := [1, T], \\ I_{1^+}^{1-\gamma}x(1) = a, \quad I_{1^+}^{1-\gamma}x(T) = b, & \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{1.1}$$

where ${}_H D_{1^+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative, $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $0 < \lambda < 1$ and let X be a Banach space, $f : J \times X \times X \times X \to X$ is given continuous function.

The outline of the paper is as follows. In Section 2, we give some basic definitions and results concerning the Hilfer-Hadamard fractional derivative. In Section 3, we present our main result by using Schaefer's fixed point theorem. In Section 4, we discuss stability analysis.

2. PRELIMINARIES

In what follows we introduce definitions, notations, and preliminary facts which are used in the sequel.

For more details, we refer to [1, 9, 10, 15, 22, 23, 24].

Definition 2.1. Let $C[J, X]$ denotes the Banach space of continuous function on $[1, T]$ with the norm

$$\|x\|_C := \sup\{x(t) : t \in J\}.$$

We denote $L^1\{R_+\}$, the space of Lebesgue integrable functions on J.

By $C_{\gamma,\log}[J,X]$ and $C^1_{\gamma,\log}[J,X]$, we denote the weighted spaces of continuous functions defined by

$$C_{\gamma,\log}[J,X] := \{f(t) : J \to X | (\log t)^\gamma f(t) \in C[J,X]\},$$

with norm

$$\|f\|_{C_{\gamma,\log}} = \|(\log t)^\gamma f(t)\|_C,$$

and

$$\|f\|_{C^n_{\gamma,\log}} = \sum_{k=0}^{n-1} \left\|f^k\right\|_C + \|f^n\|_{C_{\gamma,\log}}, \quad n \in N.$$

Moreover, $C^0_{\gamma,\log}[J,X] := C_{\gamma,\log}[J,X]$.

Now, we give some results and properties of Hadamards fractional calculus.

Definition 2.2. [2, 8] The Hadamard fractional integral of order α for a function h is defined as

$$I^\alpha_{1^+} h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds, \quad \alpha > 0,$$

provided the integral exists.

Notice that for all $\alpha, \alpha_1, \alpha_2 > 0$ and each $h \in C[J,X]$, we have $I^\alpha_{1^+} h \in C[J,X]$, and

$$(I^{\alpha_1}_{1^+} I^{\alpha_2}_{1^+} h)(t) = (I^{\alpha_1+\alpha_2}_{1^+} h)(t); \text{for a.e. } t \in J.$$

Definition 2.3. [2, 8] The Hadamard derivative of fractional order α for a function $h : [1,\infty) \to X$ is defined as

$$ {}_H D^\alpha_{1^+} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \frac{h(s)}{s} ds,$$

$$n-1 < \alpha < n, \ n = \lceil \alpha \rceil + 1,$$

where $\lceil \alpha \rceil$ denotes the integer part of real number α and $\log(\cdot) = \log_e(\cdot)$.

Let $\alpha \in (0,1]$, $\gamma \in [0,1)$ and $h \in C_{1-\gamma,\log}[J,X]$. Then the following expression leads to the left inverse operator as follows.

$$({}_H D^\alpha_{1^+} I^\alpha_{1^+} h)(t) = h(t); \quad \text{for all } t \in [1,b].$$

Moreover, if $I_{1^+}^{1-\alpha}h \in C_{1-\gamma,\log}^{1}[J,X]$, then the following composition

$$(I_{1^+}^{\alpha}\ {}_HD_{1^+}^{\alpha}h)(t) = h(t) - \frac{(I_{1^+}^{1-\alpha}h)(1^+)}{\Gamma(\alpha)}(\log t)^{\alpha-1}; \quad \text{for all } t \in [1,b].$$

In [22], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and Caputo derivatives as specific cases (see also [23, 15]).

Definition 2.4. (Hilfer-Hadamard derivative).

Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $h \in L^1\{R_+\}$, $I_{1^+}^{(1-\alpha)(1-\beta)} \in C_{\gamma,\log}^{1}[J,X]$. The Hilfer-Hadamard fractional derivative of order α and type β of h is defined as

$$({}_HD_{1^+}^{\alpha,\beta}h)(t) = \left(I_{1^+}^{\beta(1-\alpha)}\frac{d}{dt}I_{1^+}^{(1-\alpha)(1-\beta)}h\right)(t); \quad \text{for a.e. } t \in J. \tag{2.1}$$

Properties: Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, and $h \in L^1\{R_+\}$.

1. The operator $({}_HD_{1^+}^{\alpha,\beta}h)(t)$ can be written as

$$({}_HD_{1^+}^{\alpha,\beta}h)(t) = \left(I_{1^+}^{\beta(1-\alpha)}\frac{d}{dt}I_{1^+}^{1-\gamma}h\right)(t) = \left(I_{1^+}^{\beta(1-\alpha)}\ {}_HD_{1^+}^{\gamma}h\right)(t);$$
$$\text{for a.e. } t \in J.$$

   Moreover, the parameter γ satisfies

$$0 < \gamma \leq 1, \quad \gamma \geq \alpha, \quad \gamma > \beta, \quad 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (2.1) for $\beta = 0$, coincides with the Hadamard Riemann-Liouville derivative and for $\beta = 1$ with the Hadamard Caputo derivative.

$${}_HD_{1^+}^{\alpha,0} = {}_HD_{1^+}^{\alpha}, \quad \text{and} \quad {}_HD_{1^+}^{\alpha,1} = {}_H^cD_{1^+}^{\alpha}.$$

3. If ${}_HD_{1^+}^{\beta(1-\alpha)}h$ exists and in $L^1\{R_+\}$, then

$$({}_HD_{1^+}^{\alpha,\beta}I_{1^+}^{\alpha}h)(t) = \left(I_{1^+}^{\beta(1-\alpha)}\ {}_HD_{1^+}^{\beta(1-\alpha)}h\right)(t); \quad \text{for a.e. } t \in J.$$

   Furthermore, if $h \in C_{\gamma,\log}[J,X]$ and $I_{1^+}^{1-\beta(1-\alpha)}h \in C_{\gamma,\log}^{1}[J,X]$, then

$$({}_HD_{1^+}^{\alpha,\beta}I_{1^+}^{\alpha}h)(t) = h(t); \quad \text{for a.e. } t \in J.$$

4. If ${}_HD^{\gamma}_{1^+}h$ exists and in $L^1\{R_+\}$, then

$$\left(I^{\alpha}_{1^+}\ {}_HD^{\alpha,\beta}_{1^+}h\right)(t) = \left(I^{\gamma}_{1^+}\ {}_HD^{\gamma}_{1^+}h\right)(t) = h(t) - \frac{I^{1-\gamma}_{1^+}h(1^+)}{\Gamma(\gamma)}(\log t)^{\gamma-1};$$

for a.e. $t \in J$.

In order to solve our problem, the following spaces are presented

$$C^{\alpha,\beta}_{1-\gamma,\log}[J,X] = \left\{f \in C_{1-\gamma,\log}[J,X], {}_H D^{\alpha,\beta}_{1^+} f \in C_{1-\gamma,\log}[J,X]\right\},$$

and

$$C^{\gamma}_{1-\gamma,\log}[J,X] = \left\{f \in C_{1-\gamma,\log}[J,X], {}_H D^{\gamma}_{1^+} f \in C_{1-\gamma,\log}[J,X]\right\}.$$

It is obvious that

$$C^{\gamma}_{1-\gamma,\log}[J,X] \subset C^{\alpha,\beta}_{1-\gamma,\log}[J,X].$$

Lemma 2.5. *Let $\alpha > 0$, $0 \le \beta \le 1$, so the homogeneous differential equation with Hilfer-Hadamard fractional order*

$${}_HD^{\alpha,\beta}_{1^+}h(t) = 0$$

has a solution

$$h(t) = c_0(\log t)^{\gamma-1} + c_1(\log t)^{\gamma+2\beta-2} + c_2(\log t)^{\gamma+2(2\beta)-3} + \cdots + c_n(\log t)^{\gamma+n(2\beta)-(n+1)}.$$

Corollary 2.6. *Let $h \in C_{1-\gamma,\log}[J,X]$. Then the linear problem*

$${}_HD^{\alpha,\beta}_{1^+}x(t) = h(t), \quad t \in J := [1,b],$$
$$I^{1-\gamma}_{1^+}x(t)|_{t=1} = x_0, \quad \gamma = \alpha + \beta - \alpha\beta,$$

has a unique solution $x \in L^1\{R_+\}$ given by

$$x(t) = \frac{x_0}{\Gamma(\gamma)}(\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} h(s)\frac{ds}{s}.$$

Lemma 2.7. *Let $f : J \times X \times X \times X \to X$ be a function such that $f \in C_{1-\gamma,\log}[J,X]$ for any $x \in C_{1-\gamma,\log}[J,X]$. A function $x \in C^{\gamma}_{1-\gamma,\log}[J,X]$ is a solution of the integral equation*

$$x(t) = \frac{a}{\Gamma(\gamma)}(\log t)^{\gamma-1}$$

$$+\left(b-a-I_{1^+}^{1-\beta(1-\alpha)}f(T,x(T),x(\lambda T),{}_H D_{1^+}^{\alpha,\beta}x(\lambda T))\right)$$
$$\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log t)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}}$$
$$+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s),x(\lambda s),{}_H D_{1^+}^{\alpha,\beta}x(\lambda s))\frac{ds}{s}, \tag{2.2}$$

if and only if x is a solution of the Hilfer-Hadamard fractional neutral pantograph BVP

$${}_H D_{1^+}^{\alpha,\beta}x(t)=f(t,x(t),x(\lambda t),{}_H D_{1^+}^{\alpha,\beta}x(\lambda t)),\quad t\in J:=[1,T],\ \lambda\in(0,1), \tag{2.3}$$
$$I_{1^+}^{1-\gamma}x(1)=a,\quad I_{1^+}^{1-\gamma}x(T)=b,\quad \gamma=\alpha+\beta-\alpha\beta. \tag{2.4}$$

Proof. Assume x satisfies (2.2). Then Lemma 2.5 implies that

$$x(t)=c_0(\log t)^{\gamma-1}+c_1(\log t)^{\gamma+2\beta-2}$$
$$+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s),x(\lambda s),{}_H D_{1^+}^{\alpha,\beta}x(\lambda s))\frac{ds}{s}.$$

From (2.4), a simple calculation gives

$$c_0=\frac{a}{\Gamma(\gamma)},$$
$$c_1=\left(b-a-I_{1^+}^{1-\beta(1-\alpha)}f(T,x(T),x(\lambda T),{}_H D_{1^+}^{\alpha,\beta}x(\lambda T))\right)$$
$$\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{1}{(\log T)^{2\beta-1}}.$$

Hence, we get equation (2.2). Conversly, it is clear that if x satisfies equation (2.2), then equations (2.3)-(2.4) hold. □

Lemma 2.8. *[31] Suppose $1>\alpha>0$, $\overline{a}>0$ and $\overline{b}>0$ and suppose $u(t)$ is nonnegative and locally integral on $[1,+\infty)$ with*

$$u(t)\le\overline{a}+\overline{b}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}u(s)\frac{ds}{s},\quad t\in[1,+\infty).$$

Then

$$u(t)\le\overline{a}+\int_1^t\left[\sum_{n=1}^{\infty}\frac{(\overline{b}\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\log\frac{t}{s}\right)^{n\alpha-1}\overline{a}\right]\frac{ds}{s},\quad t\in[1,+\infty).$$

Remark 2.9. Under the assumptions of Lemma 2.8, let $u(t)$ be a nondecreasing function on $[1,\infty)$. Then we have

$$u(t) \leq \overline{a} E_{\alpha,1}\left(\overline{b}\Gamma(\alpha)(\log t)^{\alpha}\right),$$

where $E_{\alpha,1}$ is the Mittag-leffler function defined by

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad z \in \mathbb{C}.$$

3. MAIN RESULTS

Now we give our main existence result for problem (1.1). Before starting and proving this result, we list the following conditions:

(C1) $f : J \times X \times X \times X \to X$ is continuous function.

(C2) There exist $l, p, q, r \in C_{1-\gamma,\log}[J, X]$ with $l^* = \sup_{t\in J} l(t) < 1$ such that

$$|f(t,u,v,w)| \leq l(t) + p(t)|u| + q(t)|v| + r(t)|w|,$$

for $t \in J$, $u, v, w \in X$.

Theorem 3.1. *Let conditions (C1),(C2) hold. Then the problem* (1.1) *has at least one solution defined on* J.

Proof. Consider the operator $P : C_{1-\gamma,\log}[J, X] \to C_{1-\gamma,\log}[J, X]$ defined by

$$\begin{aligned}(Px)(t) =& \frac{a}{\Gamma(\gamma)}(\log t)^{\gamma-1} \\ &+ \left(b - a - I_{1^+}^{1-\beta(1-\alpha)} f(T, x(T), x(\lambda T), {}_H D_{1^+}^{\alpha,\beta} x(\lambda T))\right) \\ &\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{(\log t)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}} \qquad (3.1) \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s, x(s), x(\lambda s), {}_H D_{1^+}^{\alpha,\beta} x(\lambda s))\frac{ds}{s}. \qquad (3.2)\end{aligned}$$

The equation (3.2) can be written as

$$(Px)(t) = \frac{a}{\Gamma(\gamma)}(\log t)^{\gamma-1} + \left(b - a - I_{1^+}^{1-\beta(1-\alpha)} K_x(T)\right) \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}$$

$$\frac{(\log t)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}} \tag{3.3}$$
$$+\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} K_x(s)\frac{ds}{s}, \tag{3.4}$$

where

$$\begin{aligned} K_x(t) &= f(t, x(t), x(\lambda t), {}_H D_{1^+}^{\alpha,\beta} x(\lambda t)) \\ &= f(t, x(t), x(\lambda t), K_x(t)) \\ &= {}_H D_{1^+}^{\alpha,\beta} x(t). \end{aligned}$$

It is obvious that the operator P is well defined.

Step 1: P is continuous. Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma,\log}[J, X]$. Then for each $t \in J$,

$$\begin{aligned} &\left|((Px_n)(t) - (Px)(t))(\log t)^{1-\gamma}\right| \\ &\leq \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |K_{x_n}(s) - K_x(s)| \frac{ds}{s} \\ &+ \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{(\log t)^{2\beta-1}}{(\log T)^{2\beta-1}} \\ &\left(\frac{1}{\Gamma(1-\beta(1-\alpha))}\int_1^T \left(\log\frac{T}{s}\right)^{(1-\beta(1-\alpha))-1} |K_{x_n}(s) - K_x(s)| \frac{ds}{s}\right) \\ &\leq \left(\frac{(\log T)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))}\right) \|K_{x_n}(\cdot) - K_x(\cdot)\|_{C_{1-\gamma,\log}}. \end{aligned}$$

Since f is continuous(i.e. K_x is continuous), then we have

$$\|Px_n - Px\|_{C_{1-\gamma,\log}} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2: P maps bounded sets into bounded sets in $C_{1-\gamma,\log}[J, X]$.

Indeed, it is enough to show that for $\eta > 0$, there exists a positive constant l such that $x \in B_\eta \{x \in C_{1-\gamma,\log}[J, X] : \|x\| \leq \eta\}$, we have $\|(Px)\|_{C_{1-\gamma,\log}} \leq l$.

$$\begin{aligned} &\left|(Px)(t)(\log t)^{1-\gamma}\right| \\ &\leq \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{(\log t)^{2\beta-1}}{(\log T)^{2\beta-1}} \\ &\quad \left(b - a - \frac{1}{\Gamma(1-\beta(1-\alpha))}\int_1^T \left(\log\frac{T}{s}\right)^{(1-\beta(1-\alpha))-1} |K_x(s)| \frac{ds}{s}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|K_x(s)|\frac{ds}{s}\\
&:= A_1 + A_2, \qquad (3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
|K_x(t)| &= |f(t,x(t),x(\lambda t),K_x(t))|\\
&\le l(t)+p(t)\,|x(t)|+q(t)\,|x(\lambda t)|+r(t)\,|K_x(t)|\\
&\le l^*+p^*\,|x(t)|+q^*\,|x(\lambda t)|+r^*\,|K_x(t)|\\
&\le \frac{l^*+p^*\,|x(t)|+q^*\,|x(\lambda t)|}{1-r^*}. \qquad (3.6)
\end{aligned}
$$

From (3.5) and (3.6), we get

$$
\begin{aligned}
A_1 &= \frac{a}{\Gamma(\gamma)}+\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}(b-a)-\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{l^*(\log T)^{1-\beta(1-\alpha)}}{(1-r^*)\Gamma(2-\beta(1-\alpha))}\\
&\quad -\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))}\frac{(p^*+q^*)}{1-r^*}\,\|x\|_{C_{1-\gamma,\log}}.\\
A_2 &= \frac{l^*(\log T)^{\alpha+1-\gamma}}{(1-r^*)\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha+1-\gamma}}{(1-r^*)\Gamma(\alpha+1)}(p^*+q^*)\,\|x\|_{C_{1-\gamma,\log}}.
\end{aligned}
$$

Substituting A_1, A_2 in equation (3.5), we have

$$
\begin{aligned}
&\left|(Px)(t)(\log t)^{1-\gamma}\right|\\
&\le \frac{a}{\Gamma(\gamma)}+\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}(b-a)\\
&\quad+\left(\frac{(\log T)^{\alpha+1-\gamma}}{\Gamma(\alpha+1)}-\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))}\right)\frac{l^*}{1-r^*}\\
&\quad+\left(\frac{(\log T)^{\alpha+1-\gamma}}{\Gamma(\alpha+1)}-\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))}\right)\frac{(p^*+q^*)}{1-r^*}\,\|x\|_{C_{1-\gamma,\log}}\\
&:= l.
\end{aligned}
$$

Step 3: P maps bounded sets into equicontinuous set of $C_{1-\gamma,\log}[J,X]$. Let $t_1, t_2 \in J, t_1 < t_2$ and $x \in B_\eta$. Using the fact f is bounded on the compact set $J \times B_\eta$ (these $\sup_{(t,x)\in J\times B_\eta}|K_x(t)| := C < \infty$),

$$
|(Px)(t_2)-(Px)(t_2)| \le \frac{a}{\Gamma(\gamma)}\left((\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}\right)
$$

$$
\begin{aligned}
&+\left(b-a-\left|I_{1^+}^{1-\beta(1-\alpha)}K_x(T)\right|\right)\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\\
&\left[\frac{(\log t_2)^{\gamma+2\beta-2}-(\log t_1)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}}\right]\\
&+\left|\frac{1}{\Gamma(\alpha)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}K_x(s)\frac{ds}{s}-\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1}K_x(s)\frac{ds}{s}\right|\\
\leq&\frac{a}{\Gamma(\gamma)}\left((\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}\right)\\
&+\left(b-a-\frac{C(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))}\right)\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\\
&\left[\frac{(\log t_2)^{\gamma+2\beta-2}-(\log t_1)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}}\right]\\
&+\frac{C}{\Gamma(\alpha+1)}\left(\log\frac{t_2}{t_1}\right)^{\alpha}+\frac{C}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1}-\left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{ds}{s}.
\end{aligned}
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of step 1 to 3, together with Arzela-Ascoli theorem, we can conclude that $P : C_{1-\gamma,\log}[J,X] \to C_{1-\gamma,\log}[J,X]$ is continuous and completely continuous.

Step 4: A priori bounds. Now it remains to show that the set

$$\omega = \{x \in C_{1-\gamma,\log}[J,X] : x = \delta(Px), \quad 0 < \delta < 1\}$$

is bounded set.

Let $x \in \omega$, $x = \delta(Px)$ for some $0 < \delta < 1$. Thus for each $t \in J$. We have,

$$
\begin{aligned}
&x(t) =\\
&\delta\left[\frac{a}{\Gamma(\gamma)}(\log t)^{\gamma-1}+\left(b-a-I_{1^+}^{1-\beta(1-\alpha)}K_x(T)\right)\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log t)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}}\right.\\
&\left.\qquad+\frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}K_x(s)\frac{ds}{s}\right].
\end{aligned}
$$

This implies by (C2)that for each $t \in J$, we have

$$
\begin{aligned}
&\left|x(t)(\log t)^{1-\gamma}\right|\\
&\leq\left|(Px)(t)(\log t)^{1-\gamma}\right|\\
&\leq\frac{a}{\Gamma(\gamma)}+\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}(b-a)
\end{aligned}
$$

$$+\left(\frac{(\log T)^{\alpha+1-\gamma}}{\Gamma(\alpha+1)}-\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))}\right)\frac{l^*}{1-r^*}$$
$$+\left(\frac{(\log T)^{\alpha+1-\gamma}}{\Gamma(\alpha+1)}-\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))}\right)\frac{(p^*+q^*)}{1-r^*}\|x\|_{C_{1-\gamma,\log}}.$$

This shows that the set ω is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that P has a fixed point which is a solution of problem (1.1). □

4. STABILITY ANALYSIS

In this section, for the Hilfer-Hadamard type fractional neutral pantograph BVP (1.1), we adopt the definition in [20, 28] of the Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability.

Definition 4.1. The equation (1.1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^{\gamma}_{1-\gamma,\log}[J, X]$ of the inequality

$$\left|{}_HD^{\alpha,\beta}_{1^+}z(t)-f(t,z(t),z(\lambda t),{}_H D^{\alpha,\beta}_{1^+}z(\lambda t))\right|\le\epsilon,\quad t\in J,$$

there exists a solution $x \in C^{\gamma}_{1-\gamma,\log}[J, X]$ of equation (1.1) with

$$|z(t)-x(t)|\le C_f\epsilon,\quad t\in J.$$

Definition 4.2. The equation (1.1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C([0,\infty),[0,\infty)), \psi_f(0) = 0$ such that for each solution $z \in C^{\gamma}_{1-\gamma,\log}[J, X]$ of the inequality

$$\left|{}_HD^{\alpha,\beta}_{1^+}z(t)-f(t,z(t),z(\lambda t),{}_H D^{\alpha,\beta}_{1^+}z(\lambda t))\right|\le\epsilon,\quad t\in J,$$

there exists a solution $x \in C^{\gamma}_{1-\gamma,\log}[J, X]$ of equation (1.1) with

$$|z(t)-x(t)|\le\psi_f\epsilon,\quad t\in J.$$

Definition 4.3. The equation (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma,\log}[J,X]$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ of the inequality

$$\left| {}_H D^{\alpha,\beta}_{1^+} z(t) - f(t, z(t), z(\lambda t), {}_H D^{\alpha,\beta}_{1^+} z(\lambda t)) \right| \leq \epsilon\varphi(t), \quad t \in J,$$

there exists a solution $x \in C^{\gamma}_{1-\gamma,\log}[J,X]$ of equation (1.1) with

$$|z(t) - x(t)| \leq C_f \epsilon \varphi(t), \quad t \in J.$$

Definition 4.4. The equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma,\log}[J,X]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ of the inequality

$$\left| {}_H D^{\alpha,\beta}_{1^+} z(t) - f(t, z(t), z(\lambda t), {}_H D^{\alpha,\beta}_{1^+} z(\lambda t)) \right| \leq \varphi(t), \quad t \in J,$$

there exists a solution $x \in C^{\gamma}_{1-\gamma,\log}[J,X]$ of equation (1.1) with

$$|z(t) - x(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in J.$$

Remark 4.5. A function $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ is a solution of the inequality

$$\left| {}_H D^{\alpha,\beta}_{1^+} z(t) - f(t, z(t), z(\lambda t), {}_H D^{\alpha,\beta}_{1^+} z(\lambda t)) \right| \leq \epsilon, \quad t \in J,$$

if and only if there exists a function $g \in C^{\gamma}_{1-\gamma,\log}[J,X]$ (which depend on solution x) such that

1. $|g(t)| \leq \epsilon$, $t \in J$;

2. ${}_H D^{\alpha,\beta}_{1^+} z(t) = f(t, z(t), z(\lambda t), {}_H D^{\alpha,\beta}_{1^+} z(\lambda t)) + g(t)$, $t \in J$.

Remark 4.6. Clearly,

1. Definition 4.1⇒ Definition 4.2.

2. Definition 4.3⇒ Definition 4.4.

We ready to prove our stability results for problem (1.1). The arguments are based on the Banach contraction principle. First we list the following conditions:

(C3) There exist two positive constants $K > 0$ and $L > 0$ such that

$$|f(t,u,v,w) - f(t,\overline{u},\overline{v},\overline{w})| \leq K(|u-\overline{u}| + |v-\overline{v}|) + L|w-\overline{w}|,$$

for any $u, v, w, \overline{u}, \overline{v}, \overline{w} \in X$ and $t \in J$.

(C4) There exists an increasing function $\varphi \in C_{1-\gamma,\log}[J,X]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$I_{1^+}^{\alpha}\varphi(t) \leq \lambda_\varphi \varphi(t).$$

Lemma 4.7. *Let conditions (C1),(C3) hold. If*

$$\frac{2K}{1-L}\left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))} + \frac{(\log T)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\right) < 1, \tag{4.1}$$

then the problem (1.1) *has a unique solution.*

Proof. Consider the operator $P : C_{1-\gamma,\log}[J,X] \to C_{1-\gamma,\log}[J,X]$.

$$\begin{aligned}(Px)(t) =& \frac{a}{\Gamma(\gamma)}(\log t)^{\gamma-1} + \left(b - a - I_{1^+}^{1-\beta(1-\alpha)}K_x(T)\right) \\ & \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log t)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}} \\ & + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} K_x(s)\frac{ds}{s}.\end{aligned}$$

It is clear that the fixed points of P are solutions of (1.1).

Let $x, y \in C_{1-\gamma,\log}[J,X]$ and $t \in J$, then we have

$$\begin{aligned}&\left|((Px)(t) - (Py)(t))(\log t)^{1-\gamma}\right| \\ &\leq \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log t)^{2\beta-1}}{(\log T)^{2\beta-1}} \\ &\quad \left(\frac{1}{\Gamma(1-\beta(1-\alpha))}\int_1^T \left(\log\frac{T}{s}\right)^{(1-\beta(1-\alpha))-1} |K_x(s) - K_y(s)|\frac{ds}{s}\right) \\ &+ \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}|K_x(s) - K_y(s)|\frac{ds}{s}.\end{aligned} \tag{4.2}$$

and

$$|K_x(t) - K_y(t)| \leq |f(t, x(t), x(\lambda t), K_x(t)) - f(t, y(t), y(\lambda t), K_y(t))|$$

$$\leq K\left(|x(t)-y(t)|+|x(\lambda t)-y(\lambda t)|\right)+L\left|K_x(t)-K_y(t)\right|$$
$$\leq \frac{2K}{1-L}\left|x(t)-y(t)\right|. \tag{4.3}$$

By replacing (4.3) in the inequality (4.2), we get

$$\left|((Px)(t)-(Py)(t))(\log t)^{1-\gamma}\right| \leq \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}$$
$$\left(\frac{2K}{1-L}\frac{1}{\Gamma(1-\beta(1-\alpha))}\int_1^T\left(\log\frac{T}{s}\right)^{(1-\beta(1-\alpha))-1}|x(s)-y(s)|\frac{ds}{s}\right)$$
$$\frac{(\log t)^{2\beta-1}}{(\log T)^{2\beta-1}}+\frac{2K}{1-L}\frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|x(s)-y(s)|\frac{ds}{s}$$
$$\leq \frac{2K}{1-L}\left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))}+\frac{(\log T)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\right)\|x-y\|_{C_{1-\gamma,\log}}.$$

Hence,

$$\|(Px)-(Py)\|_{C_{1-\gamma,\log}}$$
$$\leq \frac{2K}{1-L}\left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))}+\frac{(\log T)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\right)\|x-y\|_{C_{1-\gamma,\log}}.$$

From (4.1), it follows that P has a unique fixed point which is solution of problem (1.1). □

Theorem 4.8. *Let conditions (C1),(C3) and* (4.1) *hold, then the problem* (1.1) *is Ulam-Hyers stable.*

Proof. Let $\epsilon>0$ and let $z\in C^{\gamma}_{1-\gamma,\log}[J,X]$ be a function which satisfies the inequality:

$$\left|{}_HD^{\alpha,\beta}_{1^+}z(t)-f(t,z(t),z(\lambda t),{}_HD^{\alpha,\beta}_{1^+}z(\lambda t))\right|\leq\epsilon,\quad \text{for any } t\in J, \tag{4.4}$$

and let $x\in C^{\gamma}_{1-\gamma,\log}[J,X]$ be the unique solution of the following Hilfer-Hadamard type pantograph BVP

$${}_HD^{\alpha,\beta}_{1^+}x(t)=f(t,x(t),x(\lambda t),{}_HD^{\alpha,\beta}_{1^+}x(\lambda t)),\ t\in J:=[1,T],$$
$$I^{1-\gamma}_{1^+}z(1)=a,\quad I^{1-\gamma}_{1^+}z(T)=b\quad \gamma=\alpha+\beta-\alpha\beta,$$

where $0 < \alpha < 1$, $0 \le \beta \le 1$ and $0 < \lambda < 1$.
Using Lemma 2.7, we obtain

$$x(t) = A_x + \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{t}{s}\right)^{\alpha-1} K_x(s) \frac{ds}{s}$$

where

$$A_x = \frac{a}{\Gamma(\gamma)} (\log t)^{\gamma-1} + \left(b - a - I_{1^+}^{1-\beta(1-\alpha)} K_x(T)\right) \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{(\log t)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}}.$$

On the other hand, if $I_{1^+}^{1-\gamma} x(T) = I_{1^+}^{1-\gamma} z(T)$ and $I_{1^+}^{1-\gamma} x(1) = I_{1^+}^{1-\gamma} x(1)$, then $A_x = A_z$.

Indeed,

$$\begin{aligned} |A_x - A_z| &\le \frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)} \frac{(\log t)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}} \left(\frac{2K}{1-L}\right) I_{1^+}^{1-\beta(1-\alpha)} |x(T) - z(T)| \\ &= 0. \end{aligned}$$

Thus, $A_x = A_z$.

Then, we have

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_x(s) \frac{ds}{s}.$$

By integration of the inequality (4.4), we obtain

$$\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{ds}{s} \right| \le \frac{\epsilon (\log T)^\alpha}{\Gamma(\alpha+1)}.$$

We have

$$\begin{aligned} |z(t) - x(t)| &\le \left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{ds}{s} \right| \\ &\quad + \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [K_z(s) - K_x(s)] \frac{ds}{s} \right| \\ &\le \frac{\epsilon (\log T)^\alpha}{\Gamma(\alpha+1)} + \left(\frac{2K}{1-L}\right) \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z(s) - x(s)| \frac{ds}{s}, \end{aligned}$$

and to apply Lemma 2.8 and Remark 2.9, we obtain

$$\begin{aligned} |z(t) - x(t)| &\le \frac{(\log T)^\alpha E_{\alpha,1}(\frac{2K}{1-L} (\log T)^\alpha)}{\Gamma(\alpha+1)} \cdot \epsilon \\ &:= C_f \epsilon. \end{aligned}$$

Thus, the equation (1.1) is Ulam-Hyers stable. □

Theorem 4.9. *Let conditions (C1), (C3), (C4) and* (4.1) *hold. Then, the problem*(1.1) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Let $z \in C^{\gamma}_{1-\gamma,\log}[J, X]$ be solution of the inequality

$$\left|{}_H D^{\alpha,\beta}_{1^+} z(t) - f(t, z(t), z(\lambda t), {}_H D^{\alpha,\beta}_{1^+} z(\lambda t))\right| \leq \epsilon\varphi(t), \quad t \in J,\ \epsilon > 0, \qquad (4.5)$$

and let $x \in C^{\gamma}_{1-\gamma,\log}[J, X]$ be the unique solution of the following Hilfer-Hadamard type BVP

$$\begin{aligned} &{}_H D^{\alpha,\beta}_{1^+} x(t) = f(t, x(t), x(\lambda t), {}_H D^{\alpha,\beta}_{1^+} x(\lambda t)),\ t \in J := [1, T],\\ &I^{1-\gamma}_{1^+} z(1) = a, \quad I^{1-\gamma}_{1^+} z(T) = b \quad \gamma = \alpha + \beta - \alpha\beta, \end{aligned}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $0 < \lambda < 1$.

Using Lemma 2.7, we obtain

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{t}{s}\right)^{\alpha-1} K_x(s) \frac{ds}{s},$$

where

$$A_z = \frac{a}{\Gamma(\gamma)} (\log t)^{\gamma-1} + \left(b - a - I^{1-\beta(1-\alpha)}_{1^+} K_z(T)\right) \frac{\Gamma(2\beta)}{\Gamma(\gamma + 2\beta - 1)} \frac{(\log t)^{\gamma+2\beta-2}}{(\log T)^{2\beta-1}}.$$

By integration of the inequality (4.5), we get

$$\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{ds}{s} \right| \leq \epsilon\lambda_\varphi \varphi(t). \qquad (4.6)$$

On the other hand, we have

$$\begin{aligned} |z(t) - x(t)| \leq& \left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{ds}{s} \right| \\ &+ \left(\frac{2K}{1-L}\right) \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z(s) - x(s)| \frac{ds}{s} \\ \leq& \epsilon\lambda_\varphi \varphi(t) + \left(\frac{2K}{1-L}\right) \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |z(s) - x(s)| \frac{ds}{s}. \end{aligned}$$

By applying Lemma 2.8 and Remark 2.9, we get

$$|z(t) - x(t)| \leq \epsilon\lambda_\varphi \varphi(t) E_{\alpha,1}\left(\frac{2K}{1-L} (\log T)^\alpha\right), \quad t \in [1, T].$$

Thus, the equation (1.1) is generalized Ulam-Hyers-Rassias stable. □

5. EXAMPLE

In this section, we give an example to illustrate the usefulness of our main results.

Example 5.1. Let us consider consider the following Hilfer-Hadamard type fractional neutral pantograph BVP

$$ {}_HD^{\alpha,\beta}_{1^+}x(t) = \frac{1}{4} + \frac{1}{20}\left(x(t) + x\left(\frac{1}{2}\right) +_H D^{\alpha,\beta}_{1^+}x\left(\frac{1}{2}\right)\right), \quad t \in J := [1,e], \tag{5.1}$$

$$ I^{1-\gamma}_{1^+}x(1) = 1, \quad I^{1-\gamma}_{1^+}x(e) = 2, \quad \gamma = \alpha + \beta - \alpha\beta. \tag{5.2}$$

Notice that this problem is a particular case of (1.1).

Set

$$ f(t,u,v,w) = \frac{1}{4} + \frac{1}{20}u + \frac{1}{20}v + \frac{1}{20}w, $$

for $u, v, w \in X$, and $t \in J$.

Clearly, the function f satisfies condition of Theorem 3.1.

For each $u, v, w, \overline{u}, \overline{v}, \overline{w} \in X$ and $t \in J$.

$$ |f(t,u,v,w) - f(t,\overline{u},\overline{v},\overline{w})| \leq \frac{1}{20}\left(|u-\overline{u}| + |v-\overline{v}|\right) + \frac{1}{20}|w-\overline{w}|. $$

Hence, the condition (C3) is satisfied with $K = L = \frac{1}{20}$. Here $T = e$.

If $\alpha = \frac{2}{3}$, $\beta = \frac{1}{2}$ and choose $\gamma = \frac{5}{6}$.

Thus, condition from (4.1)

$$ \frac{2K}{1-L}\left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))} + \frac{(\log T)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\right) \approx 0.1925 < 1, $$

If $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and choose $\gamma = \frac{3}{4}$.

$$ \frac{2K}{1-L}\left(\frac{\Gamma(2\beta)}{\Gamma(\gamma+2\beta-1)}\frac{(\log T)^{1-\beta(1-\alpha)}}{\Gamma(2-\beta(1-\alpha))} + \frac{(\log T)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\right) \approx 0.2123 < 1, $$

It follows from Lemma 4.7 that the problem (5.1)-(5.2) has a unique solution. Moreover, Theorem 4.8 implies that the problem (5.1)-(5.2) is Ulam-Hyers stable.

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